Exact-Four-Colorability, Exact Domatic Number Problems, and the Boolean Hierarchy*

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Abstract. This paper surveys some of the work that was inspired by Wagner's general technique to prove completeness in the levels of the boolean hierarchy over NP. In particular, we show that it is DP-complete to decide whether or not a given graph can be colored with exactly four colors. DP is the second level of the boolean hierarchy. This result solves a question raised by Wagner in his 1987 TCS paper; its proof uses a clever reduction by Guruswami and Khanna. Similar results on various versions of the exact domatic number problem are also discussed. The result on Exact-Four-Colorability appeared in IPL, 2003, see [1]. The results on exact domatic number problems, obtained jointly with Tobias Riege, are to appear in TOCS, see [2].

Keywords. Exact colorability, exact domatic number, boolean hierarchy completeness

1 Introduction, Historical Notes, and Definitions

In the 1970s, Meyer and Stockmeyer [3,4] noted that the problem

$$\mathtt{MEE} = \left\{ \langle \varphi, k \rangle \;\middle|\; \begin{array}{l} \varphi \text{ is a boolean formula in DNF, } k \geq 0, \text{ and} \\ \text{there exists a boolean formula } \psi \text{ with at} \\ \text{most } k \text{ literals such that } \psi \text{ is equivalent to } \varphi \end{array} \right\}$$

is coNP-hard but seems to be not coNP-complete. Motivated by this observation, they introduced the polynomial hierarchy, which is inductively defined by:

$$\begin{aligned} &\Delta_0^p = \Sigma_0^p = \Pi_0^p = \mathrm{P}; \\ &\Delta_{i+1}^p = \mathrm{P}^{\Sigma_i^p}, \quad \Sigma_{i+1}^p = \mathrm{NP}^{\Sigma_i^p}, \text{ and } \quad \Pi_{i+1}^p = \mathrm{co}\Sigma_{i+1}^p \quad \text{for } i \geq 0; \\ &\mathrm{PH} = \bigcup_{k \geq 0} \Sigma_k^p. \end{aligned}$$

They observed that MEE is in Σ_2^p. Figure 1 shows the inclusion structure of the polynomial hierarchy.

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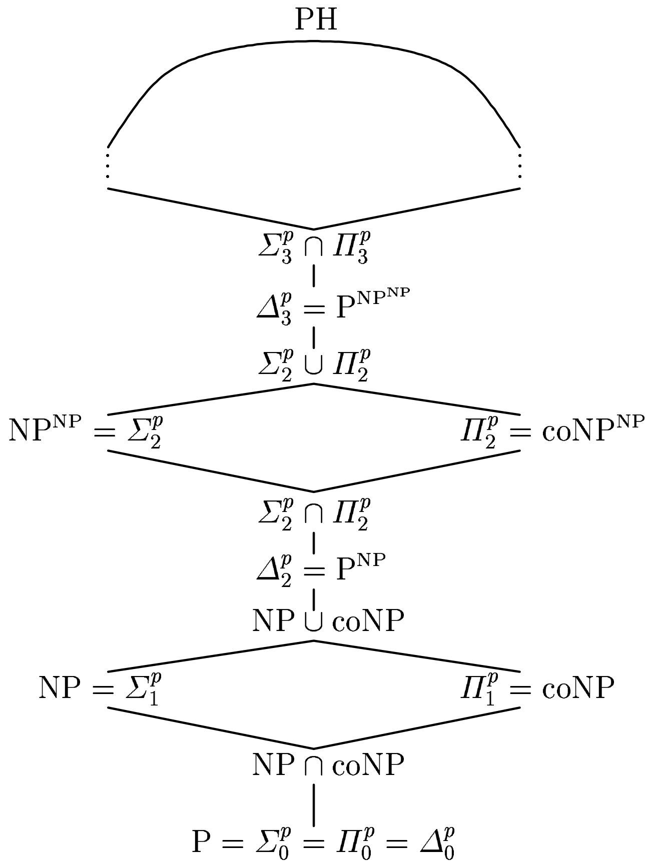

Fig. 1. Polynomial hierarchy

Papadimitriou and Zachos [5] introduced the complexity class $\mathrm{P}^{\mathrm{NP}[\mathcal{O}(\log n)]}$, the class of problems solvable by $\mathcal{O}(\log n)$ sequential Turing queries to NP. Hemaspaandra [6] and Köbler, Schöning, and Wagner [7] proved that $\mathrm{P}^{\mathrm{NP}[\mathcal{O}(\log n)]}$ equals $\mathrm{P}_{||}^{\mathrm{NP}}$, the class of problems solvable by parallel (a.k.a. truth-table) access to NP. Wagner [8] provided about half a dozen other characterizations of this class, and he introduced the notation Θ_2^p for it. By definition, $\mathrm{NP} \subseteq \Theta_2^p \subseteq \Delta_2^p$. It is known that if NP contains some problem that is hard for Θ_2^p, then the polynomial hierarchy collapses to NP. The class Θ_2^p is also closely related to the question of whether NP has sparse Turing-hard sets [9], and to various other topics; see, e.g., [10,11,12].

In the 1980s, Papadimitriou and Yannakakis [13] noted that certain NP-hard and coNP-hard problems seem to be not complete for NP or coNP:

– *Critical Problems* such as `Minimal-3-Uncolorability`: Given a graph G, is it true that G is not 3-colorable but deleting any of its vertices yields a

3-colorable graph?

– *Exact problems* such as `Exact-4-Colorability` (to be defined in Definition 1 below).
– *Unique solution problems* such as `Unique-SAT`: Given a boolean formula, is it true that it has exactly one satisfying assignment?

Motivated by this observation, they introduced the class of differences of NP sets:

$$\mathrm{DP} = \{A - B \mid A, B \in \mathrm{NP}\}.$$

All the above problems are in DP.

For any graph G, $\chi(G)$ is the *chromatic number of* G, i.e., the smallest number of colors needed to legally color G. For each k, define

$$k\texttt{-Colorability} = \{G \mid G \text{ is a graph with } \chi(G) \leq k\}.$$

The problem `2-Colorability` is in P, yet `3-Colorability` is NP-complete, see Stockmeyer [14]. We now define the exact versions of colorability problems.

Definition 1 (Exact Colorability Problems). *Let M_k be a set that consists of k noncontiguous integers, and let t be a positive integer. Define*

$$\texttt{Exact-}M_k\texttt{-Colorability} = \{G \mid G \textit{ is a graph with } \chi(G) \in M_k\},$$
$$\texttt{Exact-}t\texttt{-Colorability} = \{G \mid G \textit{ is a graph with } \chi(G) = t\}.$$

Merging, unifying, and expanding the results that originally were obtained independently by Cai and Hemaspaandra [15] and by Gundermann, Wagner, and Wechsung [16,17], Cai et al. [18,19] generalized DP by introducing the boolean hierarchy over NP. The symbols $\wedge$ and $\vee$, respectively, denote the *complex intersection* and the *complex union* of set classes: $\mathcal{C} \wedge \mathcal{D} = \{A \cap B \mid A \in \mathcal{C} \text{ and } B \in \mathcal{D}\}$ and $\mathcal{C} \vee \mathcal{D} = \{A \cup B \mid A \in \mathcal{C} \text{ and } B \in \mathcal{D}\}$.

Definition 2 (Boolean Hierarchy over NP). *The* boolean hierarchy over NP *is inductively defined by:*

$$\mathrm{BH}_0(\mathrm{NP}) = \mathrm{P}, \quad \mathrm{BH}_1(\mathrm{NP}) = \mathrm{NP}, \quad \mathrm{BH}_2(\mathrm{NP}) = \mathrm{NP} \wedge \mathrm{coNP} = \mathrm{DP},$$
$$\mathrm{BH}_k(\mathrm{NP}) = \mathrm{BH}_{k-2}(\mathrm{NP}) \vee \mathrm{BH}_2(\mathrm{NP}) \quad \textit{for } k \geq 3, \textit{ and}$$
$$\mathrm{BH}(\mathrm{NP}) = \bigcup_{k \geq 1} \mathrm{BH}_k(\mathrm{NP}).$$

Figure 2 shows the inclusion structure of the boolean hierarchy. Note further that $\mathrm{BH}(\mathrm{NP}) \subseteq \Theta_2^p \subseteq \Delta_2^p \subseteq \Sigma_2^p \subseteq \mathrm{PH}$. Kadin [20] was the first to show that a collapse of the boolean hierarchy implies a collapse of the polynomial hierarchy.

Theorem 1 (Kadin). *If $\mathrm{BH}_k(\mathrm{NP}) = \mathrm{coBH}_k(\mathrm{NP})$ for some $k \geq 1$, then the polynomial hierarchy collapses down to its third level:* $\mathrm{PH} = \Sigma_3^p \cap \Pi_3^p$.

The collapse consequence of Theorem 1 has been strenghtened later on; see the survey by Hemaspaandra, Hemaspaandra, and Hempel [21].

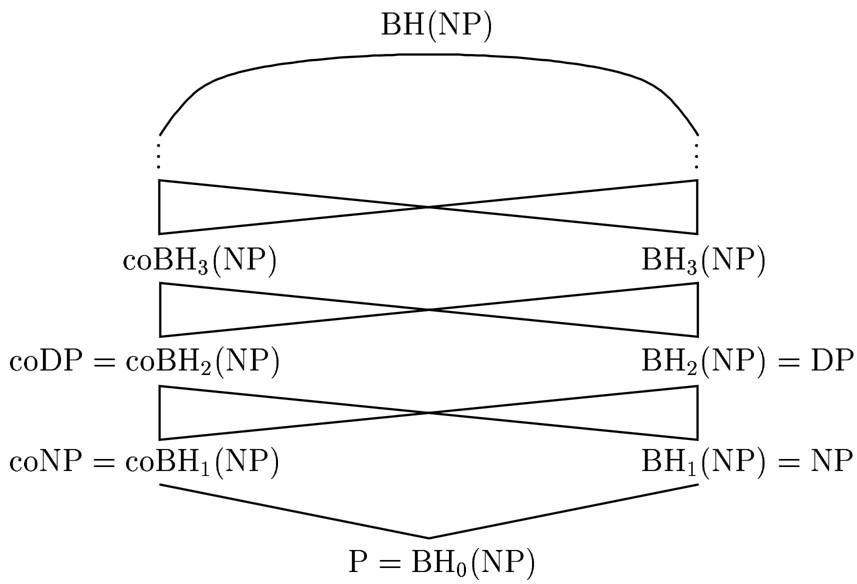

Fig. 2. Boolean hierarchy over NP

2 Some Results Obtained by Wagner's Technique

Wagner [22] established conditions sufficient to prove hardness for Θ_2^p and for the levels of the boolean hierarchy over NP. We first state his sufficient condition for proving Θ_2^p-hardness.

Lemma 1 (Wagner). *Let A be some NP-complete set, and let B be any set. If there exists a polynomial-time computable function g such that for all $\varphi_1, \ldots, \varphi_k$ in Σ^* with $(\forall j : 1 \leq j < k)\,[\varphi_{j+1} \in A \implies \varphi_j \in A]$ it holds that*

$$||\{i \mid \varphi_i \in A\}|| \text{ is odd } \iff g(\varphi_1, \ldots, \varphi_k) \in B, \tag{2.1}$$

then B is Θ_2^p-hard.

Using Lemma 1, Wagner proved dozens of problems Θ_2^p-complete, including the following variants of the colorability problem:

$$\begin{aligned}
\texttt{Color}_{\texttt{odd}} &= \{G \mid G \text{ is a graph such that } \chi(G) \text{ is odd}\},\\
\texttt{Color}_{\texttt{equ}} &= \{\langle G, H\rangle \mid G \text{ and } H \text{ are graphs with } \chi(G) = \chi(H)\},\\
\texttt{Color}_{\texttt{leq}} &= \{\langle G, H\rangle \mid G \text{ and } H \text{ are graphs with } \chi(G) \leq \chi(H)\}.
\end{aligned}$$

Wagner's technique has been applied to prove further natural problems, which arise in a variety of contexts, Θ_2^p-hard or even Θ_2^p-complete. For example, Lemma 1 was useful in determining the complexity of the winner problem for certain voting systems, including Carroll elections [23], Young elections [24], and

Kemeny elections [25,26]; see Hemaspaandra and Hemaspaandra [25] for more background on computational politics. Wagner's technique was also useful for showing that recognizing those graphs for which certain efficient approximation heuristics for the independent set and the vertex cover problem do well is Θ_2^p-complete [27,28]; see also the survey [29]. Moreover, Lemma 1 is the key lemma for raising the trivial coNP-hardness of `MEE` to Θ_2^p-hardness, see Hemaspaandra and Wechsung [30]. Note that Umans [31] proved this problem even Σ_2^p-complete using a different technique.

In what follows, we focus on completeness for exact colorability and exact domatic number problems in the even levels of the boolean hierarchy. The following lemma, which is also due to Wagner [22], is the key lemma to establish these results.

Lemma 2 (Wagner). *Let A be some NP-complete set, let B be any set, and let $k \geq 1$ be fixed. If there exists a polynomial-time computable function g such that for all $\varphi_1, \ldots, \varphi_{2k}$ in Σ^* with $(\forall j : 1 \leq j < 2k)\,[\varphi_{j+1} \in A \implies \varphi_j \in A]$ it holds that*

$$||\{i \mid \varphi_i \in A\}|| \text{ is odd } \iff g(\varphi_1, \ldots, \varphi_{2k}) \in B, \tag{2.2}$$

then B is $\mathrm{BH}_{2k}(\mathrm{NP})$-hard.

3 Exact Colorability Problems

In this section, we turn to the exact colorability problems defined in Definition 1. Using Lemma 2, Wagner [22] proved the following result.

Theorem 2 (Wagner). `Exact-`M_k`-Colorability` *is $\mathrm{BH}_{2k}(\mathrm{NP})$-complete for $M_k = \{6k+1, 6k+3, \ldots, 8k-1\}$. In particular, for $k = 1$, it is DP-complete to determine whether or not $\chi(G) = 7$.*

In [22], Wagner raised the following questions: How small can the numbers in a k-element set M_k be chosen so as to ensure that `Exact-`M_k`-Colorability` still is $\mathrm{BH}_{2k}(\mathrm{NP})$-complete? In particular, for $k = 1$, is it DP-complete to determine whether or not $\chi(G) = 4$? That is, for which threshold $t \in \{4, 5, 6, 7\}$ exactly does `Exact-`t`-Colorability` jump from NP to DP-complete? These questions have been answered recently in [1]. Note that `Exact-3-Colorability` is in NP and thus cannot be DP-complete, unless the boolean hierarchy over NP (and by Theorem 1 the polynomial hierarchy as well) collapses.

Theorem 3 (Rothe). `Exact-`M_k`-Colorability` *is $\mathrm{BH}_{2k}(\mathrm{NP})$-complete for $M_k = \{3k+1, 3k+3, \ldots, 5k-1\}$. In particular, for $k = 1$, it is DP-complete to determine whether or not $\chi(G) = 4$.*

A proof sketch for Theorem 3 is presented in the remainder of this section. Crucially, this proof uses:

– Wagner's tool for proving $\mathrm{BH}_{2k}(\mathrm{NP})$-hardness stated as Lemma 2 above,

– the standard reduction σ from 3-SAT to 3-Colorability satisfying

$$\varphi \in \texttt{3-SAT} \implies \chi(\sigma(\varphi)) = 3, \tag{3.3}$$
$$\varphi \notin \texttt{3-SAT} \implies \chi(\sigma(\varphi)) = 4, \tag{3.4}$$

– and Guruswami and Khanna's reduction ρ from 3-SAT to 3-Colorability satisfying

$$\varphi \in \texttt{3-SAT} \implies \chi(\rho(\varphi)) = 3, \tag{3.5}$$
$$\varphi \notin \texttt{3-SAT} \implies \chi(\rho(\varphi)) = 5. \tag{3.6}$$

Among the above three items, the Guruswami–Khanna reduction is the technically most challenging one. Originally, Guruswami and Khanna's seminal result is not motivated by the issue of proving the hardness of exact colorability. Rather, it was motivated by issues related to the hardness of approximating the chromatic number of 3-colorable graphs. Intuitively, their result says that it is NP-hard to 4-color a 3-colorable graph. This result had been obtained earlier on by Khanna, Linial, and Safra [32] using the PCP theorem, which is due to Arora, Lund, Motwani, Sudan, and Szegedy [33]. Guruswami and Khanna [34] gave a novel proof of this result, which does not rely on the PCP theorem. Their direct transformation in fact consists of the following two subsequent reductions:

$$\texttt{3-SAT} \leq_{\mathrm{m}}^{\mathrm{p}} \texttt{IS} \leq_{\mathrm{m}}^{\mathrm{p}} \texttt{3-Colorability},$$

where IS is the independent set problem. Figure 3 shows the standard reduction 3-SAT $\leq_{\mathrm{m}}^{\mathrm{p}}$ IS, for the specific formula

$$\varphi(x, y, z) = (x \vee y \vee z) \wedge (\neg x \vee y \vee z) \wedge (\neg x \vee y \vee \neg z) \wedge (x \vee \neg y \vee z).$$

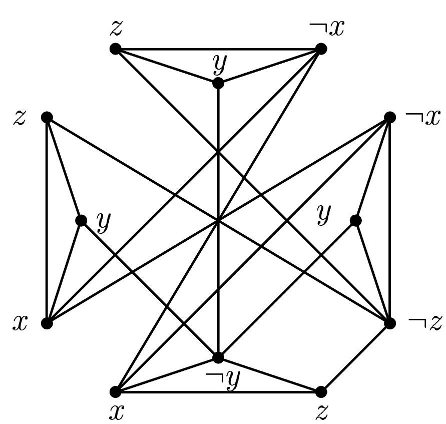

Fig. 3. Graph G in the reduction 3-SAT $\leq_{\mathrm{m}}^{\mathrm{p}}$ IS

Clauses in the formula correspond to triangles in the graph constructed, and corners of two distinct triangles are connected by an edge if and only if they correspond to some literal and its negation. Suppose the given formula has m clauses, and denote the corresponding m triangles in G by $T_1, T_2, \ldots, T_m$. To each T_i in G, there corresponds a tree-like structure S_i shown in Figure 4:

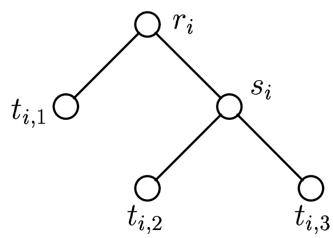

Fig. 4. Tree-like structure S_i in the Guruswami–Khanna reduction

The three "leaves" $t_{i,1}$, $t_{i,2}$, and $t_{i,3}$ in S_i correspond to the three corners of the triangle T_i. Every "vertex" of S_i has the form of the basic template, which is a 3×3 grid such that the vertices in each row and column induce a 3-clique as shown in Figure 5:

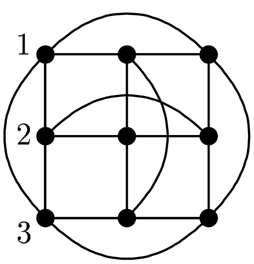

Fig. 5. Basic template in the Guruswami–Khanna reduction

The "ground vertices" in the first column of any such basic template in fact are shared among all basic templates in each of the tree-like structures. Since these "ground vertices" form a 3-clique, every legal coloring assigns three distinct colors to them, say 1, 2, and 3.

Figure 6 shows the connection pattern between the "vertices" r_i, $t_{i,1}$, and s_i of S_i and two additional triangles. An analogous pattern applies to s_i, $t_{i,2}$, and $t_{i,3}$. Every vertex of the templates and the triangles is labeled by a triple of colors, and the vertices are connected according to the following simple rule: Two vertices are adjacent if and only if their labels differ in each coordinate.

The intuition of how to connect S_i and S_j, for distinct i and j, is as follows. A "vertex" in some S_i (with respect to some coloring) is said to be *selected* if

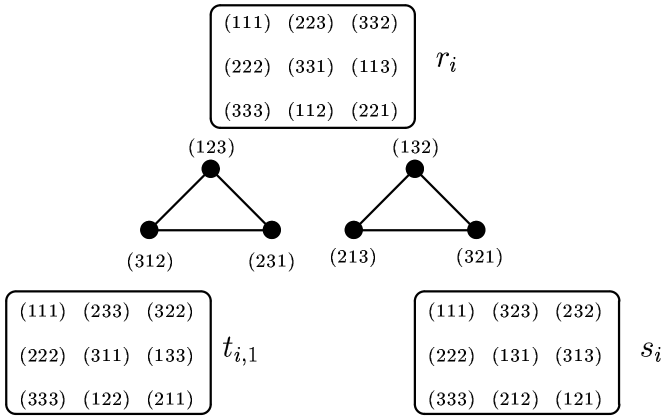

Fig. 6. Connection pattern between the templates of a tree-like structure

and only if at least one of the three rows in its basic template receives colors that form an even permutation of $\{1, 2, 3\}$. That is, a "vertex" is selected if and only if

- the first row has colors $1, 2, 3$ from left to right, or
- the second row has colors $2, 3, 1$, or
- the third row has colors $3, 1, 2$.

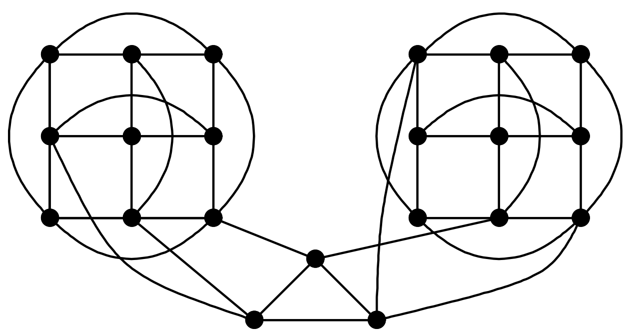

Fig. 7. Gadget connecting two "leaves" of the "same row" kind

Clearly, for each legal 4-coloring of S_i, every "vertex" is either selected or not selected. For each pair of "vertices," $t_{i,k}$ and $t_{j,\ell}$, that are adjacent in graph G, appropriate gadgets are inserted to prevent that both these "leaves" are selected simultaneously. (This is necessary, since otherwise any 4-coloring of the graph constructed would imply that G has an independent set of size m.) To this end, two kinds of gadgets are used, the *"same row" gadget* and the *"different rows" gadget*. Figure 7 shows the *"same row" gadget*, which prevents that $t_{i,k}$ and $t_{j,\ell}$ are simultaneously selected because of the same row. Figure 8 shows the *"different rows" gadget*, which prevents that $t_{i,k}$ and $t_{j,\ell}$ are selected simultaneously because of different rows.

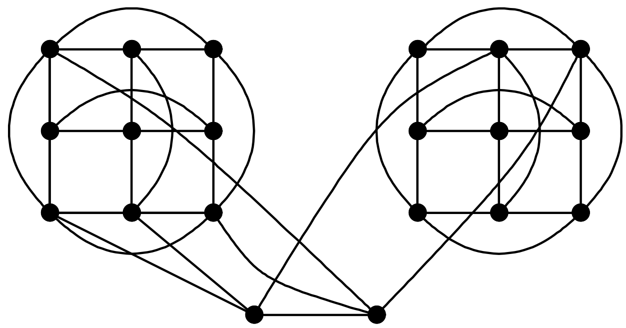

Fig. 8. Gadget connecting two "leaves" of the "different rows" kind

This completes the reduction ρ that transforms the formula φ via graph G to graph $H = \rho(\varphi)$. We omit the detailed argument of why this reduction works to prove (3.5) and (3.6), referring to [34] instead. We merely mention that it can be shown that:

(a) For each i with $1 \leq i \leq m$, there exists a 3-coloring of the vertices in S_i such that exactly one of the three "leaves" $t_{i,1}$, $t_{i,2}$, and $t_{i,3}$ is selected.

(b) Every legal 4-coloring of S_i selects at least one of $t_{i,1}$, $t_{i,2}$, or $t_{i,3}$.

Implications (3.5) and (3.6) follow from (a) and (b).

Note that Guruswami and Khanna claimed in their conference paper [34] that $\varphi \notin$ 3-SAT implies $5 \leq \chi(H) \leq 6$. However, as has been observed in [1], the Guruswami–Khanna reduction even yields the stronger implication (3.6), which we need in order to apply Wagner's Lemma 2.

We are now ready to apply Lemma 2 with $k = 1$, $A =$ 3-SAT, and $B =$ Exact-4-Colorability. Given two formulas φ_1 and φ_2 satisfying

$$\varphi_2 \in \text{3-SAT} \implies \varphi_1 \in \text{3-SAT}, \tag{3.7}$$

define the graphs $H_1 = \rho(\varphi_1)$ and $H_2 = \sigma(\varphi_2)$, where ρ is the Guruswami–Khanna reduction, which satisfies (3.5) and (3.6), and σ is the standard reduction from 3-SAT to 3-Colorability, which satisfies (3.3) and (3.4).

Let D be the disjoint union of H_1 and H_2. Thus,

$$\chi(D) = \max\{\chi(H_1), \chi(H_2)\}.$$

Consider the following three cases:

– If $\varphi_1 \in$ 3-SAT and $\varphi_2 \in$ 3-SAT, then $\chi(\varphi_1) = 3$ and $\chi(\varphi_2) = 3$, so $\chi(D) = 3$.
– If $\varphi_1 \in$ 3-SAT and $\varphi_2 \notin$ 3-SAT, then $\chi(\varphi_1) = 3$ and $\chi(\varphi_2) = 4$, so $\chi(D) = 4$.
– If $\varphi_1 \notin$ 3-SAT and $\varphi_2 \notin$ 3-SAT, then $\chi(\varphi_1) = 5$ and $\chi(\varphi_2) = 4$, so $\chi(D) = 5$.

By (3.7), the case distinction is complete. It follows that (2.2) is satisfied. By Lemma 2, Exact-4-Colorability is DP-hard. Since Exact-4-Colorability is in DP, it is DP-complete. Completeness of Exact-M_k-Colorability in $\mathrm{BH}_{2k}(\mathrm{NP})$ for the k-element set $M_k = \{3k + 1, 3k + 3, \ldots, 5k - 1\}$ is proven analogously.

4 Exact Domatic Number Problems

For any graph G, a *dominating set of* G is a subset $D \subseteq V(G)$ such that each vertex $u \in V(G) - D$ is adjacent to some vertex $v \in D$. Let $\delta(G)$ denote the *domatic number of* G, i.e., the maximum number of disjoint dominating sets. For each k, define the problem

$$k\text{-DNP} = \{G \mid G \text{ is a graph with } \delta(G) \geq k\}.$$

It is known that 3-DNP is NP-complete, whereas 2-DNP is in P; see Garey and Johnson [35]. As suggested by Gasarch during the talk, there is no need to present any motivation for the domatic number problem. We thus omit the motivation and merely mention that this problem is related, for example, to the tasks of distributing resources in a computer network or of locating facilities in a communication network. More details can be found in, e.g., [36,2]. We now define the exact versions of domatic number problems.

Definition 3 (Exact Domatic Number Problems). *Let M_k be a set that consists of k noncontiguous integers, and let t be a positive integer. Define*

$$\text{Exact-}M_k\text{-DNP} = \{G \mid G \textit{ is a graph with } \delta(G) \in M_k\},$$
$$\text{Exact-}t\text{-DNP} = \{G \mid G \textit{ is a graph with } \delta(G) = t\}.$$

4.1 A General Framework for Dominating Set Problems

In order to investigate exact domatic number problems, we adopt Heggernes and Telle's general, uniform approach to define graph problems by partitioning the vertex set of a graph into generalized dominating sets [37]. These are subsets of the vertex set of a given graph, parameterized by two sets of nonnegative integers, σ and ρ, which restrict the number of neighbors for each vertex in the partition. Let $\mathbb{N} = \{0, 1, 2, \ldots\}$ denote the set of nonnegative integers, and let $\mathbb{N}^+ = \{1, 2, 3, \ldots\}$ denote the set of positive integers.

Definition 4 (Heggernes and Telle). *Let G be a given graph, let $\sigma \subseteq \mathbb{N}$ and $\rho \subseteq \mathbb{N}$ be given sets, and let $k \in \mathbb{N}^+$. Let $N(v) = \{w \in V(G) \mid \{v, w\} \in E(G)\}$ be the neighborhood of any vertex v in G.*

1. *A subset $U \subseteq V(G)$ of the vertices of G is said to be a* (σ, ρ)-set *if and only if*
   - *for each $u \in U$, $||N(u) \cap U|| \in \sigma$, and*
   - *for each $u \notin U$, $||N(u) \cap U|| \in \rho$.*
2. *A* (k, σ, ρ)-partition *of G is a partition of $V(G)$ into k pairwise disjoint subsets $V_1, V_2, \ldots, V_k$ such that V_i is a (σ, ρ)-set for each i, $1 \leq i \leq k$.*
3. *Define the problem*

$$(k, \sigma, \rho)\text{-}\mathtt{Partition} = \{G \mid G \textit{ is a graph that has a } (k, \sigma, \rho)\textit{-partition}\}.$$

Note that $(k, \{0\}, \mathbb{N})$-`Partition` is nothing other than k-`Colorability`, and $(k, \mathbb{N}, \mathbb{N}^+)$-`Partition` is nothing other than k-`DNP`. This observation is illustrated by the following example. Note further that $(k, \{0\}, \mathbb{N})$-`Partition` is a minimum problem, whereas $(k, \mathbb{N}, \mathbb{N}^+)$-`Partition` is a maximum problem.

Example 1 (Generalized Dominating Sets). Figure 9 shows two copies of some graph G with five vertices. Vertices labeled by the same number belong to the same (σ, ρ)-set, where either $\sigma = \{0\}$ and $\rho = \mathbb{N}$ (i.e., k-`Colorability`), or $\sigma = \mathbb{N}$ and $\rho = \mathbb{N}^+$ (i.e., k-`DNP`). According to the partition into (σ, ρ)-sets shown on the left-hand side of Figure 9, G is in $(4, \{0\}, \mathbb{N})$-`Partition`. That is, G is a 4-colorable graph and the partition indicated corresponds to the four color classes of G. In contrast, the partition into (σ, ρ)-sets on the right-hand side of Figure 9 shows that G is in $(3, \mathbb{N}, \mathbb{N}^+)$-`Partition`. That is, G has a domatic number of 3.

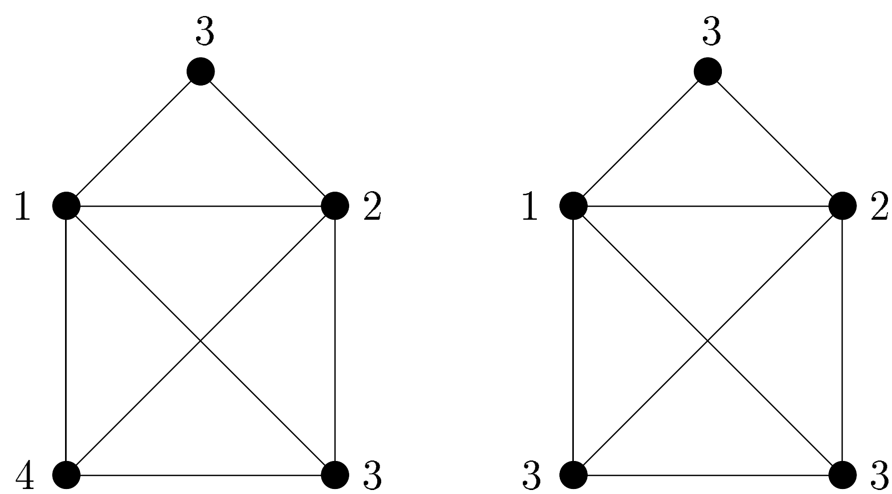

Fig. 9. A graph in $(4, \{0\}, \mathbb{N})$-`Partition` (left) and $(3, \mathbb{N}, \mathbb{N}^+)$-`Partition` (right)

4.2 Summary of Results and Proof Ideas

Heggernes and Telle [37] obtained the NP-completeness results for the problems (k, σ, ρ)-`Partition` that are shown in Table 1. Here is the key: Table 1 gives the smallest value of k for which (k, σ, ρ)-`Partition` is NP-complete. Here, "∞" means that this problem is efficiently solvable for all values of k; a superscript "+" indicates a *maximum problem*: For all $k \geq 1$,

$$(k+1, \sigma, \rho)\text{-}\mathtt{Partition} \subseteq (k, \sigma, \rho)\text{-}\mathtt{Partition};$$

and a superscript "−" indicates a *minimum problem*: For all $k \geq 1$,

$$(k, \sigma, \rho)\text{-}\mathtt{Partition} \subseteq (k+1, \sigma, \rho)\text{-}\mathtt{Partition}.$$

σ \ ρ	$\mathbb{N}$	$\mathbb{N}^+$	$\{1\}$	$\{0,1\}$
$\mathbb{N}$	∞^-	3^+	2	∞^-
$\mathbb{N}^+$	∞^-	2^+	2	∞^-
$\{1\}$	2^-	2	3	3^-
$\{0,1\}$	2^-	2	3	3^-
$\{0\}$	3^-	3	4	4^-

Table 1. NP-completeness for the problems (k, σ, ρ)-`Partition`

We now define the exact versions of generalized dominating set problems.

Definition 5. *Define* `Exact`-(k, σ, ρ)-`Partition`, *the* exact version of the problem (k, σ, ρ)-`Partition`, *to be either*

- (k, σ, ρ)-`Partition`$\cap\overline{(k-1, \sigma, \rho)\text{-}\mathtt{Partition}}$ *if* $k \geq 2$ *and* (k, σ, ρ)-`Partition` *is a minimum problem, or*
- (k, σ, ρ)-`Partition`$\cap\overline{(k+1, \sigma, \rho)\text{-}\mathtt{Partition}}$ *if* $k \geq 1$ *and* (k, σ, ρ)-`Partition` *is a maximum problem.*

Note that all `Exact`-(k, σ, ρ)-`Partition` problems are in DP. Note further that `Exact`-$(k, \{0\}, \mathbb{N})$-`Partition` is nothing other than `Exact`-k-`Colorability`, and `Exact`-$(k, \mathbb{N}, \mathbb{N}^+)$-`Partition` is nothing other than `Exact`-k-`DNP`. Table 2 gives the best known values of $j \mid k$ for which `Exact`-(k, σ, ρ)-`Partition` is (NP-complete or coNP-complete) | DP-complete. Again, "∞" means that this problem is efficiently solvable for all values of k. Here, a dash "—" indicates that this problem is neither a maximum nor a minimum problem and thus is not considered.

Except the DP-completeness of `Exact`-$(k, \{0\}, \mathbb{N})$-`Partition`, which is proven in [1] (see Theorem 3), all DP-completeness results in Table 2 are due to Riege and Rothe [2]. We state the results from Table 2 in Theorem 4 below and provide the proof ideas. We do not attempt to give full, detailed proofs, though, referring to the original source [2] instead.

σ \ ρ	$\mathbb{N}$	$\mathbb{N}^+$	$\{0,1\}$
$\mathbb{N}$	∞	$2 \mid 5$	∞
$\mathbb{N}^+$	∞	$1 \mid 3$	∞
$\{1\}$	$2 \mid 5$	—	$3 \mid ?$
$\{0,1\}$	$2 \mid 5$	—	$3 \mid ?$
$\{0\}$	$3 \mid 4$	—	$4 \mid ?$

Table 2. DP-completeness for the problems $\texttt{Exact-}(k,\sigma,\rho)\texttt{-Partition}$

Theorem 4 (Riege and Rothe).

1. *For each* $i \geq 5$, $\texttt{Exact-}i\texttt{-DNP} = \texttt{Exact-}(i,\mathbb{N},\mathbb{N}^+)\texttt{-Partition}$ *is* DP*-complete. In contrast,* $\texttt{Exact-2-DNP} = \texttt{Exact-}(2,\mathbb{N},\mathbb{N}^+)\texttt{-Partition}$ *is* coNP*-complete.*
2. *For each* $i \geq 3$, $\texttt{Exact-}(i,\mathbb{N}^+,\mathbb{N}^+)\texttt{-Partition}$ *is* DP*-complete. In contrast,* $\texttt{Exact-}(1,\mathbb{N}^+,\mathbb{N}^+)\texttt{-Partition}$ *is* coNP*-complete.*
3. *For each* $i \geq 5$, $\texttt{Exact-}(i,\{0,1\},\mathbb{N})\texttt{-Partition}$ *is* DP*-complete. In contrast,* $\texttt{Exact-}(2,\{0,1\},\mathbb{N})\texttt{-Partition}$ *is* NP*-complete.*
4. *For each* $i \geq 5$, $\texttt{Exact-}(i,\{1\},\mathbb{N})\texttt{-Partition}$ *is* DP*-complete. In contrast,* $\texttt{Exact-}(2,\{1\},\mathbb{N})\texttt{-Partition}$ *is* NP*-complete.*

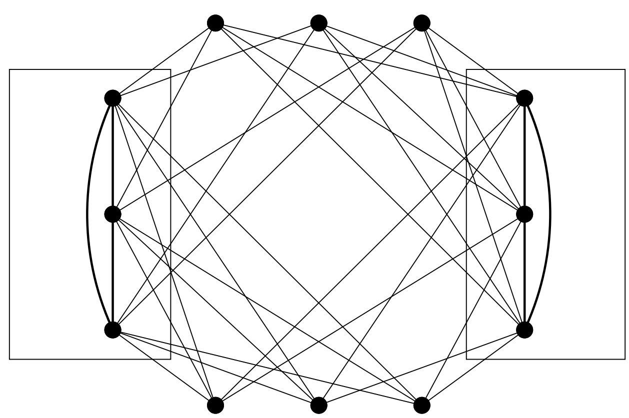

Fig. 10. Gadget for proving $\texttt{Exact-5-DNP}$ DP-complete

The proof of the first part of Theorem 4 uses the gadget shown in Figure 10 to provide a reduction from 3-Colorability that satisfies the hypothesis (2.2) of

Wagner's Lemma 2. The construction in Figure 10 extends Kaplan and Shamir's reduction from 3-`Colorability` to 3-`DNP` with useful properties [38]; see also [2].

The proof of the second part of Theorem 4 uses the gadget shown in Figure 11 to provide a reduction from `NAE-3-SAT` that satisfies the hypothesis (2.2) of Wagner's Lemma 2. The problem `NAE-3-SAT` ("not-all-equal satisfiability") asks whether a given boolean formula φ can be satisfied such that in none of the clauses of φ all literals are true. Schaefer proved that `NAE-3-SAT` is NP-complete [39]. The construction in Figure 11 is inspired by Heggernes and Telle's reduction from `NAE-3-SAT` to $(2, \mathbb{N}^+, \mathbb{N}^+)$-`Partition`; see [37] and also [2].

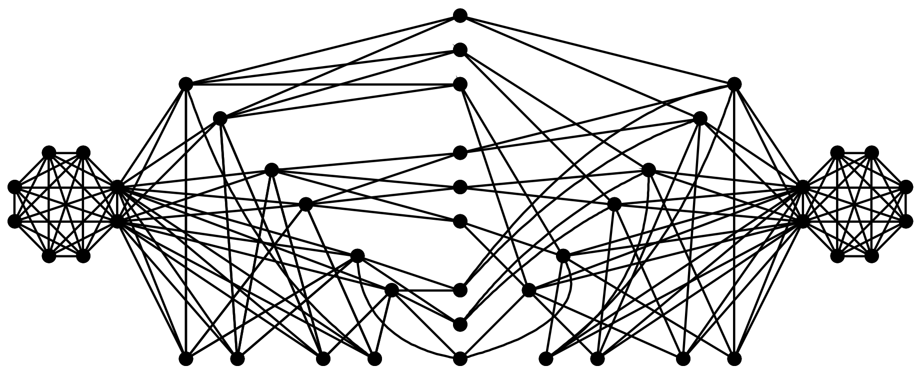

Fig. 11. Gadget for proving `Exact`-$(3, \mathbb{N}^+, \mathbb{N}^+)$-`Partition` DP-complete

The proof of the third part of Theorem 4 uses a reduction from 1-3-`SAT` that satisfies the hypothesis (2.2) of Wagner's Lemma 2. The problem 1-3-`SAT` ("one-in-three satisfiability") asks whether, given a boolean formula φ, there exists a subset T of the literals of φ with $||T \cap C_i|| = 1$ for each clause C_i. Schaefer proved that 1-3-`SAT` is NP-complete, even if all literals in the given boolean formula are positive [39].

Figure 12 shows this construction, which is based on Heggernes and Telle's reduction from 1-3-`SAT` to $(2, \{0, 1\}, \mathbb{N})$-`Partition`, see [37]. The symbol $\oplus$ in Figure 12 denotes the *join operation on graphs*, i.e., for any two graphs G_1 and G_2, $G_1 \oplus G_2$ is the graph with vertex set $V(G_1 \oplus G_2) = V(G_1) \cup V(G_2)$ and edge set $E(G_1 \oplus G_2) = E(G_1) \cup E(G_2) \cup \{\{a, b\} \mid a \in V(G_1) \text{ and } b \in V(G_2)\}$.

The proof of the fourth part of Theorem 4 is obtained by suitably modifying the proof of the third part of Theorem 4.

Generalizing the results on exact generalized dominating set problems from Theorem 4, we obtain completeness results in the higher levels of the boolean hierarchy. In Theorem 5, we state this generalization for the problem `Exact`-M_k-`DNP` only, where $M_k = \{4k+1, 4k+3, \ldots, 6k-1\}$. Analogously, the completeness

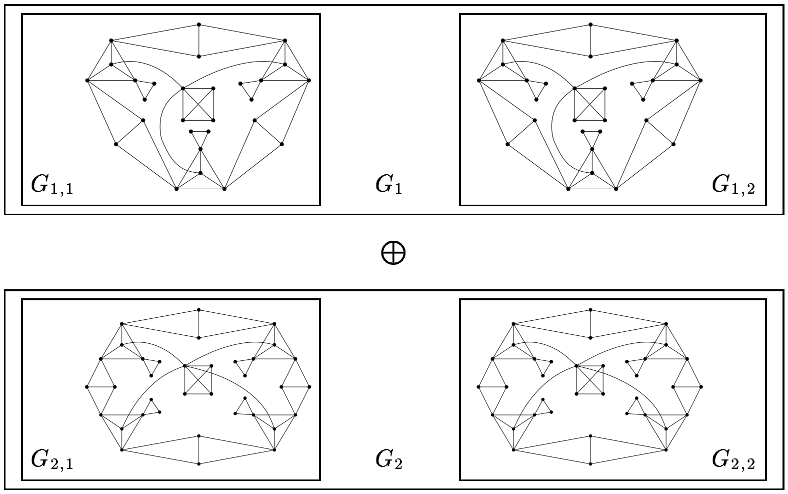

Fig. 12. Reduction to prove Exact-$(5, \{0, 1\}, \mathbb{N})$-Partition DP-complete

results for Exact-(k, σ, ρ)-Partition stated in the second, third, and fourth part of Theorem 4 can be lifted to the higher levels of the boolean hierarchy over NP.

Theorem 5 (Riege and Rothe). *Exact-M_k-DNP is $\mathrm{BH}_{2k}(\mathrm{NP})$-complete for $M_k = \{4k+1, 4k+3, \ldots, 6k-1\}$.*

Finally, define the following variants of the domatic number problem:

$$\mathtt{DNP}_{\mathtt{odd}} = \{G \mid G \text{ is a graph such that } \delta(G) \text{ is odd}\},$$
$$\mathtt{DNP}_{\mathtt{equ}} = \{\langle G, H\rangle \mid G \text{ and } H \text{ are graphs with } \delta(G) = \delta(H)\},$$
$$\mathtt{DNP}_{\mathtt{leq}} = \{\langle G, H\rangle \mid G \text{ and } H \text{ are graphs with } \delta(G) \leq \delta(H)\}.$$

Theorem 6 (Riege and Rothe). *$\mathtt{DNP}_{\mathtt{odd}}$, $\mathtt{DNP}_{\mathtt{equ}}$, and $\mathtt{DNP}_{\mathtt{leq}}$ each are Θ_2^p-complete.*

5 Conclusions and Open Questions

This survey paper has presented some of the results that were inspired by Wagner's general technique [22] to prove completeness in the levels of the boolean hierarchy over NP and in Θ_2^p, the class of problems solvable via parallel access to NP. In particular, Θ_2^p-completeness results were obtained for a variety of natural problems arising in computational politics [23,24,25,26] and for problems related to certain approximation heuristics for hard graph problems [27,28,29]. In addition, Wagner's technique was useful to prove Θ_2^p-hardness of MEE, the minimum equivalent expression problem, see Hemaspaandra and Wechsung [30] and also Umans [31].

Turning to completeness in the levels of the boolean hierarchy, Theorem 3 answered a question raised by Wagner in [22]: It is DP-complete to decide whether or not a given graph can be colored with exactly four colors. Guruswami and Khanna's clever reduction [34] that is central to this proof was sketched, and it was shown how this reduction can be employed by Wagner's technique.

Theorem 4 in particular has shown that `Exact-5-DNP` is DP-complete. In contrast, `Exact-2-DNP` is coNP-complete, and thus this problem cannot be DP-complete unless the boolean hierarchy collapses. For $i \in \{3,4\}$, the question of whether or not the problems Exact-i-DNP are DP-complete remains open. To close this gap, one would have to find a reduction from some NP-complete problem to the exact domatic number problem that yields graphs having never a domatic number of three.

In addition, we have studied the exact versions of Heggernes and Telle's generalized dominating set problems [37], denoted by Exact-(k,σ,ρ)-Partition, where the parameters σ and ρ specify the number of neighbors that are allowed for each vertex in the partition. Theorem 4 presented DP-completeness results for a number of such problems that are summarized in Table 2, which gives the best values of k for which the problems Exact-(k,σ,ρ)-Partition are known to be DP-complete. This value of k is not yet optimal in many cases. For example, by Theorem 4, Exact-$(5,\{0,1\},\mathbb{N})$-Partition is DP-complete and Exact-$(2,\{0,1\},\mathbb{N})$-Partition is NP-complete. What about the complexity of Exact-$(i,\{0,1\},\mathbb{N})$-Partition for $i \in \{3,4\}$? It would also be interesting to obtain DP-completeness results for those cases in Table 2 that currently have only question marks.

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References

1. Rothe, J.: Exact complexity of Exact-Four-Colorability. Information Processing Letters **87** (2003) 7–12
2. Riege, T., Rothe, J.: Complexity of the exact domatic number problem and of the exact conveyor flow shop problem. Theory of Computing Systems (2004) On-line publication DOI 10.1007/s00224-004-1209-8. Paper publication to appear.
3. Meyer, A., Stockmeyer, L.: The equivalence problem for regular expressions with squaring requires exponential space. In: Proceedings of the 13th IEEE Symposium on Switching and Automata Theory. (1972) 125–129
4. Stockmeyer, L.: The polynomial-time hierarchy. Theoretical Computer Science **3** (1977) 1–22

5. Papadimitriou, C., Zachos, S.: Two remarks on the power of counting. In: Proceedings of the 6th GI Conference on Theoretical Computer Science, Springer-Verlag *Lecture Notes in Computer Science #145* (1983) 269–276
6. Hemachandra, L.: The strong exponential hierarchy collapses. In: Proceedings of the 19th ACM Symposium on Theory of Computing, ACM Press (1987) 110–122
7. Köbler, J., Schöning, U., Wagner, K.: The difference and truth-table hierarchies for NP. R.A.I.R.O. Informatique théorique et Applications **21** (1987) 419–435
8. Wagner, K.: Bounded query classes. SIAM Journal on Computing **19** (1990) 833–846
9. Kadin, J.: $\mathrm{P}^{\mathrm{NP}[\log n]}$ and sparse Turing-complete sets for NP. Journal of Computer and System Sciences **39** (1989) 282–298
10. Long, T., Sheu, M.: A refinement of the low and high hierarchies. Mathematical Systems Theory **28** (1995) 299–327
11. Krentel, M.: The complexity of optimization problems. Journal of Computer and System Sciences **36** (1988) 490–509
12. Hemachandra, L., Wechsung, G.: Kolmogorov characterizations of complexity classes. Theoretical Computer Science **83** (1991) 313–322
13. Papadimitriou, C., Yannakakis, M.: The complexity of facets (and some facets of complexity). Journal of Computer and System Sciences **28** (1984) 244–259
14. Stockmeyer, L.: Planar 3-colorability is NP-complete. SIGACT News **5** (1973) 19–25
15. Cai, J., Hemachandra, L.: The boolean hierarchy: Hardware over NP. In: Proceedings of the 1st Structure in Complexity Theory Conference. Springer-Verlag *Lecture Notes in Computer Science #223* (1986) 105–124
16. Wechsung, G.: On the boolean closure of NP. In: Proceedings of the 5th Conference on Fundamentals of Computation Theory, Springer-Verlag *Lecture Notes in Computer Science #199* (1985) 485–493 (An unpublished precursor of this paper was coauthored by K. Wagner).
17. Gundermann, T., Wechsung, G.: Counting classes with finite acceptance types. Computers and Artificial Intelligence **6** (1987) 395–409
18. Cai, J., Gundermann, T., Hartmanis, J., Hemachandra, L., Sewelson, V., Wagner, K., Wechsung, G.: The boolean hierarchy I: Structural properties. SIAM Journal on Computing **17** (1988) 1232–1252
19. Cai, J., Gundermann, T., Hartmanis, J., Hemachandra, L., Sewelson, V., Wagner, K., Wechsung, G.: The boolean hierarchy II: Applications. SIAM Journal on Computing **18** (1989) 95–111
20. Kadin, J.: The polynomial time hierarchy collapses if the boolean hierarchy collapses. SIAM Journal on Computing **17** (1988) 1263–1282 Erratum appears in the same journal, 20(2):404, 1991.
21. Hemaspaandra, E., Hemaspaandra, L., Hempel, H.: What's up with downward collapse: Using the easy-hard technique to link boolean and polynomial hierarchy collapses. SIGACT News **29** (1998) 10–22
22. Wagner, K.: More complicated questions about maxima and minima, and some closures of NP. Theoretical Computer Science **51** (1987) 53–80
23. Hemaspaandra, E., Hemaspaandra, L., Rothe, J.: Exact analysis of Dodgson elections: Lewis Carroll's 1876 voting system is complete for parallel access to NP. Journal of the ACM **44** (1997) 806–825
24. Rothe, J., Spakowski, H., Vogel, J.: Exact complexity of the winner problem for Young elections. Theory of Computing Systems **36** (2003) 375–386

25. Hemaspaandra, E., Hemaspaandra, L.: Computational politics: Electoral systems. In: Proceedings of the 25th International Symposium on Mathematical Foundations of Computer Science, Springer-Verlag *Lecture Notes in Computer Science #1893* (2000) 64–83
26. Spakowski, H., Vogel, J.: The complexity of Kemeny's voting system. In: Proceedings of the 5th Argentinian Workshop on Theoretical Computer Science. (2001) 157–168
27. Hemaspaandra, E., Rothe, J.: Recognizing when greed can approximate maximum independent sets is complete for parallel access to NP. Information Processing Letters **65** (1998) 151–156
28. Hemaspaandra, E., Rothe, J., Spakowski, H.: Recognizing when heuristics can approximate minimum vertex covers is complete for parallel access to NP. In: Proceedings of the 28th International Workshop on Graph-Theoretic Concepts in Computer Science (WG 2002), Springer-Verlag *Lecture Notes in Computer Science #2573* (2002) 258–269
29. Hemaspaandra, E., Hemaspaandra, L., Rothe, J.: Raising NP lower bounds to parallel NP lower bounds. SIGACT News **28** (1997) 2–13
30. Hemaspaandra, E., Wechsung, G.: The minimization problem for boolean formulas. SIAM Journal on Computing **31** (2002) 1948–1958
31. Umans, C.: The Minimum Equivalent DNF problem and shortest implicants. In: Proceedings of the 39th IEEE Symposium on Foundations of Computer Science, IEEE Computer Society Press (1998) 556–563
32. Khanna, S., Linial, N., Safra, S.: On the hardness of approximating the chromatic number. Combinatorica **20** (2000) 393–415
33. Arora, S., Lund, C., Motwani, R., Sudan, M., Szegedy, M.: Proof verification and intractability of approximation problems. Journal of the ACM **45** (1998) 501–555
34. Guruswami, V., Khanna, S.: On the hardness of 4-coloring a 3-colorable graph. In: Proceedings of the 15th Annual IEEE Conference on Computational Complexity, IEEE Computer Society Press (2000) 188–197
35. Garey, M., Johnson, D.: Computers and Intractability: A Guide to the Theory of NP-Completeness. W. H. Freeman and Company, New York (1979)
36. Feige, U., Halldórsson, M., Kortsarz, G., Srinivasan, A.: Approximating the domatic number. SIAM Journal on Computing **32** (2002) 172–195
37. Heggernes, P., Telle, J.: Partitioning graphs into generalized dominating sets. Nordic Journal of Computing **5** (1998) 128–142
38. Kaplan, H., Shamir, R.: The domatic number problem on some perfect graph families. Information Processing Letters **49** (1994) 51–56
39. Schaefer, T.: The complexity of satisfiability problems. In: Proceedings of the 10th ACM Symposium on Theory of Computing, ACM Press (1978) 216–226
40. Rothe, J.: Complexity Theory and Cryptology. An Introduction to Cryptocomplexity. EATCS Texts in Theoretical Computer Science. Springer-Verlag, Berlin, Heidelberg, New York (2005) To appear.