The Hofmann-Mislove Theorem for general topological structures

Martin Maria Kovár*

University of Technology, Faculty of Electrical Engineering and Communication, Department of Mathematics, Technická 8, Brno, 616 69, Czech Republic
kovar@feec.vutbr.cz

Abstract. In this paper we prove a modification of Hofmann-Mislove theorem for a topological structure similar to the minusspaces of de Groot, in which the empty set "need not be open". This will extend, in a slightly relaxed form, the validity of the classical Hofmann-Mislove theorem also to some of those spaces, whose underlying topology need not be (quasi-) sober.

Date: 11. 2. 2004. Last revision: 14. 10. 2004
Keywords: compact saturated set, Scott open filter, (quasi-) sober space.

1 Preliminaries and Introduction

In the previous paper [10] we studied the posets satisfying Hofmann-Mislove theorem in some sense. More precisely, we defined a poset set $(X, \leq)$ to be Hofmann-Mislove, if the properly generalized Scott open filters in $(X, \leq)$ were in a one-to-one correspondence with the properly generalized compact saturated sets in some $P \subseteq X$, equipped with the topology ω_P induced from upper-interval topology ω on X. We proved that if such P exists, it is unique. We called P a generalized spectrum of $(X, \leq)$. In particular, we proved that if $(X, \leq)$ is a distributive lattice, it is Hofmann-Mislove. Applying this on a topological space (X, τ), we studied topology bases $\mathcal{T} \subseteq \tau$ which were distributive lattices under the set inclusion. We were interested in a condition in which the meaning the points of the topological structure $(X, \mathcal{T})$ was the same as of $(\mathcal{P}, \omega_{\mathcal{P}})$, where $\mathcal{P} \subseteq \mathcal{T}$ was the generalized spectrum of $(\mathcal{T}, \subseteq)$, consisting of all prime open sets of $(\mathcal{T}, \subseteq)$. We found out that if we wanted to keep the traditional meaning of compactness, the only working cases were $\mathcal{T} = \tau$ or $\mathcal{T} = \tau \smallsetminus \{\varnothing\}$. Further, any $P \in \mathcal{P}$ should be a complement of a closure of a unique point $p \in X$. While the case $\mathcal{T} = \tau$ led naturally to the sobriety of (X, τ) and already had been fully described by the (well-known) classical theory, the possibility $\mathcal{T} =$

* The author acknowledges support from Grant no. 201/03/0933 of the Grant Agency of the Czech Republic and from the Research Intention MSM 262200012 of the Ministry of Education of the Czech Republic

$\tau \smallsetminus \{\varnothing\}$ required some further study, which we now intend to realize in this paper. However, for more the detailed explanation why we choose exactly that structure as we define below as 'spaces', we refer the reader to the previous paper [10] and especially to its last section. The topology-like structures in which the empty set "need not be open" were already studied, for example, by de Groot, who pointed out some of their interesting properties related to co-compactness. It comes out that such structures slightly differs from topological spaces in some aspects and therefore it constitutes a legitimate object of study. Note that in the class of topological T_0 spaces, the classical Hofmann-Mislove theorem holds if and only if the considered topological space is sober [3].

As we have already premised above, throughout of this paper, we refer the term 'space' as the following structure. Let X be a set, $\mathcal{T} \subseteq 2^X$. We say that the pair $(X, \mathcal{T})$ is a space if $\mathcal{T}$ is closed under all finite intersections and all non-empty unions. The elements of $\mathcal{T}$ we call open sets, the elements of $\mathcal{C} = \{H | X \smallsetminus H \in \mathcal{T}\}$ are called closed sets of $(X, \mathcal{T})$. Note that we also admit the empty intersection in the previous definition, so we have $\cap\varnothing = X \in \mathcal{T}$ but in a contrast to the topological spaces, we do not necessarily assume that $\cup\varnothing = \varnothing \in \mathcal{T}$. Our notion 'space' is also close to the notion of minusspace of [4] and [5]. However, in a minusspace, $\mathcal{C}$ is supposed to be a cover of X and it need not contain $\varnothing$ as a complement of $X = \cap\varnothing$. The terms base and subbase of open (closed, respectively) sets or briefly, an open (closed, respectively) base are defined in the same way as in topological spaces, i.e. as a subfamily of the family of open (closed, respectively) sets such that every open (closed, respectively) set is a union (intersection, respectively) of some elements of the subfamily. Let $(X, \mathcal{T})$ be a space. A closed set in $(X, \mathcal{T})$ is irreducible if it is non-empty and it cannot be represented as a union of two proper subsets. A complement of a closed irreducible set is a prime open set. Thus an open set is prime iff it cannot be expressed as an intersection of two strictly bigger sets. A space $(X, \mathcal{T})$ is called anti-Hausdorff if every two non-empty open sets have a non-empty intersection. By a filter in $\mathcal{T}$ we mean a family $\mathcal{F} \subseteq \mathcal{T}$ such that

(i) $X \in \mathcal{F}$,
(ii) $U \in \mathcal{F}, U \subseteq V \in \mathcal{T} \implies V \in \mathcal{F}$, and
(iii) $U, V \in \mathcal{F} \implies U \cap V \in \mathcal{F}$.

Note that in a contrast to the usual convention, we admit $\varnothing$ to be an element of a filter if $\varnothing \in \mathcal{T}$. The filter $\mathcal{F} \subseteq \mathcal{T}$ is said to be Scott open, if for every collection $\Omega \subseteq \mathcal{T}$ such that $\bigcup \Omega \in \mathcal{F}$ there exist $U_1, U_2, \dots U_k \in \Omega$ such that $\bigcup_{i=1}^{k} U_i \in \mathcal{F}$. It is easy to see that if $(X, \mathcal{T})$ is a space, then $\tau = \mathcal{T} \cup \{\varnothing\}$ is a topology on X. If not otherwise specified, all topological notions will be related to this topology. We say that open set $P \in \mathcal{T}$ is principal, if $P = X \smallsetminus \mathrm{cl}\{p\}$ for some $p \in X$. Recall that a topological space is called sober, if every irreducible closed set is a closure of a unique point. If the uniqueness of that point is not required, we get the definition of a quasisober space. The preorder of specialization is a reflexive and transitive binary relation on X defined by $x \leqslant y$ iff $x \in \mathrm{cl}\,\{y\}$. This relation is antisymmetric and hence a partial ordering iff X is a T_0 space. For any set $A \subseteq X$ we denote $\uparrow A = \{x | \, x \geqslant y \text{ for some } y \in A\}$

and $\downarrow A = \{x|\, x \leqslant y \text{ for some } y \in A\}$. A set A is saturated in $(X, \mathcal{T})$ if $A = \uparrow A$. One can easily check that a non-empty set is saturated iff it is an intersection of open sets. Let ψ be a family of sets. We say that ψ has the finite intersection property, or briefly, that ψ has f.i.p., if for every $P_1, P_2, \dots, P_k \in \psi$ it follows $P_1 \cap P_2 \cap \cdots \cap P_k \neq \varnothing$. For the definition of compactness, we do not assume any additional separation axiom. We say that a subset S of a space $(X, \mathcal{T})$ is compact if every open cover of S has a finite subcover. A topological space (X, τ) is said to be maximal compact if τ is maximal among all compact topologies on X. Let $\mathcal{D} \subseteq 2^X$. Recall that $S \subseteq X$ is compact with respect (or relative) to the family $\mathcal{D}$, if for every family $\mathcal{F} \subseteq \mathcal{D}$ such that the family $\{S\} \cup \mathcal{F}$ has f.i.p. it follows $S \cap (\bigcap \mathcal{F}) \neq \varnothing$, or equivalently, iff every filter base $\mathcal{F} \subseteq \mathcal{D}$ such that every element of $\mathcal{F}$ meets S the filter base $\mathcal{F}$ has a cluster point in S. Let $\mathcal{D} \subseteq \mathcal{C}$ be a closed base of $(X, \mathcal{T})$. It follows from Alexander's theorem that $S \subseteq X$ is compact iff it is compact with respect to $\mathcal{D}$. The notion 'compact relative to $\mathcal{D}$', however, has a limitation that it always works with all subsets of $\mathcal{D}$. Thus we slightly refine this notion to the following. Let X be a set and $\varXi \subseteq 2^{2^X}$. That means $\varXi$ is some collection of families of subsets of X. We say that a set $K \subseteq X$ is $\cap$-compact with respect to $\varXi$, if for every family $\mathcal{F} \in \varXi$ such that $\{K\} \cup \mathcal{F}$ has f.i.p. the set $K \cap (\bigcap \mathcal{F})$ is non-empty. A complementary notion is $\cup$-compactness, which can be defined as follows. We say that a set $K \subseteq X$ is $\cup$-compact with respect to $\varXi$, if for every family $\mathcal{O} \in \varXi$ such that $K \subseteq \bigcup \mathcal{O}$ there exist $U_1, U_2, \dots, U_k \in \mathcal{O}$ such that $K \subseteq \bigcup_{i=1}^{k} U_i$. We leave to the reader to consider how the less general modifications of compactness can be expressed by the more general notions.

2 The Hofmann-Mislove Theorem

Let us start with the following definition.

Definition 2.1. *Let $(X, \mathcal{T})$ be a space, $\mathcal{F} \subseteq \mathcal{T}$ a filter. We say that $\mathcal{F}$ is wide, if for every $U \in \mathcal{T}$, $U \in \mathcal{F}$ iff $P \in \mathcal{F}$ for every principal open superset P of U.*

Some equivalent conditions for the wideness of a filter are given by the following theorem. It can be seen that the wideness of a filter simplifies its behavior. Then it is determined by properties of the closure operator on singletons (condition (ii)). However, a compactness-like characterization is also possible (condition (iv)).

Proposition 2.1. *Let $(X, \mathcal{T})$ be a space, $\mathcal{C}$ be the family of its closed sets. Let $\mathcal{F} \subseteq \mathcal{T}$ a be filter and denote $\mathcal{F}^{\mathcal{C}} = \{X \setminus U|\, U \in \mathcal{F}\}$. The following conditions are equivalent.*

(i) *$\mathcal{F}$ is wide.*
(ii) *For each closed $C \in \mathcal{C}$, $\{\mathrm{cl}\{p\}|\, p \in C\} \subseteq \mathcal{F}^{\mathcal{C}}$ implies $C \in \mathcal{F}^{\mathcal{C}}$.*
(iii) *For every $U \in \mathcal{T}$, $\bigcap \mathcal{F} \subseteq U$ implies $U \in \mathcal{F}$.*

(iv) *For every* $C \in \mathcal{C}$ *which meets every* $F \in \mathcal{F}$, $C \cap (\bigcap \mathcal{F}) \neq \varnothing$.

Proof. Suppose (i). Let $C \in \mathcal{C}$ such that $\{\mathrm{cl}\{p\} | p \in C\} \subseteq \mathcal{F}^{\mathcal{C}}$. Denote $U = X \setminus C$. Let $P \supseteq U$, where $P = X \setminus \mathrm{cl}\{p\}$. Then $\mathrm{cl}\{p\} \subseteq C$, which means that $\mathrm{cl}\{p\} \in \mathcal{F}^{\mathcal{C}}$ and so $P \in \mathcal{F}$. Since $\mathcal{F}$ is wide, we have $U \in \mathcal{F}$, which yields $C \in \mathcal{F}^{\mathcal{C}}$, that is, (ii) holds.

Suppose (ii). Let $C \in \mathcal{C}$ and suppose that $C \cap F \neq \varnothing$ for every $F \in \mathcal{F}$. Then, in particular, $C \notin \mathcal{F}^{\mathcal{C}}$, which means that there exists $p \in \mathcal{C}$ such that $X \setminus \mathrm{cl}\{p\} \notin \mathcal{F}$. Then, if $U \in \mathcal{F}$, it follows $U \nsubseteq X \setminus \mathrm{cl}\{p\}$. Hence, $U \cap \mathrm{cl}\{p\} \neq \varnothing$, which implies $p \in U$. Therefore, $p \in C \cap (\bigcap \mathcal{F})$. Hence, it holds (iv).

The implication (iv) $\to$ (iii) is clear. Suppose (iii). Let $U \in \mathcal{T}$ such that for every $p \in X$ such that $U \subseteq X \setminus \mathrm{cl}\{p\}$ it holds $X \setminus \mathrm{cl}\{p\} \in \mathcal{F}$. Let $x \in \bigcap \mathcal{F}$. Then $X \setminus \mathrm{cl}\{x\} \notin \mathcal{F}$, which gives $U \nsubseteq X \setminus \mathrm{cl}\{x\}$, so $U \cap \mathrm{cl}\{x\} \neq \varnothing$. But then $x \in U$, which implies that $\bigcap \mathcal{F} \subseteq U$. From (iii) now it follows that $U \in \mathcal{F}$, so $\mathcal{F}$ is wide and (i) consequently holds.

For filters which are Scott open, their behavior with respect to prime open sets (or dually, irreducible closed sets) is sufficient to ensure the wideness of the filter.

Proposition 2.2. *Let* $(X, \mathcal{T})$ *be a space,* $\mathcal{C}$ *be the family of its closed sets,* $\mathcal{F} \subseteq \mathcal{T}$ *be a Scott open filter. Denote* $\mathcal{F}^{\mathcal{C}} = \{X \setminus U | U \in \mathcal{F}\}$. *The following statements are equivalent.*

(i) $\mathcal{F}$ *is wide.*
(ii) *For every prime* $U \in \mathcal{T}$, $U \in \mathcal{F}$ *iff* $P \in \mathcal{F}$ *for every principal open* $P \supseteq U$.
(iii) *For every prime* $U \in \mathcal{T}$, $\bigcap \mathcal{F} \subseteq U$ *implies* $U \in \mathcal{F}$.
(iv) *For every irreducible* $H \in \mathcal{C}$, $\{\mathrm{cl}\{p\} | p \in H\} \subseteq \mathcal{F}^{\mathcal{C}}$ *implies* $H \in \mathcal{F}^{\mathcal{C}}$.
(v) *For every irreducible* $H \in \mathcal{C}$ *which meets every* $F \in \mathcal{F}$, $H \cap (\bigcap \mathcal{F}) \neq \varnothing$.

Proof. The implication (i)$\to$(ii) is clear, as well as (ii) $\leftrightarrow$ (iv) and (iii) $\leftrightarrow$ (v). Suppose (ii). Let $U \in \mathcal{T}$ be prime and suppose that $\bigcap \mathcal{F} \subseteq U$. Let $U \subseteq X \setminus \mathrm{cl}\{p\}$ for some $p \in X$. Then $p \notin \bigcap \mathcal{F}$, so there is some $V \in \mathcal{F}$ such that $p \notin V$. Hence, $\mathrm{cl}\{p\} \subseteq X \setminus V$, which means that $V \subseteq X \setminus \mathrm{cl}\{p\}$. Consequently, $X \setminus \mathrm{cl}\{p\} \in \mathcal{F}$. By (ii), $U \in \mathcal{F}$. Hence, (iii) holds.

Suppose (iii). Let $U \in \mathcal{T}$ and let $\mathcal{F} \subseteq \mathcal{T}$ be a Scott open filter such that $\bigcap \mathcal{F} \subseteq U$. Suppose that $U \notin \mathcal{F}$. We put $\mathcal{M} = \{V | V \in \mathcal{T}, U \subseteq V \notin \mathcal{F}\}$. Let $\mathcal{L} \subseteq \mathcal{M}$ be a chain. Suppose that $\bigcup \mathcal{L} \in \mathcal{F}$. Since $\mathcal{F}$ is Scott open, there exists $V \in \mathcal{L}$ such that $V \in \mathcal{F}$, which is not possible, so $\bigcup \mathcal{L} \notin \mathcal{F}$. Then $\bigcup \mathcal{L} \in \mathcal{M}$ is an upper-bound of $\mathcal{L}$. By Zorn's Lemma, there exists a maximal element of $\mathcal{M}$, say W. Let $P, Q \in \mathcal{T}$ such that $W = P \cap Q$. It is not possible that $P \in \mathcal{F}$ and $Q \in \mathcal{F}$ since $\mathcal{F}$ is a filter. Then one of the sets P, Q, say P, is a superset of W such that $P \notin \mathcal{F}$. Then $P \in \mathcal{M}$ and from the maximality of W it follows that $P = W$. Hence, W is a prime set such that $\bigcap \mathcal{F} \subseteq W$. Then $W \in \mathcal{F}$ by (iii), which contradicts to the fact that W is a maximal element of $\mathcal{M}$. Therefore, $U \in \mathcal{F}$, which implies (i) by Proposition 2.1.

Finally, we can characterize the spaces in which all Scott open filters are wide.

Proposition 2.3. *Let $(X,\mathcal{T})$ be a space, $\mathcal{C}$ be the family of its closed sets. The following statements are equivalent.*

(i) *Every Scott open filter is wide.*
(ii) *Every irreducible closed set is a closure of a (not necessarily unique) singleton.*

Proof. Suppose (i). Let $H \in \mathcal{C}$ be irreducible. We put $\mathcal{F} = \{U \mid U \in \mathcal{T}, U \cap H \neq \varnothing\}$. Clearly $\varnothing \notin \mathcal{F}$ and it is also clear that $\mathcal{F}$ is an upper set. Let $U, V \in \mathcal{F}$ and suppose that $U \cap V \notin \mathcal{F}$. Then $U \cap V \cap H = \varnothing$, which means that $H \subseteq X \smallsetminus (U \cap V) = (X \smallsetminus U) \cup (X \smallsetminus V)$. We put $A = H \cap (X \smallsetminus U)$ and $B = H \cap (X \smallsetminus V)$. Then $A \cup B = H$. From the irreducibility of H it follows that $A = H$ or $B = H$. Suppose, for the certainty, that $A = H$. Then $H = H \cap (X \smallsetminus U)$, which implies that $H \subseteq X \smallsetminus U$. But this contradicts to the assumption that $U \in \mathcal{F}$. Hence, $U \cap V \in \mathcal{F}$, which means that $\mathcal{F}$ is a filter. Let $\mathcal{O} \subseteq \mathcal{T}$ be a collection such that $\bigcup \mathcal{O} \in \mathcal{F}$. Then $H \cap (\bigcap O) \neq \varnothing$, so there exists $U \in \mathcal{O}$ such that $U \in \mathcal{F}$. Now we know that $\mathcal{F}$ is a Scott open filter whose every element meets H. Then, by Proposition 2.2, we have $H \cap (\bigcap \mathcal{F}) \neq \varnothing$. Choose $x \in H \cap (\bigcap \mathcal{F})$. Then $\mathrm{cl}\{x\} \subseteq H$. Let $t \in H$ and let $U \in \mathcal{T}$ be such that $t \in U$. Then $U \cap H \neq \varnothing$, so $U \in \mathcal{F}$ and, consequently, $x \in U$. Hence, $t \in \mathrm{cl}\{x\}$, so $H = \mathrm{cl}\{x\}$, which implies (ii).

Suppose (ii). Let $H \in \mathcal{C}$ be irreducible, $\mathcal{F} \subseteq \mathcal{T}$ be a Scott open filter and suppose that $H \cap F \neq \varnothing$ for every $F \in \mathcal{F}$. There exists $x \in X$ such that $H = \mathrm{cl}\{x\}$, which implies that $x \in F$ for every $F \in \mathcal{F}$. Hence, $x \in H \cap (\bigcap \mathcal{F}) \neq \varnothing$. It follows from Proposition 2.2 that $\mathcal{F}$ is wide. That is, (i) holds.

If the space $(X,\mathcal{T})$ is topological, then the condition (ii) says that the space is quasisober. However, there may exist non-topological spaces satisfying this condition whose underlying topologies are not quasisober. So, it seems to be better and less confusing to choose another name for that property rather than to extend quasisobriety also to non-topological spaces. The implication (iii)$\rightarrow$(i) in Proposition 2.2 constitutes the core of the Keimel-Paseka's proof [6] of the Hofmann-Mislove theorem. Hence, we connect the name of the new property described by Proposition 2.3 with the names of Klaus Keimel and Jan Paseka.

Definition 2.2. *Let $(X,\mathcal{T})$ be a space. We say that it is a Keimel-Paseka space if every Scott open filter in $(X,\mathcal{T})$ is wide.*

Hence, by Proposition 2.3, every quasisober topological space is a Keimel-Paseka space. On the other hand, the only case in which the property of being Keimel-Paseka essentially differs from quasisobriety there are the non-topological spaces $(X,\mathcal{T})$ in which $X \notin \mathcal{C}$ is irreducible in the underlying topology but it is not a closure of a singleton (in the underlying topology). This is the only case when $\mathcal{T}$ cannot be extended to a quasisober topology on X. We can illustrate it by the following two examples.

Example 2.1. Let X be an infinite set, $\mathcal{T} = \{U | U \subseteq X, X \smallsetminus U \text{ is finite}\}$. Then $(X, \mathcal{T})$ is a non-topological Keimel-Paseka space, such that the topology $\tau = \mathcal{T} \cup \{\varnothing\}$ generated by $\mathcal{T}$ on X is not quasisober.

The construction of the next example is originally due to E. van Douwen [2], who however constructed it for another purpose.

Example 2.2. Let X be a set, $\mathcal{A} \subseteq 2^X$. Let $l : \mathcal{A} \to X$ be a mapping such that $l(A) \in X \smallsetminus A$ for every $A \in \mathcal{A}$. We put $\mathcal{S} = \{B \cup \{l(A)\} | B \subseteq A \in \mathcal{A}\}$ and use $\mathcal{S}$ as a subbase for the family $\mathcal{C}$ of the closed sets of a space $(X, \mathcal{T})$. Then $(X, \mathcal{T})$ is a Keimel-Paseka space.

Proof. Note that we do not automatically assume $X \in \mathcal{C}$, so $(X, \mathcal{T})$ is not topological if X cannot be generated from $\mathcal{S}$. Let us prove that $(X, \mathcal{T})$ has the declared property. Let $H \in \mathcal{C}$ be a closed set. Then it can be expressed in the form

$$H = \bigcap_{i \in I} \bigcup_{P_i(B,A)} (B \cup \{l(A)\}),$$

where I is some index set and for each $i \in I$, $P_i(B, A)$ is a condition that is only satisfied by finitely many pairs (B, A) such that $B \subseteq A \in \mathcal{A}$. Suppose that H is infinite. Then the set $D = H \smallsetminus \bigcap_{i \in I} \bigcup_{P_i(B,A)} \{l(A)\}$ is also infinite. Let $x \in D$. We put

$$H(x) = \bigcap_{i \in I} \bigcup_{P_i(B,A)} ((B \smallsetminus \{x\}) \cup \{l(A)\}).$$

One can easily check that $H(x)$ is closed and $H(x) = H \smallsetminus \{x\}$. If $a, b \in D$ are two distinct elements, then $H = H(a) \cup H(b)$ is a union of two strictly smaller closed sets, so H is not irreducible. Then each irreducible closed set in $(X, \mathcal{T})$ is finite and hence, $\cap$-compact with respect to $2^{\mathcal{T}}$. Now, it follows from Proposition 2.2 that $(X, \mathcal{T})$ is a Keimel-Paseka space.

Remark that a very special and interesting case of the previous example we get if we take $X = \mathbb{N}$ and $\mathcal{A}$ an ω-MAD subfamily of 2^X, that is, an infinite maximal subfamily of $[\mathbb{N}]^\omega = \{A | A \subseteq \mathbb{N}, |A| = \omega\}$ such that for each distinct $A, B \in \mathcal{A}$ the intersection $A \cap B$ is finite. It is shown in [2] that then the topology τ on X generated by $\mathcal{T}$ is anti-Hausdorff, Fréchet and the convergent sequences have unique limits. Moreover, by [11], (X, τ) is maximal compact. Then its de Groot dual (see [4], [5], [8] and [9]), which is the topology generated by compact (saturated) sets used as its closed subbase, equals to the original topology. Hence, we have an example which shows that the de Groot dual of a compact T_1 topological space need not be sober. However, as we will see next, since this space is Keimel-Paseka, it satisfies the Hofmann-Mislove theorem, even in a formulation that formally agrees with its classical version.

Now, we can formulate the generalized Hofmann-Mislove theorem which holds for any space. The cost of this improvement, however, is that we need to restrict the class of the considered Scott open filters.

Theorem 2.1. *Let $(X,\mathcal{T})$ be a space. There exists a (order-reversing) bijection between the wide Scott open filters and the compact saturated sets in $(X,\mathcal{T})$, which is given by the following construction:*

(i) *If $\mathcal{F}$ is a wide Scott open filter, then $\bigcap\mathcal{F}$ is the corresponding compact saturated set.*
(ii) *If K is a compact saturated set, then $\mathcal{F}=\{U \mid U\in\mathcal{T}, K\subseteq U\}$ is the corresponding wide Scott open filter.*

Proof. Let $\mathcal{F}\subseteq\mathcal{T}$ be a wide Scott open filter, $\mathcal{O}\subseteq\mathcal{T}$ be an open cover of $K=\bigcap\mathcal{F}$. Then $K\subseteq\bigcup\mathcal{O}$. It follows from Proposition 2.2 that $\bigcup\mathcal{O}\in\mathcal{F}$ and since $\mathcal{F}$ is Scott open, there exist $U_1,U_2,\dots,U_k\in\mathcal{O}$ such that $K\subseteq\bigcup_{i=1}^k U_i$. Hence, K is compact. Note that K is also saturated as an intersection of open sets.

Conversely, if K is compact, then $\mathcal{F}=\{U \mid U\in\mathcal{T}, K\subseteq U\}$ is a Scott open filter and $K\subseteq\bigcap\mathcal{F}$. Moreover, if K is saturated and $x\notin K$, then there exists $U\in\mathcal{T}$ such that $K\subseteq U$ and $x\notin U$. Then $x\notin\bigcap\mathcal{F}$, so $K=\bigcap\mathcal{F}$. Now, if $\bigcap\mathcal{F}\subseteq U$ for some $U\in\mathcal{T}$, we have $U\in\mathcal{F}$, so $\mathcal{F}$ is wide by Proposition 2.2.

Of course, in a Keimel-Paseka space, the assumption of wideness of the considered Scott open filters may be omitted. Let $(X,\mathcal{T})$ be a space, $\mathcal{F}\subseteq\mathcal{T}$ a wide filter. Note that if $\bigcap\mathcal{F}=\varnothing$, then $\mathcal{F}=\mathcal{T}$. In particular, if $(X,\mathcal{T})$ is a minusspace, $\bigcap\mathcal{F}=\varnothing$ iff $\mathcal{F}=\mathcal{T}$. If $(X,\mathcal{T})$ is a topological space, $\bigcap\mathcal{F}=\varnothing$ iff $\varnothing\in\mathcal{F}$.

Corollary 2.1. *Let $(X,\mathcal{T})$ be a Keimel-Paseka space. If $\mathcal{K}$ is a non-empty filtered family of compact saturated sets, then $\bigcap\mathcal{K}$ is compact and saturated.*

Proof. We put $\mathcal{F}=\{U \mid U\in\mathcal{T}$, there exists $K\in\mathcal{K}$ such that $K\subseteq U\}$. Then $\mathcal{F}$ is a wide Scott open filter, so $\bigcap\mathcal{F}$ is compact saturated and we have $\bigcap\mathcal{K}\subseteq\bigcap\mathcal{F}$. Suppose that $x\notin\bigcap\mathcal{K}$. Then there exists $K\in\mathcal{K}$ such that $x\notin K$. Since K is saturated and hence an intersection of open sets, there exists $U\in\mathcal{T}$ with $K\subseteq U$ and $x\notin U$. Then $U\in\mathcal{F}$ and so $x\notin\bigcap\mathcal{F}$. It means that $\bigcap\mathcal{K}=\bigcap\mathcal{F}$. Moreover, if $(X,\mathcal{T})$ is topological, then $\bigcap\mathcal{K}=\varnothing$ implies $\varnothing\in\mathcal{F}$, which gives $\varnothing\in\mathcal{K}$.

The following result is already known as a corollary of the classical Hofmann-Mislove theorem.

Corollary 2.2. *Let (X,τ) be a quasisober topological space. If $\mathcal{K}$ is a non-empty filtered family of non-empty compact saturated sets, then $\bigcap\mathcal{K}$ is compact, saturated and non-empty.*

Corollary 2.3. *Let $(X,\mathcal{T})$ be a Keimel-Paseka space. Then every $C\in\mathcal{C}$ is $\cap$-compact with respect to the filtered families of compact saturated sets.*

Proof. Let $C\in\mathcal{C}$ and let $\mathcal{K}$ be a filtered family of compact saturated sets, such that $C\cap K\neq\varnothing$ for every $K\in\mathcal{K}$. Similarly as in proof of Corollary 2.1 we put $\mathcal{F}=\{U \mid U\in\mathcal{T}$, there exists $K\in\mathcal{K}$ such that $K\subseteq U\}$. Then $\mathcal{F}\subseteq\mathcal{T}$ is a Scott open filter such that $C\cap U\neq\varnothing$ for every $U\in\mathcal{F}$. Since $(X,\mathcal{T})$ is Keimel-Paseka, we have $C\cap(\bigcap\mathcal{F})\neq\varnothing$. Since the elements of $\mathcal{K}$ are saturated sets, $\bigcap\mathcal{K}=\bigcap\mathcal{F}$, which completes the proof.

In a general space, there can exist Scott open filters that are not wide but which, however, have compact intersections. The following theorem describes the relationship of the wide Scott open filters to the other Scott open filters with the same compact intersection.

Theorem 2.2. *Let $(X, \mathcal{T})$ be a space, $\mathcal{F} \subseteq \mathcal{T}$ a filter. The following statements are equivalent:*

(i) *$\mathcal{F}$ is wide and Scott open.*
(ii) *$\mathcal{F}$ is wide and $\bigcap \mathcal{F}$ is compact.*
(iii) *$\mathcal{F}$ is maximal among all Scott open filters with the same compact intersection.*
(iv) *$\mathcal{F}$ is the greatest Scott open filter having the same compact intersection.*

Proof. Suppose (i). It follows from Theorem 2.1 and its proof that $\bigcap \mathcal{F}$ is compact. We have (ii). Let $\mathcal{F}' \subseteq \mathcal{T}$ be a Scott open filter such that $\bigcap \mathcal{F} = \bigcap \mathcal{F}'$. Then for every $U \in \mathcal{F}'$, $\bigcap \mathcal{F} \subseteq U$ and since $\mathcal{F}$ is wide, by Proposition 2.2 we have $U \in \mathcal{F}$. Hence, $\mathcal{F}' \subseteq \mathcal{F}$, which also gives (iv). The implication (iv)→(iii) is clear. Suppose (iii). We put $K = \bigcap \mathcal{F}$. Then K is a compact saturated set and $\mathcal{G} = \{U | U \in \mathcal{T}, K \subseteq U\}$ is a wide Scott open filter containing $\mathcal{F}$. From maximality of $\mathcal{F}$ it follows $\mathcal{F} = \mathcal{G}$. Hence, $\mathcal{F}$ is wide and we have (i). Suppose (ii). Let $K = \bigcap \mathcal{F}$. Then $\mathcal{G} = \{U | U \in \mathcal{T}, K \subseteq U\}$ is a wide Scott open filter containing $\mathcal{F}$ and since $\mathcal{F}$ is wide, $\mathcal{G} = \mathcal{F}$. Then $\mathcal{F}$ is a wide Scott open filter and we have (i).

Now, we have the implications (i)→(ii), (i)→(iv), (iv)→(iii), (iii)→(i) and (ii)→(i). Hence, all the statements (i), (ii), (iii) and (iv) are equivalent.

Remark that Theorem 2.1 and Theorem 2.2 together with Proposition 2.2 connects two kinds of compactness – the $\cap$-compactness of the closed (irreducible) sets with respect to the Scott open filters, which is responsible for their wideness, and $\cup$-compactness of their intersections with respect to the families of open sets.

3 Some closing remarks

Originally, the study and development of the modified Hofmann-Mislove Theorem was motivated by the effort of solving the question of D. E. Cameron, wether every compact topology is contained in some maximal compact topology [1]. It should be noted that H-P. Künzi and D. van der Zypen proved in [11], among other very interesting results, that any compact T_1 sober topology is contained in some maximal compact topology. The version of the Hofmann-Mislove Theorem presented here allows to extend this result to Keimel-Paseka spaces (some advanced properties of de Groot dual are also needed, see e.g. [8] and [9]). However, the author recently get a much better and more general result, which also uses the techniques and ideas developed in this paper, but in their final form they are too special and too modified to be able presented here without some essential changes of the paper. Since these changes would probably make the theory more complicated, less straightforward and worse understandable, the author decided to present the solution of the Cameron's question separately from the main theory. Unfortunately, the author so far has no alternative example able to demonstrate the utility of the developed theory similarly as the previous contribution to the Cameron's question. Nevertheless, the author believes that extending well-known theorems (as Hofmann-Mislove's) to more general setting is still interesting and potentially useful anyway.

References

1. Cameron, D. E., *A survey of maximal topological spaces*, Topology Proc. 2 (1977), 11-60.
2. van Douwen, E. K., *An anti-Hausdorff Fréchet space in which convergent sequences have unique limits*, Topology Appl. 51 (1993), 147-158.
3. Gierz G., Hofmann K. H., Keimel K., Lawson J. D., Scott D. S., *Continuous Lattices and Domains*, Cambridge University Press, Cambridge (2003), pp. 1-591.
4. de Groot J.,*An Isomorphism Principle in General Topology*, Bull. Amer. Math. Soc. 73 (1967) 465-375.
5. de Groot J., Herrlich H., Strecker G. E., Wattel E. *Compactness as an Operator*, Compositio Mathematica 21 (4) (1969) 349-375.
6. Keimel K., Paseka J., *A direct proof of the Hofmann-Mislove theorem*, Proc. Amer. Math. Soc. 120 (1994), 301-303.
7. Kopperman R., *Asymetry and Duality in Topology*, Topology Appl. 66 (1) (1995) 1-39.
8. Kovár, M. M., *At most 4 topologies can arise from iterating the de Groot dual*, Topology Appl. 130 (2003) 175-182.
9. Kovár, M. M., *On Iterated De Groot Dualizations of Topological Spaces*, Topology Appl. (to appear), 1-7.
10. Kovár, M. M., *Hofmann-Mislove Theorem for general posets*, Preprint (2004), 1-16.
11. Künzi, H-P. A., van der Zypen, D. *Maximal (sequentially) compact topologies*, Categorial Structures and Their Applications (W. Gähler, G. Preuss, eds.), World Scientific Publishing Company, 2004, pp. 173-187.