On Maximality of Compact Topologies

Martin Maria Kovár⋆

University of Technology, Faculty of Electrical Engineering
and Communication, Department of Mathematics,
Technická 8, Brno, 616 69, Czech Republic
kovar@feec.vutbr.cz

Dedicated to author's wife and their newborn daughter.

Abstract. Using some advanced properties of the de Groot dual and some generalization of the Hofmann-Mislove theorem, we solve in the positive the question of D. E. Cameron: Is every compact topology contained in some maximal compact topology?

Date: 29. 8. 2004. Last revision: 14. 10. 2004
Keywords: de Groot dual, compact saturated set, wide Scott open filter, maximal compact topology.
2000 Mathematics Subject Classification: 54A10, 54D30.

1 Introduction

The intention of the paper is to prove the conjecture of D. E. Cameron [1] that every compact topology is contained in some maximal compact topology. However, the genesis of the author's solution of this Cameron's problem was not straightforward at all. At the very beginning there was a solution of another old problem, due to J. D. Lawson and M. Mislove. They asked wether the process of iterating the de Groot dual will stop, after finitely many steps, with two topologies which are duals of each other. This question now is known as one of two partial questions of Problem 540 of well-known monograph Open problems in topology [12]. The iteration process of the de Groot dual seems to be rather far from the properties of maximal compact topologies and the question of D. E. Cameron. However, the authors's result that for any topological space (X, τ) it holds $\tau^{dd} = \tau^{dddd}$ (where d stands for the operation of taking the de Groot dual) [8] and the techniques developed for its proof naturally led to the solution of another difficult problem: For which topological spaces (X, τ) it holds $\tau = \tau^{dd}$? It seems to be almost certain that without knowing its solution [9], the Cameron's question could not be positively answered in the presented way. In 2003, J. D. Lawson, in a communication with the author, stated an interesting question

⋆ The author acknowledges support from Grant no. 201/03/0933 of the Grant Agency of the Czech Republic and from the Research Intention MSM 262200012 of the Ministry of Education of the Czech Republic

Dagstuhl Seminar Proceedings 04351
Spatial Representation: Discrete vs. Continuous Computational Models
http://drops.dagstuhl.de/opus/volltexte/2005/118

wether the de Groot dual of a compact T_1 topological space is always a sober space. After some effort spent on trying to prove the conjecture, the author found a counterexample in an advanced paper of H.-P. A. Künzi and D. van der Zypen [11]. They cited an example originally due to E. van Douwen [2] who, using properties of almost disjoint families, constructed a compact T_1, Fréchet, anti-Hausdorff US space. H.-P. A. Künzi and D. van der Zypen proved that the space is maximal compact, so $\tau = \tau^d$ holds, but this topology obviously is not sober. In that paper, H.-P. A. Künzi and D. van der Zypen revived the Cameron's question and among other interesting results, they proved that any compact T_1 sober topology is contained in some maximal compact topology. Because of sobriety needed as the essential assumption, their theorem unfortunately did not cover some simple spaces such as the cofinite space or some spaces with too "bad" behavior, such as the previously mentioned space due to E. van Douwen.

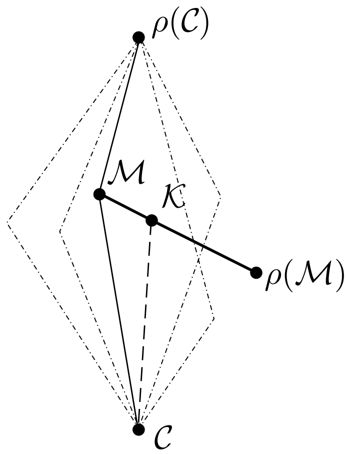

Figure 1.

The principle of the result of H.-P. A. Künzi and D. van der Zypen is based on a construction of a maximal ring of sets, say $\mathcal{M}$, whose existence is ensured by Zorn's Lemma, which contains the family of closed sets $\mathcal{C}$ and is contained in the family $\rho(\mathcal{C})$ of all compact sets of the given space (see Figure 1). If the given space is sober, Hofmann-Mislove theorem ensures compactness of the topology generated by the ring $\mathcal{M}$ (used as a closed base), while its maximality is a consequence of the maximality of the ring $\mathcal{M}$. Unfortunately, if the given space is not sober, the topology generated by the ring $\mathcal{M}$ may fail be compact. For example, if the given topology is cofinite, then the maximal ring generates the discrete topology. However, by a good fortune, the author noticed that the de Groot dual of the topology generated by the ring is always compact. Moreover, the topology generated by $\mathcal{M}$ and the topology of its de Groot dual $\rho(\mathcal{M})$ are duals of each other and they form a pair of antispaces of de Groot. To prove this fact which still works for any space, the previously mentioned analysis of de the Groot dual in the papers [8], and especially [9], is needed. Even more luckily, the author found out that between these two antispaces there always exists a maximal compact topology with the family of closed sets, say $\mathcal{K}$. In case that the given space is sober, the antispaces coincide with the maximal compact topology between. The only problem of the more general case was how to ensure the maximal compact topology to contain the topology of the given space. Especially for that purpose, but not only for that one, a modified version of Hofmann-Mislove theorem was developed and referred by the author during the 2004 Summer Conference on Topology and Its Applications in Cape Town and in the Daghstuhl seminar "Spatial representation: Discrete vs. Continuous Computational Models" in 2004. The early version of the presented result solved the Cameron's question positively for Keimel-Paseka spaces (which are not necessarily topological, see [10] for the precise definition). In topology it means for those topological spaces (X, τ), in which every closed set, if appropriate with exception of X, is a closure of (not necessarily unique) singleton (see [10]). After incorporating some new ideas (having their sources on author's return trip from the seminar, somewhere in the train between Brno and Dagstuhl), we present the final and general solution, which works for any space.

2 Definitions and Terminology

Throughout this paper, the term 'space' is always referred as a topological space. Let (X, τ) be a space. We say that the two distinct points $x, y \in X$ are T_0 separable, if there exists an open set $U \in \tau$ which meets the set $\{x, y\}$ at exactly one point. A closed set in (X, τ) is irreducible if it is non-empty and it cannot be represented as a union of two proper subsets. A complement of a closed irreducible set is a prime open set. Thus an open set is prime iff it cannot be expressed as an intersection of two strictly bigger sets. A space (X, τ) is called anti-Hausdorff if every two non-empty open sets have a non-empty intersection. By a filter in τ we mean a family $\mathcal{F} \subseteq \tau$ such that

(i) $X \in \tau$,
(ii) $U \in \mathcal{F}$, $U \subseteq V \in \tau \implies V \in \mathcal{F}$, and
(iii) $U, V \in \mathcal{F} \implies U \cap V \in \mathcal{F}$.

Let (X, τ) be a space, $\mathcal{F} \subseteq \tau$ a filter, $\mathcal{O} \subseteq \tau$ a nonempty subfamily. We say that $\mathcal{F}$ is wide with respect to $\mathcal{O}$ if for every $U \in \mathcal{O}$, $\bigcap \mathcal{F} \subseteq U$ implies $U \in \mathcal{F}$. If $\mathcal{O} = \tau$, we simply say that the filter $\mathcal{F}$ is wide. The filter $\mathcal{F} \subseteq \tau$ is said to be Scott open, if for every collection $\mathcal{O} \subseteq \tau$ such that $\bigcup \mathcal{O} \in \mathcal{F}$ there exist $U_1, U_2, \dots U_k \in \mathcal{O}$ such that $\bigcup_{i=1}^{k} U_i \in \mathcal{F}$. Recall that a topological space is sober, if every irreducible closed subset of the space is a closure of a unique point. A set in a topological space is called saturated iff it is an intersection of open sets. Let ψ be a family of sets. We say that ψ has the finite intersection property, or briefly, that ψ has f.i.p., if for every $P_1, P_2, \dots, P_k \in \psi$ it holds $P_1 \cap P_2 \cap \dots \cap P_k \neq \varnothing$. For the definition of compactness, we do not assume any additional separation axiom. We say that a subset S of a space (X, τ) is compact if every open cover of S has a finite subcover. A space (X, τ) is said to be maximal compact if τ is maximal among all compact topologies on X. A space (X,τ) is called a KC-space if each compact subset of X is closed. It is not difficult to show that a space is maximal compact iff it is a compact KC-space. Let $f : X \to Y$ be a continuous mapping. We say that f is perfect if it is closed and every fibre $f^{-1}(y)$, where $y \in Y$, is compact. In the contrary of some classical resources, we do not assume X to be Hausdorff.

Let (X,τ) be a space, $\mathcal{C}$ be the family of all closed sets of (X,τ) and $\mathcal{D} \subseteq \mathcal{C}$ its closed base. There are two versions of the de Groot dual established in the literature. If we use the family of all compact sets $\rho(\mathcal{C})$ as the closed subbase of a new topology on X, we obtain the original de Groot construction of the dual. Let us denote this dual topology by τ^ρ. Applying the dual operation once more, we obtain the family $\rho^2(\mathcal{C})$ of all square compact subsets of (X,τ). There is an important property of $\rho^2(\mathcal{C})$ – in a contrast to $\rho(\mathcal{C})$, $\rho^2(\mathcal{C})$ is always closed under arbitrary intersections (see [3]). Another important property of the iterated de Groot duals which we will use in this paper is the inclusion $\rho(\mathcal{C}) \subseteq \rho^3(\mathcal{C})$. For more detail, the reader is referred to [3], [4] and [5].

If we use the family of all compact saturated sets instead of $\rho(\mathcal{C})$, we get a modified construction, sometimes also called de Groot dual (see [7]) as a natural extension of the original notion. The original de Groot's construction has a sense especially for T_1 spaces but it is less convenient for topologies motivated by computer science, where the modified dual seems to be more applicable. The main reason is because the modified dual switches the preorder of specialization – a binary relation that is given by setting $x \le y$ iff $x \in \mathrm{cl}\{y\}$ for every $x, y \in X$. The dual topology following the modified construction we denote by τ^d. The properties of the dual operator d were studied by the author in [8] and [9]. Because the closed bases appeared to be more important than the corresponding topologies in these studies, the author associated the family of all compact saturated sets of (X,τ) rather with its closed base $\mathcal{D}$ than with $\mathcal{C}$ or τ and denoted it by $\mathcal{D}^d$. Probably the most general known relationship between the iterated de Groot duals of this modified version is the identity $(\mathcal{D} \cup \mathcal{D}^{dd})^d = \mathcal{D}^d$, which holds for any space (see [8], [9]). This identity also implies that $\mathcal{D}^d \subseteq \mathcal{D}^{ddd}$, which is a counterpart of the similar inclusion that was obtained by J. de Groot, G. E. Strecker and E. Wattel in [3], as mentioned above. Since in T_1 spaces all sets are saturated, both dual constructions coincide in this setting. Finally, let $\mathcal{A}, \mathcal{B} \subseteq 2^X$. In the following, both arrows $\downarrow$ and $\uparrow$ are related to the preorder of specialization. Recall that the family $\mathcal{A}$ is $\mathcal{B}$-down-conservative if for every $A \in \mathcal{A}$ and $B \in \mathcal{B}$ it follows $\downarrow (A \cap B) \in \mathcal{A}$. The family $\mathcal{B}$ is said to be up-compact if every $B \in \mathcal{B}$ is compact with respect to the family $\{\uparrow \{x\} \,|\, x \in X\}$. For more detail related to these notions, the reader is referred to [9].

3 The proof of the conjecture

An alternative of the presented proof it could be one long, technical proof, without noticing any connections that are behind. We do not prefer this way, rather than this we tried to follow the genesis of the solution, that was described in the introductory section. We will use the following theorem as our starting point.

Theorem 1. *Let $\mathcal{D}$ be a closed subbase of the space (X, τ). The following statements are equivalent:*

(i) $\tau = \tau^{dd}$.
(ii) *τ has an up-compact and $\mathcal{D}^d$-down-conservative closed subbase.*
(iii) *τ has a maximal up-compact and $\mathcal{D}^d$-down-conservative closed subbase.*
(iv) *$\mathcal{D}^{dd}$ is the greatest up-compact and $\mathcal{D}^d$-down-conservative closed base of the topology τ.*

For the proof, the reader is referred to [9]. The binary relation that we will define in the next definition is in some properties similar to the preorder of specialization. However, in a contrast to the specialization preorder, it can distinguish between points also in a T_1 space. There is not enough room for investigating the behavior of that preorder in this paper, however, it should be done later.

Definition 1. *Let (X, τ) be a space and let $x, y \in X$. We put $x \preccurlyeq y$ if for every $U \in \tau$, $x \in U$ implies $U \cup \{y\} \in \tau$. If, in addition, $x \neq y$, we write $x \prec y$.*

The following lemma contains a well-known result. However, the classical topological literature assumes (X, τ) to be Hausdorff in the definition of the perfect mapping, and for our purpose, such a limitation could not be accepted. Hence, we prefer to repeat the result with the complete proof.

Lemma 1. *Let (X, τ), (Y, σ) be spaces and $f : X \to Y$ a perfect mapping. If $K \subseteq Y$ is compact, then also $f^{-1}(K)$ is compact.*

Proof. Let $\mathcal{O} \subseteq \tau$ be an open directed cover of $f^{-1}(K)$. We will show that the family $\mathcal{O}' = \{Y \smallsetminus f(X \smallsetminus U) | U \in \mathcal{O}\}$ is an open cover of K. Let $y \in K$. Then $f^{-1}(y) \subseteq f^{-1}(K) \subseteq \bigcup_{U \in \mathcal{O}} U$, and since $f^{-1}(y)$ is compact, there exists $U \in \mathcal{O}$ such that $f^{-1}(y) \subseteq U$. Then $f^{-1}(y) \cap (X \smallsetminus U) = \varnothing$, which means that $y \notin f(X \smallsetminus U)$. Hence, $y \in Y \smallsetminus f(X \smallsetminus U)$. Then the family $\mathcal{O}'$ is an open cover of K. Since K is compact, there exist $U_1, U_2, \ldots, U_k$, such that

$$K \subseteq \bigcup_{i=1}^{k} (Y \smallsetminus f(X \smallsetminus U_i)).$$

Let $x \in f^{-1}(K)$. Then $f(x) \in K$. In particular, there exists $i \in \{1, 2, \ldots, k\}$ such that $f(x) \in Y \smallsetminus f(X \smallsetminus U_i)$. Then $f(x) \notin f(X \smallsetminus U_i)$, so $x \notin X \smallsetminus U_i$ and so $x \in U_i$. Hence, $f^{-1}(K) \subseteq \bigcup_{i=1}^{k} U_i$. Therefore, $f^{-1}(K)$ is compact.

The next lemma, which is inspired by a paper of K. Keimel and J. Paseka [6], is related to the Hofmann-Mislove theorem. However, we consider its a very special modification, which localize all the considerations to the neighborhood base of a single point. Note that in a sober space (X, τ), all Scott open filters are wide (with respect to τ), which ensures the classical, one-to-one correspondence between the Scott open filters and the compact saturated sets (see, e.g. [6]). In our approach, for some technical reasons, the counterpart of that correspondence is implicitly contained in the proof of another lemma below, Lemma 5.

Lemma 2. *Let (X,τ) be a T_1 space containing two points $m \prec q$. Then every Scott open filter $\mathcal{F} \subseteq \tau$ is wide with respect to $\tau(m) = \{U | \, m \in U \in \tau, q \notin U\}$.*

Proof. Let $\mathcal{F} \subseteq \tau$ be a Scott open filter such that for some $U \in \tau(m)$, we have $\bigcap \mathcal{F} \subseteq U$. Suppose that $U \notin \mathcal{F}$. We put $\mathcal{M} = \{V | \, V \in \tau(m), U \subseteq V \notin \mathcal{F}\}$. Let $\mathcal{L} \subseteq \mathcal{M}$ be a linearly ordered chain. If $\bigcup \mathcal{L} \in \mathcal{F}$, then, since $\mathcal{F}$ is Scott open, there exists $V \in \mathcal{L}$ such that $V \in \mathcal{F}$. But this is impossible, so we have $\bigcup \mathcal{L} \notin \mathcal{F}$. We also have $m \in U \subseteq \bigcup \mathcal{L}$ and, of course, $q \notin \bigcup \mathcal{L}$, which together means that $\bigcup \mathcal{L} \in \tau(m)$ and, consequently, $\bigcup \mathcal{L} \in \mathcal{M}$. But then $\bigcup \mathcal{L}$ is an upper bound of the chain $\mathcal{L}$, so by Zorn's Lemma, there exists $W \in \mathcal{M}$ which is a maximal element of $\mathcal{M}$ with respect to the set inclusion. Suppose that $P, Q \in \tau$ such that $P \cap Q = W$. Since $q \notin W$ and $\{q\}$ is a closed set, then also $(P \smallsetminus \{q\}) \cap (Q \smallsetminus \{q\}) = W$. Hence, without loss of generality, we may assume that $P, Q \in \tau(m)$. Further, it is not possible for both of the sets P, Q to belong to $\mathcal{F}$ since $\mathcal{F}$ is a filter and $W \notin \mathcal{F}$. Say, for certainty, that $P \notin \mathcal{F}$. Then $P \in \mathcal{M}$ and, because of the maximality of W, it holds $P = W$. Therefore, W is prime. Now, let $S = W \cup \{q\}$ and $T = X \smallsetminus \{q\}$. The set S is open since $m \prec q$ and we have $W = S \cap T$. Since $W \subsetneq S$ is prime, it follows that $W = T$. Let $V \in \mathcal{F}$. Then $V \nsubseteq W = X \smallsetminus \{q\}$, which means that $q \in V$. Then $q \in \bigcap \mathcal{F} \subseteq U \subseteq W$, which is a contradiction. Hence, the assumption of $U \notin \mathcal{F}$ is false and we can conclude that $U \in \mathcal{F}$. But this means that $\mathcal{F}$ is wide with respect to $\tau(m)$.

A source of inspiration for the next lemma was a simple observation of H-P. A. Künzi and D. vander Zypen [11], that the cofinite topology is contained in some compact Hausdorff topology which one can obtain as the Alexandroff one-point compactification of the infinite discrete space. In fact, in the general case which is described by our lemma, the neighborhoods belonging to the different iterations of the de Groot dual of the space are involved.

Lemma 3. *Let (X,τ) be a compact space. Suppose that there exists $m \in X$ such that every closed $C \in \mathcal{C}$ which does not contain m is square compact. Then τ is contained in some maximal compact topology κ on X, which is generated by its closed subbase*

$$\mathcal{K} = \{A | \, A \in \rho(\mathcal{C}), m \in A\} \cup \{B | \, B \in \rho^2(\mathcal{C}), m \notin B\}.$$

Proof. Let $\mathcal{C}$ be the family of all closed sets in (X,τ). Since (X,τ) is compact, $\mathcal{C} \subseteq \rho(\mathcal{C})$, which implies $\rho^2(\mathcal{C}) \subseteq \rho(\mathcal{C})$. Let κ be the topology generated by $\mathcal{K}$ as a subbase for the closed sets. We will show that (X,κ) is compact. Let $\mathcal{E} \subseteq \mathcal{K} \subseteq \rho(\mathcal{C})$ be a family which has f.i.p. Suppose that $m \notin \bigcap \mathcal{E}$. Then there exists $B \in \mathcal{E} \cap \rho^2(\mathcal{C})$. Then $\bigcap \mathcal{E} \neq \varnothing$, since B is compact with respect to $\rho(\mathcal{C})$. By Alexander's subbase lemma, (X,κ) is compact. We will show that $\tau \subseteq \kappa$.

Let $C \in \mathcal{C}$. If $m \in \mathcal{C}$, it is clear that $C \in \mathcal{K}$. Let $m \notin C$. By the assumption, $C \in \rho^2(\mathcal{C})$, which implies that $\mathcal{C} \in \mathcal{K}$. Hence, $\mathcal{C} \subseteq \mathcal{K}$ and $\tau \subseteq \kappa$.

Let $K \subseteq X$ be compact with respect to κ. Since $\tau \subseteq \kappa$, $K \in \rho(\mathcal{C})$. If $m \in K$, then $K \in \mathcal{K}$. Let $m \notin K$. We will show that $K \in \rho^2(\mathcal{C})$. Let $\varphi \subseteq \rho(\mathcal{C})$ be a family that $\{K\} \cup \varphi$ has f.i.p. We put $\chi = \{F \cup \{m\} | \, F \in \varphi\}$. Then also $\chi \subseteq \rho(\mathcal{C})$ and since all elements of χ contain m, we have $\chi \subseteq \mathcal{K}$. The family $\{K\} \cup \chi$ has f.i.p., so $K \cap (\bigcap \varphi) = K \cap (\bigcap \chi) \neq \varnothing$. Hence, $K \in \rho^2(\mathcal{C})$. Since $m \notin K$, we have $K \in \mathcal{K}$. In any case, K is closed in (X, κ). Now, we can see that (X, κ) is a compact KC-space, which implies that it is maximal compact.

Lemma 4. *Let (X, τ) be a T_1 space, $\mathcal{C}$ be the family of all closed sets and $m, q \in X$ be two distinct points. Then $\mathcal{C}_0 = \{C | \, C \in \mathcal{C}, \{m, q\} \cap C \neq \varnothing\}$ is a closed base for (X, τ).*

Proof. The set $\mathcal{C}_0$ is closed under binary unions. Let $D \in \mathcal{C}$. We will express D as an intersection of elements of $\mathcal{C}_0$. If $m \in D$ or $q \in D$, then $D \in \mathcal{C}_0$ and we are done. Suppose that $D \cap \{m, q\} = \varnothing$. We put $D_1 = D \cup \{m\}$ and $D_2 = D \cup \{q\}$. We have $D = D_1 \cap D_2$ and $D_1, D_2 \in \mathcal{C}_0$, which completes the proof.

After the next step we will be almost done. The core of the whole proof is contained in the lemma below. We will use the modified construction of the maximal ring of sets (due to H-P. A. Künzi and D. van der Zypen, [11]), some advanced properties of the de Groot dual, here expressed in Theorem 1, the gain of the previous lemma and we will finish by an implicit use of some modification of the Hofmann-Mislove theorem, prepared for use in Lemma 2. Note that we actually do not need to have a maximal ring of sets (that is, a maximal family which is closed under finite unions and finite intersections), the maximality with respect to the set inclusion and the closeness under the binary intersections is fully sufficient.

Lemma 5. *Let (X, τ) be a compact T_1 space containing two points $m \prec q$. Then τ is contained in some maximal compact topology.*

Proof. Let (X, τ) be compact and $\mathcal{C}$ let be the family of the closed sets in (X, τ). Then $\mathcal{C} \subseteq \rho(\mathcal{C})$. Let $\mathcal{M}$ be a maximal collection (with respect to the set inclusion) such that $\mathcal{C} \subseteq \mathcal{M} \subseteq \rho(\mathcal{C})$ among all collections closed under binary intersections. The existence of such a collection is a consequence of the Zorn's lemma. Let μ be a topology generated by $\mathcal{M}$ as a closed subbase. Let $D \in \rho(\mathcal{M})$. We put

$$\mathcal{M}_D = \mathcal{M} \cup \{D\} \cup \{D \cap M | \, M \in \mathcal{M}\}.$$

It is clear that $\mathcal{M}_D$ is closed under binary intersections. If $M \in \mathcal{M}$, then M is closed in (X, μ) and so $D \cap M \in \rho(\mathcal{M}) \subseteq \rho(\mathcal{C})$. Hence $\mathcal{C} \subseteq \mathcal{M}_D \subseteq \rho(\mathcal{C})$. From the maximality of $\mathcal{M}$ it follows that $\mathcal{M}_D = \mathcal{M}$. In particular, $\rho(\mathcal{M}) \subseteq \mathcal{M}$, which also implies that $\rho(\mathcal{M}) \subseteq \rho^2(\mathcal{M})$. Since $\mathcal{M}$ is closed under binary intersections, it is $\rho(\mathcal{M})$-conservative, i.e. for every $D \in \rho(\mathcal{M})$ and $M \in \mathcal{M}$ we have $D \cap M \in \mathcal{M}$. By Theorem 1 we have $\mu = \mu^{\rho\rho}$ and $\rho^2(\mathcal{M})$ is the greatest $\rho(\mathcal{M})$-conservative closed subbase of (X, μ). Then $\mathcal{M} \subseteq \rho^2(\mathcal{M})$. Denote by λ the topology generated by $\rho(\mathcal{M})$ as a closed subbase. Then $\lambda \subseteq \mu$, $\lambda^\rho = \mu^{\rho\rho} = \mu$ and $\mu^\rho = \lambda^{\rho\rho} = \lambda$. Let $\mathcal{E} \subseteq \rho(\mathcal{M}) \subseteq \rho^2(\mathcal{M})$ be a non-empty family with f.i.p. Let $P \in \mathcal{E}$. Then also $P \in \rho^2(\mathcal{M})$, so $\varnothing \neq P \cap (\bigcap \mathcal{E}) = \bigcap \mathcal{E}$. Hence, λ is compact.

Since $\mathcal{M} \subseteq \rho^2(\mathcal{M})$, we have $\rho^3(M) \subseteq \rho(\mathcal{M})$. Because always $\rho(\mathcal{M}) \subseteq \rho^3(\mathcal{M})$, it follows that $\rho(\mathcal{M}) = \rho^2(\rho(\mathcal{M})) = \rho^3(\mathcal{M})$. Then $\rho(\mathcal{M})$ is the family of all closed sets of the topology λ. We put

$$\mathcal{K} = \{A|\, A \in \rho^2(\mathcal{M}), m \in A\} \cup \{B|\, B \in \rho(\mathcal{M}), m \notin B\},$$

where $\rho(\mathcal{M})$ stands in place of $\mathcal{C}$ and λ in place of τ in Lemma 3. Notice that if $m \notin P \in \rho(\mathcal{M})$, it follows from the identity $\rho(\mathcal{M}) = \rho^2(\rho(\mathcal{M})) = \rho^3(\mathcal{M})$ that P is square compact with respect to $\rho(\mathcal{M})$ and the conditions of Lemma 3 are satisfied. Hence, it follows from this lemma that $\mathcal{K}$ generates some maximal compact topology, say κ.

We will show that $\tau \subseteq \kappa$. Let $C \in \mathcal{C}_0 = \{C|\, C \in \mathcal{C}, \{m, q\} \cap C \neq \varnothing\}$. If $m \in C$, from $\mathcal{C} \subseteq \mathcal{M} \subseteq \rho^2(\mathcal{M})$ it follows that $C \in \mathcal{K}$. Let $m \notin C$. Then $X \setminus C \in \tau(m) = \{U|\, m \in U \in \tau, q \notin U\}$. Suppose that $\psi \subseteq \mathcal{M} \subseteq \rho(\mathcal{C})$ is a family such that $\{C\} \cup \psi$ has f.i.p. Since $\mathcal{M}$ is closed under binary intersections, we may assume, without loss of generality, the same assumption regarding the family ψ. We put $\mathcal{F} = \{U|\, U \in \tau$, there exists $K \in \psi$, such that $K \subseteq U\}$. The elements of ψ are compact and sice (X, τ) is a T_1 space, also saturated. Obviously, $\bigcap \psi \subseteq \bigcap \mathcal{F}$. Conversely, let $x \notin \bigcap \psi$. Then $x \notin K$ for some $K \in \psi$. But K is an intersection of open sets since it is saturated. Then there exists some $U \in \tau$ with $K \subseteq U$ but $x \notin U$. It means that $U \in \mathcal{F}$ and so $x \notin \bigcap \mathcal{F}$. Thus $\bigcap \mathcal{F} = \bigcap \psi$. Let $\mathcal{O} \subseteq \tau$ be a family with $\bigcup \mathcal{O} \in \mathcal{F}$. Then $K \subseteq \bigcup \mathcal{O}$ for some compact $K \in \psi$, so there exist $U_1, U_2, \ldots U_k \in \mathcal{O}$ such that $K \subseteq \bigcup_{i=1}^k U_i \in \mathcal{F}$. Then $\mathcal{F}$ is a Scott open filter whose every element meets C. In particular, we have $X \setminus C \notin \mathcal{F}$. By Lemma 2, $\bigcap \mathcal{F} \nsubseteq X \setminus C$. Then $C \cap (\bigcap \psi) = C \cap (\bigcap \mathcal{F}) \neq \varnothing$, which means that $C \in \rho(\mathcal{M})$, and consequently, $C \in \mathcal{K}$. Now, we can conclude that $\mathcal{C}_0 \subseteq \mathcal{K}$, and by Lemma 4, it holds $\tau \subseteq \kappa$.

Every compact topology is contained in some compact T_1 topology – it is sufficient to take the join of the given topology with the cofinite topology. Hence, the only problem now it could be, how to ensure the existence of two points $m \prec q$ in the given space. No problem if one point is doubled, as we can see from the next lemma.

Lemma 6. *Let (X, τ) be a compact space with $a, b \in X$, $a \neq b$, which are not T_0-separable. Then there exists a compact space (X, τ') which is T_1, $\tau \subseteq \tau'$ and $a \prec b$ in (X, τ').*

Proof. We put $\tau_0' = \{U \setminus F|\, U \in \tau, F \subseteq X$ is finite$\}$. The family τ_0' covers X and it is closed under finite intersections. Then τ_0' is an open base of some topology on X, say τ'. It is an easy exercise to verify that (X, τ') is a compact T_1 space. Now, let $a \in W = U \setminus F$ for some $U \in \tau$ and $F \subseteq X$ finite. Then also $b \in U$. If $b \notin F$, then $b \in W = W \cup \{b\} \in \tau$ and we are done. Let $b \in F$. Then $W \cup \{b\} = (U \setminus F) \cup \{b\} = U \setminus (F \setminus \{b\})$, which is an element of $\tau_0' \subseteq \tau'$ by its definition. Hence, for the space (X, τ'), we have $a \prec b$.

But, after all, if some point is missing, why not to add it to the space? We just only need a way, how we can return back to the original underlying set. A proper quotient mapping could be that way.

Lemma 7. *Let* (Y, λ) *be a maximal compact space,* $a, b \in Y$, $a \neq b$. *Denote* $X = Y \smallsetminus \{b\}$,

$$f(y) = \begin{cases} y \text{ for } y \in Y \smallsetminus \{a, b\} \\ a \text{ for } y \in \{a, b\} \end{cases} ,$$

and $\kappa = \{V \mid V \subseteq X, f^{-1}(V) \in \lambda\}$. *Then* (X, κ) *is also a maximal compact space.*

Proof. Obviously, (X, κ) is a quotient space of (Y, λ). We will show that the quotient mapping $f : Y \to X$ is perfect. Since every fibre $f^{-1}(x)$, $x \in X$ is finite, it is sufficient to show that f is closed. Let $H \subseteq Y$ be closed. If $\{a, b\} \subseteq H$, then $f^{-1}(X \smallsetminus f(H)) = f^{-1}(Y \smallsetminus H) = Y \smallsetminus H \in \lambda$. Then $f(H)$ is closed in (X, κ). Now suppose that exactly one of the points a, b is contained in H. We put $G = H \cup \{a, b\}$. Then $f(H) = f(G)$. Since (Y, λ) is maximal compact, it is T_1 and so G is closed. Hence, we can reduce this case to the previous one. Then $f(H) = f(G)$ is closed in (X, κ). Finally, let $\{a, b\} \subseteq Y \smallsetminus H$. Then $f(H) = H$ and $f^{-1}(X \smallsetminus f(H)) = f^{-1}(X \smallsetminus H) = Y \smallsetminus H \in \lambda$, which again means that $f(H)$ is closed in (X, κ). Hence, f is a perfect mapping.

Now, let $K \subseteq X$ be compact. Then $f^{-1}(K) \subseteq Y$ is compact by Lemma 1. But (Y, λ) is maximal compact and hence a KC space, so $f^{-1}(K)$ is closed and then so $K = f(f^{-1}(K))$, since f is perfect. We can see that (X, κ) is a compact KC-space, which is equivalent to its maximal compactness.

Now, it remains to make the last step and verify, that all the elements of the "puzzle" will joint together.

Theorem 2. *Let* (X, τ) *be a compact space. Then* τ *is contained in some maximal compact topology.*

Proof. Of course, we may suppose that $X \neq \varnothing$. Let $a \in X$ be some element and let $b \notin X$. We put $Y = X \cup \{b\}$ and $\sigma = \{U \mid a \notin U \in \tau\} \cup \{U \cup \{b\} \mid a \in U \in \tau\}$. Then (Y, σ) is a compact space almost like (X, τ) but $a, b \in Y$ share the same open neighborhoods and they are not T_0-separable. By Lemma 6 and Lemma 5 there exists a maximal compact topology on Y, say λ, which contains σ. Finally, from Lemma 7 it follows that there exists a maximal compact topology κ, which is a quotient of λ and it is given by the corresponding construction of Lemma 7.

Let $U \in \tau$. If $a \notin U$, then $U \in \sigma \subseteq \lambda$. The quotient mapping $f : Y \to X$ equals to the identity on U, so $f^{-1}(U) = U$ and, consequently, U is also open in the quotient space (X, κ). That is, $U \in \kappa$. Now, suppose that $a \in U$. Then $V = U \cup \{b\} \in \sigma \subseteq \lambda$. Of course, we have $f(V) = U$ and $f^{-1}(U) = V$. Hence, $U \in \kappa$ by the definition of the quotient topology. Together we have $\tau \subseteq \kappa$, which completes the proof.

References

1. Cameron, D. E., *A survey of maximal topological spaces*, Topology Proc. 2 (1977), 11-60.
2. van Douwen, E. K., *An anti-Hausdorff Fréchet space in which convergent sequences have unique limits*, Topology Appl. 51 (1993), 147-158.
3. de Groot J., Strecker G.E., *Wattel E., The Compactness Operator in General Topology*, Proceedings of the Second Prague Topological Symposium, Prague, 1966, pp. 161-163.
4. de Groot J.,*An Isomorphism Principle in General Topology*, Bull. Amer. Math. Soc. 73 (1967) 465-375.
5. de Groot J., Herrlich H., Strecker G. E., Wattel E. *Compactness as an Operator*, Compositio Mathematica 21 (4) (1969) 349-375.
6. Keimel K., Paseka J., *A direct proof of the Hofmann-Mislove theorem*, Proc. Amer. Math. Soc. 120 (1994), 301-303.
7. Kopperman R., *Asymetry and Duality in Topology*, Topology Appl. 66 (1) (1995) 1-39.
8. Kovár, M. M., *At most 4 topologies can arise from iterating the de Groot dual*, Topology Appl. 130 (2003) 175-182.
9. Kovár, M. M., *On Iterated De Groot Dualizations of Topological Spaces*, Topology Appl. (to appear), 1-7.
10. Kovár, M. M., *The Hofmann-Mislove Theorem for general topological structures*, Preprint (2004), 1-9.
11. Künzi, H-P. A., van der Zypen, D. *Maximal (sequentially) compact topologies*, Categorial Structures and Their Applications (W. Gähler, G. Preuss, eds.), World Scientific Publishing Company, 2004, pp. 173-187.
12. Lawson J.D., Mislove M., *Problems in domain theory and topology*, Open Problems in Topology (van Mill J., Reed G. M., eds.), North-Holland, Amsterdam, 1990, pp. 349-372.