Compactness in apartness spaces

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Abstract. In this note, we establish some results which suggest a possible solution to the problem of finding the right constructive notion of *compactness* in the context of a not–necessarily–uniform apartness space.

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1 Introduction

Let X be a nonempty set. We assume that there is a *set–set apartness* relation $\bowtie$ between pairs of subsets of X, such that the following axioms hold, where

$$-S = \{x \in X : \{x\} \bowtie S\}, \tag{1}$$

denotes the the *apartness complement* of S.

B1 $X \bowtie \emptyset$.
B2 $S \bowtie T \Rightarrow S \cap T = \emptyset$.
B3 $R \bowtie (S \cup T) \Leftrightarrow R \bowtie S \wedge R \bowtie T$
B4 $S \bowtie T \Rightarrow T \bowtie S$.
B5 $x \in -S \Rightarrow \exists_T (x \in -T \wedge \forall_y (y \in -S \vee y \in T))$.

We then call X an *apartness space,* or, if clarity demands, a *set–set apartness space.*

Our work on apartness spaces corresponds to the classical proximity spaces discussed in [4] (Part II) and developed more fully in [5]. What particularly distinguishes our theory is that it is constructive: we use intuitionistic logic throughout.

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Defining

$$x \neq y \Leftrightarrow \{x\} \bowtie \{y\}$$

and

$$x \bowtie S \Leftrightarrow \{x\} \bowtie S,$$

we obtain an inequality and a so–called *point–set apartness* associated with the given set–set one. For future reference, we note the following point–set consequence of axioms **B1–B5:**

A5 $\forall_{x \in X} \forall_S \left(x \bowtie S \Rightarrow \forall_y \left(x \neq y \vee y \bowtie S \right) \right).$

The canonical example of an apartness space is a uniform space $(X, \mathcal{U})$, where for constructive purposes we require the uniform structure $\mathcal{U}$ to satisfy the following condition which holds automatically under classical logic:

$$\forall_{u \in \mathcal{U}} \exists_{V \in \mathcal{U}} \forall_{\mathbf{x} \in X \times X} \left(\mathbf{x} \in U \vee \mathbf{x} \notin V \right).$$

We define the inequality and the apartness on $(X, \mathcal{U})$ by

$$x \neq y \Leftrightarrow \exists_{U \in \mathcal{U}} \ \left((x, y) \notin U \right),$$

and

$$S \bowtie T \Leftrightarrow \exists_{U \in \mathcal{U}} \ \left(S \times T \subset \sim U \right). \tag{2}$$

Note that in any set E with an inequality $\neq$, the complement of a subset S is defined as

$$\sim S = \{ x \in E : \forall_{y \in S} \left(x \neq y \right) \}.$$

Every apartness space $(X, \bowtie)$ has a natural *apartness topology* $\tau_{\bowtie}$ in which a base of open sets is formed by the apartness complements. The elements of the apartness topology are called *nearly open sets.* The apartness topology corresponding to the apartness defined by (2) for a uniform space $(X, \mathcal{U})$ is precisely the standard uniform topology $\tau_{\mathcal{U}}$, in which a base of neighbourhoods of x consists of sets of the form

$$U[x] := \{ y \in X : (x, y) \in U \}$$

with $U \in \mathcal{U}$.

We are interested in finding a good constructive notion of compactness that can be applied to general apartness spaces. We believe that such a notion should fulfil at least the following conditions:

c1 For a uniform space it should be equivalent to the space being totally bounded and complete.

c2 An apartness space should be compact if and only if, classically, its apartness topology has the Heine–Borel covering property.

The first problem we face is that the classical notions of Heine–Borel compactness and sequential compactness are of limited and no constructive use, respectively: the first condition holds for the interval $[0,1]$ in two models of constructive mathematics (Brouwer's intuitionism and classical mathematics) but fails in the recursive model; the second condition, although true for $[0,1]$ in the classical model, is false in both the intuitionistic and the recursive models. For these reasons, Bishop [1] adopted total boundedness plus completeness as the defining conditions for a compact metric (and, by extension, uniform) space. The problem with these notions is that they are definitely tied to the context of uniform spaces, whereas (in contrast to the classical situation with proximity spaces—see pages 71–73 of [5]), there are significant apartness spaces whose apartness relations are induced classically *but not constructively* by uniform structures [3].

Let $(X, \bowtie)$ be an apartness space. For nonempty subsets A, B of X we say that A is *well contained* in B, and we write $A \ll B$, if there exists $C \subset X$ such that $X = B \cup \sim C$ and $A \bowtie \sim C$; this definition is classically equivalent to the one given on page 15 of [5]. Let $\mathcal{B}_w$ be the class of all sets of the form

$$\bigcup_{i=1}^{m} B_i \times B_i$$

where there exist subsets $A_1, \ldots, A_m$ of X such that $A_i \ll B_i$ for each i, and $X = \bigcup_{i=1}^{m} A_i$. The class $\mathcal{B}_w$ is nonempty: we have $X = X \cup \emptyset = X \cup \sim X$ and $X \bowtie \sim X$, so $X \times X \in \mathcal{B}_w$.

In this paper we prove some general results about the family $\mathcal{B}_w$ and then use these as the basis of a proposal for a notion of compactness in apartness spaces.

2 Properties of $\mathcal{B}_w$

The *apartness class of uniformities* for a given apartness space $(X, \bowtie)$ is the set $\mathcal{A} = \mathcal{A}(X, \bowtie)$ of uniform structures $\mathcal{U}$ that induce the given apartness on X, in the sense that (2) holds. Classically, $\mathcal{A}$ is nonempty and contains a unique totally bounded member, for which $\mathcal{B}_w$ is a basis of entourages ([5], page 73, (12.3)); whence

$$S \bowtie_w T \Leftrightarrow \exists_{U \in \mathcal{B}_w} (S \times T \subset \sim U) \tag{3}$$

Constructively we cannot prove that $\mathcal{A}$ is nonempty in general (see [3]). Nevertheless, as we aim to show, under reasonable conditions on the apartness, we can prove that (3) defines a set–set apartness on X.

Lemma 1. *Let $(X, \bowtie)$ be an apartness space satisfying the condition*

A4$_s$ $(A \bowtie B \wedge -B \subset \sim C) \Rightarrow A \bowtie C$,

and let S, T be subsets of X such that $S \bowtie T$. Then $S \bowtie \sim\sim T$.

Every uniform apartness space satisfies **A4**$_s$; see [8].

Proposition 1. *If X satisfies $\boldsymbol{A4}_s$, then $\mathcal{B}_w$ is a filter base of symmetric sets containing the diagonal Δ of $X \times X$. Moreover, $\mathcal{B}_w$ is closed under finite intersections.*

Proposition 2. *If $(X, \bowtie)$ satisfies $\boldsymbol{A4}_s$, then $\bowtie_w$ satisfies axioms **B1**–**B4**, and if $x \bowtie_w S$, then $x \bowtie S$. Moreover,*

$$\forall_{S,T \subset X} \left(S \bowtie_w T \Rightarrow S \bowtie T \right). \tag{4}$$

if and only if the following condition holds:

B1$_s$ $\forall_{S,T \subset X} \left(S \times T = \emptyset \Rightarrow S \bowtie T \right).$

Property **B1**$_s$ holds in any uniform apartness space.

Here is an example of a set–set apartness that cannot be given by a uniform structure yet satisfies **A4**$_s$. Starting with a point–set apartness space $(X, \bowtie)$, define a set–set relation on subsets of X by

$$S \bowtie T \Leftrightarrow \forall_{x \in X} \left(x \bowtie S \vee x \bowtie T \right). \tag{5}$$

It is shown in [3] that this definition provides us with a set–set apartness on X, and that if this apartness is induced by a uniform structure, then the weak law of excluded middle,

$$\neg P \vee \neg\neg P,$$

holds. To prove that the set–set apartness on X satisfies **A4**$_s$, let $A \bowtie B$ and $-B \subset {\sim}C$. Then for each $x \in X$, either $x \bowtie A$ or else $x \in -B \subset {\sim}C$; in the latter case, property

A4 $x \in -S \subset {\sim}T \Rightarrow x \bowtie T$

of a point–set apartness shows that $x \bowtie C$. Thus

$$\forall_{x \in X} \left(x \bowtie A \vee x \bowtie C \right)$$

—that is, $A \bowtie C$.

What about axiom **B1**$_s$ in this case? Let S, T be subsets of X such that $S \times T = \emptyset$. It seems unlikely that we can prove that

$$\forall_{x \in X} \left(x \bowtie S \vee x \bowtie T \right),$$

since the latter condition contains a disjunction. In fact, we can produce a Brouwerian example, as follows. Take $X = \{0, 1\}$ with the set–set apartness defined by (5). Consider any syntactically correct proposition P, and let

$$S := \{x : x = 0 \wedge P\},$$
$$T := \{x : x = 0 \wedge \neg P\}.$$

Then $S \times T = \emptyset$. But if $S \bowtie T$, then either $0 \in S$, and therefore P holds, or else $0 \in T$, and therefore $\neg P$ holds.

This example also shows that we cannot derive (4), equivalent to $\mathbf{B1}_s$, for every apartness space.

We now introduce the strongest of the separation properties normally studied in apartness–space theory: the *Efremovič condition,*

$$S \bowtie T \Rightarrow \exists_U \left(S \bowtie U \wedge T \bowtie\sim U \right). \tag{6}$$

This condition implies $\mathbf{A4}_s$; see [8].

We are also interested in a weaker property than (6), the *Efremovič–point condition:*

$$x \in -T \Rightarrow \exists_U \left(x \in -U \wedge T \bowtie\sim U \right).$$

We need the following result, first proved in [7]

Proposition 3. *Let X be an apartness space, and $x \in X$. Then for each $S \in \mathcal{B}_w$, there exists $U \subset X$ such that $x \in - \sim U \subset S[x]$. If also X satisfies the Efremovič–point condition and $x \in -U$, then there exist V, W such that $x \in -W$, $\sim W \ll -U$, $x \in -V$, $X = -W \cup V$, and $S[x] \subset -V$, where*

$$S = (-\{x\} \times -\{x\}) \cup (-V \times -V) \in \mathcal{B}_w.$$

Corollary 1. *If X satisfies the Efremovič–point condition, then for each $x \in X$ the sets $S[x]$ with $S \in \mathcal{B}_w$ generate the neighbourhood filter of x in the apartness topology.*

A uniform structure $\mathcal{U}$ on a topological space (X, τ) is said to be *compatible with the topology* on X if the corresponding uniform topology coincides with τ.

Corollary 2. *If an apartness space $(X, \bowtie)$ satisfies the Efremovič–point condition and $\mathcal{B}_w$ generates a uniform structure $\mathcal{U}_w$ on X, then $\mathcal{U}_w$ is compatible with $\tau_\bowtie$.*

We can now give conditions under which the point–set apartness $\bowtie_w$ coincides with $\bowtie$.

Proposition 4. *If X satisfies the Efremovič–point condition, then*

$$\forall_{x \in X} \forall_A \left(x \bowtie A \Rightarrow x \bowtie_w A \right).$$

We see from Corollary 1 and Proposition 4 that under the Efremovič–point condition, even if $\bowtie_w$ is not an apartness on X —and hence even if $\mathcal{B}_w$ does not generate a uniform structure (let alone one that is compatible with the apartness $\bowtie$) —both $\bowtie_w$ and $\mathcal{B}_w$ are related nicely to the topology $\tau_\bowtie$ on X.

Proposition 5. *If a set–set apartness space $(X, \bowtie)$ satisfies the Efremovič condition, then the set–set relation defined by*

$$S \bowtie_w T \Leftrightarrow \exists_{B \in \mathcal{B}_w} \left(S \times T \subset \sim B \right)$$

*satisfies axioms **B1–B5**, and the corresponding point–set apartness coincides with the point–set apartness induced by $\bowtie$.*

3 $\mathcal{B}_w$ and $\mathcal{A}(X, \bowtie)$

We next consider the connection between $\mathcal{B}_w$ and uniform structures that induce the apartness on X.

Proposition 6. *Let $(X,\mathcal{U})$ be a uniform apartness space, and define $\mathcal{B}_w$ relative to the uniform apartness on X as above. Then*

(i) *for each $V \in \mathcal{B}_w$ there exists $U \in \mathcal{U}$ such that $U \subset V$;*
(ii) *if $\mathcal{B}_w$ generates $\mathcal{U}$, then $\mathcal{U}$ is totally bounded.*

Let n be a positive integer. By an *n–chain* in a uniform space $(X,\mathcal{U})$ we mean an n–tuple $(U_1, \ldots, U_n)$ of entourages such that for each k, n,

$$U_{k+1}^2 \subset U_k \quad \text{and} \quad X \times X = U_k \cup \sim U_{k+1}.$$

The constructive axioms for a uniform structure ensure that for each entourage U and each positive integer n there exists an n–chain $(U_1, \ldots, U_n)$ with $U_1 = U$; see [3].

Proposition 7. *Let $(X,\mathcal{U})$ be a totally bounded uniform apartness space. Then $\mathcal{U}$ is generated by $\mathcal{B}_w$.*

Corollary 3. *Let $(X, \bowtie)$ be an apartness space. Then there is at most one uniform structure on X that is totally bounded and induces the given apartness.*

Corollary 4. *Let $(X, \bowtie)$ be an apartness space such that $\mathcal{A}(X, \bowtie)$ contains a totally bounded member $\mathcal{T}$. Then $\mathcal{T} \subset \mathcal{U}$ for each $\mathcal{U} \in \mathcal{A}$.*

4 A proposal for compactness

Since every uniform apartness space satisfies not just the Efremovič–point condition, but also the full Efremovič condition (6), in view of all the foregoing it seems reasonable to define an apartness space $(X, \bowtie)$ to be *compact* if

(a) it satisfies the Efremovič condition and
(b) $\mathcal{B}_w$, as defined above, generates a uniform structure $\mathcal{U}_w$ that is complete[1] in the usual uniform–space sense.

[1] A possible weakening of this condition goes like this. We might require not that $\mathcal{B}_w$ generate a complete *uniform* structure, but simply that every net $(x_n)_{n\in D}$ in X that satisfies the obvious quasi–Cauchy condition,

$$\forall_{U\in B_w} \exists_{N\in D} \forall_{m,nN} \ \ ((x_m, x_n) \in U),$$

converges to a limit in X.

In that case, by Proposition 6, $\mathcal{U}_w$ is totally bounded. Also, by Corollary 2, the associated uniform topology $\tau_{\mathcal{U}_w}$ is just $\tau_{\bowtie}$; whence, classically, the topological space $(X, \tau_{\bowtie})$ is compact in any of the usual equivalent senses. So our definition fulfils the requirement c2 made earlier.

We would dearly like to show that it fulfils the requirement c1. This would mean proving that a uniform apartness space $(X,\mathcal{U})$ is compact in our sense if and only if the uniform structure $\mathcal{U}$ is complete and totally bounded. If $(X,\mathcal{U})$ is totally bounded, then by Proposition 7, $\mathcal{U} = \mathcal{U}_w$; so if also $\mathcal{U}$ is complete, then so is $\mathcal{U}_w$, and therefore X is compact in our proposed sense. However, to prove that if X is compact in that sense, then $(X,\mathcal{U})$ is both complete and totally bounded seems to be hard, if not impossible under the exclusion of contradiction arguments (cf. [6], page 142).

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