The de Groot dual for general collections of sets

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Abstract. A topology is de Groot dual of another topology, if it has a closed base consisting of all its compact saturated sets. Until 2001 it was an unsolved problem of J. Lawson and M. Mislove whether the sequence of iterated dualizations of a topological space is finite. In this paper we generalize the author's original construction to an arbitrary family instead of a topology. Among other results we prove that for any family $\mathfrak{C} \subseteq 2^X$ it holds $\mathfrak{C}^{dd} = \mathfrak{C}^{dddd}$. We also show similar identities for some other similar and topology-related structures.

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1 Preliminaries and Introduction

J. Lawson and M. Mislove stated a problem [7], which topologies can arise as de Groot dual topologies and whether the process of taking duals terminates by two topologies which are dual to each other. The partial solutions were given by G. E. Strecker, J. de Groot and E. Wattel (1966, [2]) – about 30 years before the question was stated – for T_1 spaces, and in 2000 by B. Burdick for certain topologies on hyperspaces. The final and general solution of the second part of the question was given by the author in 2001 [5]. The first part of the problem, that is, which topologies can arise as de Groot duals, still seems to be unsolved.

The author experimented with various topology-related structures to obtain analogous results as for topological spaces in [5]. Some mathematicians commented the proof presented in [5] as a very technical one, and still it is not completely clear, whether there exists some more illuminating theory behind. However, surprisingly it seems that the topology is not essential for the main results. Certainly, any family of sets can be used as a closed subbase for a topology which one can dualize by taking the compact saturated sets as the closed

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subbase for the new, dual topology. But the topological link of that chain can be skipped if one can, in some alternative and proper way, define or replace the notions of compactness and saturation. This approach is new in this paper in the comparison with [5] or [6] although the main results can also be found in [6]. The author believes that the initial step for solving also the remaining part of the problem of J. Lawson and M. Mislove, that is, finding out which topologies can arise as duals, is understanding the process of creating the dual in detail. In this paper we reduce the process of constructing the dual by skipping topology. The iteration properties of the dual now are consequences of some algebra and combinatorics of infinite sets, while the topology need not be mentioned during the proof at all. This is the main difference from the approach of the previous papers [5] and [6].

Secondly, if the properties of the topological dual are consequences derived from something more general, there is no essential reason why not mention also some other, topology-related structures, like the closure spaces, the nearness spaces or the topological systems. All of them are, less or more, in relationships with some topologies in their background. If we want to define properly the corresponding de Groot dual for these structures, the construction should respect these topologies. Hence, the iteration properties could not differ too much from the topological case and of course, these results could not be very deep. However, we mention them in this paper because of completeness. But first, let us make some denotations and recall some notions.

Let X be a set. We say that a binary relation $\leq$ on X is a *preorder* if it is reflexive and transitive. Let $A \subseteq X$. We denote $\uparrow A = \{x|\, x \in X, \exists a \in A : a \leq x\}$ and $\downarrow A = \{x|\, x \in X, \exists a \in A : x \leq a\}$. Let ψ be a family of sets. We say that ψ has the finite intersection property, or briefly, that ψ has f.i.p., if for every $P_1, P_2, \dots, P_k \in \psi$ it follows $P_1 \cap P_2 \cap \cdots \cap P_k \neq \varnothing$. Let $\mathfrak{C} \subseteq 2^X$. We say that a set $K \subseteq X$ is *compact with respect to* $\mathfrak{C}$ if for every $\zeta \subseteq \mathfrak{C}$ such that the family $\{K\} \cup \zeta$ has f.i.p. it follows $K \cap (\bigcap \zeta) \neq \varnothing$. Let $\mathrm{cl} : 2^X \to 2^X$. We say that cl is a *closure operator* and (X, cl) is a *closure space* if $\mathrm{cl}\, \varnothing = \varnothing$ and for every $A, B \subseteq X$ it holds $A \subseteq \mathrm{cl}\, A$ and $\mathrm{cl}(A \cup B) = \mathrm{cl}\, A \cup \mathrm{cl}\, B$. The closure cl is topological iff for every $A \subseteq X$, $\mathrm{cl}\, A = \mathrm{cl}\,\mathrm{cl}\, A$. The family $\xi \subseteq 2^{2^X}$ is called a *nearness* and (X, ξ) is said to be a *nearness space* [4] if the following five conditions are satisfied:

- (N_1) If $\mathfrak{A}, \mathfrak{B} \subseteq 2^X$ such that for every $A \in \mathfrak{A}$ there is $B \in \mathfrak{B}$ such that $B \subseteq A$ and $\mathfrak{B} \in \xi$, then $\mathfrak{A} \in \xi$.
- (N_2) If $\bigcup \mathfrak{A} \neq \varnothing$, then $\mathfrak{A} \in \xi$.
- (N_3) $\varnothing \neq \xi \neq 2^{2^X}$.
- (N_4) If $\{A \cup B|\, A \in \mathfrak{A}, B \in \mathfrak{B}\} \in \xi$ then $\mathfrak{A} \in \xi$ or $\mathfrak{B} \in \xi$.
- (N_5) If $\{\{x|\, x \in X, \{A, \{x\}\} \in \xi\} \,|\, A \in \mathfrak{A}\} \in \xi$, then $\mathfrak{A} \in \xi$.

Note that in the literature one can also find some other, slightly modified axioms for the nearness structure. A poset $(A, \leq)$ is a *frame* if every subset has a join, every finite subset has a meet and for every $a \in A$, $B \subseteq A$ it holds $a \wedge (\bigvee B) = \bigvee_{b \in B} (a \wedge b)$. A *topological system* is a triple $(X, A, \vdash)$, where X is a set, A a frame and $\vdash \subseteq X \times A$ a binary relation such that for every $B \subseteq A$, finite $C \subseteq A$

and $x \in X$ it holds $(x \vdash \bigvee B) \Leftrightarrow (x \vdash b)$ for some $b \in B$ and $(x \vdash \bigwedge C) \Leftrightarrow (x \vdash c)$ for each $c \in C$. Let (X, τ) be a topological space. The preorder $\leqslant$ on X given by $x \leqslant y \Leftrightarrow x \in \operatorname{cl}\{y\}$ is called the *preorder of specialization* on X. A set $A \subseteq X$ is *saturated* if A is upper-closed in the specialization preorder. The topology τ^d is said to be *(de Groot) dual* to τ or *cocompact* if τ^d has a closed subbase consisting of all compact saturated sets in (X, τ). A topological space (X, τ) is said to be *symmetric* or R_0 if the specialization preorder is a symmetric binary relation. All topological spaces and other topology-related structures are assumed without any additional separation axioms.

2 Main Results

Given a family $\mathfrak{C} \subseteq 2^X$, we need to construct an analogue of the specialization preorder on X. We put $\leq_{\mathfrak{C}} = \{(x, y) | \, \forall C \in \mathfrak{C} : y \in C \implies x \in C\}$. Obviously, $\leq_{\mathfrak{C}}$ is a preorder on X and for every $C \in \mathfrak{C}$ it follows $C = \downarrow_{\mathfrak{C}} C$. The family $\mathfrak{C}$ is called *down-complete* if $\{\downarrow_{\mathfrak{C}} \{x\} \,|\, x \in X\} \subseteq \mathfrak{C}$. We say that $\mathfrak{C}$ is a *up-compact* if every $C \in \mathfrak{C}$ is compact with respect to the family $\{\uparrow_{\mathfrak{C}} \{x\} \,|\, x \in X\}$. We denote $\mathfrak{C}^* = \mathfrak{C} \cup \{\downarrow_{\mathfrak{C}} \{x\} \,|\, x \in X\}$. Obviously, $\mathfrak{C}^*$ is a down-complete family. We also denote $\mathfrak{C}^d = \{P | \, P \subseteq X, P = \uparrow_{\mathfrak{C}} P, P$ is compact with respect to $\mathfrak{C}\}$. One of the most important properties of the dual is that it switches the corresponding (specialization) preorder similarly as in topological spaces.

Proposition 2.1. *Let $\mathfrak{C} \subseteq 2^X$. Then $\leq_{\mathfrak{C}^d}$ is an inverse preorder relation to $\leq_{\mathfrak{C}}$.*

Proof. Suppose that for some $x, y \in X$ we have $x \leq_{\mathfrak{C}} y$. Let $x \in P$ for some $P \in \mathfrak{C}^d$. Then $y \in \uparrow_{\mathfrak{C}} P = P$ and so $y \leq_{\mathfrak{C}^d} x$. Conversely, suppose $y \leq_{\mathfrak{C}^d} x$. Then $\uparrow_{\mathfrak{C}} \{x\}$ is compact with respect to $\mathfrak{C}$ and so $y \in \uparrow_{\mathfrak{C}} \{x\}$. Hence $x \leq_{\mathfrak{C}} y$.

Let us give several examples of duals in various structures which have some topology in their background. The construction of the dual is possible either because the topological space is simply a special case of that structure (e.g. the closure spaces) or because the topology generated by that structure is special in some sense as well as its dual topology, which allows us to return back to the original structure (e.g. the nearness spaces and the symmetry of the generated topology).

Example 2.1. Let (X, τ) be a topological space, $\mathfrak{C} \subseteq 2^X$ the family of all closed sets or some closed base for τ. Then $\downarrow_{\mathfrak{C}} \{x\} = \operatorname{cl}\{x\}$ and $\leq_{\mathfrak{C}}$ is a preorder of specialization on X. The family $\mathfrak{C}$ is down-complete, $\mathfrak{C}^d$ is the family of all compact saturated sets, which generates the dual topology τ^d on X as its closed subbase.

Example 2.2. Let (X, cl) be a closure space whose closure is not necessarily topological. We put $\mathfrak{C} = \{\operatorname{cl} A | \, A \subseteq X\}$. As the dual closure cl^d we take the (topological) closure in the topological space (X, τ^d), where τ^d is generated by its closed subbase $\mathfrak{C}^d$.

Example 2.3. Let (X,ξ) be a nearness space. We put $\mathfrak{C}=\{\mathrm{cl}_\xi A|\, A\subseteq X\}$, where $\mathrm{cl}_\xi A=\{x|\, x\in X, \{A,\{x\}\}\in\xi\}$. The closure cl_ξ is symmetric and topological. Hence $\downarrow_{\mathfrak{C}}\{x\}=\mathrm{cl}_\xi\{x\}$ for every $x\in X$. Then the relation $\leq_{\mathfrak{C}}$ is symmetric as well as $\leq_{\mathfrak{C}^d}$ by Proposition 2.1. Therefore, the dual topological space (X,τ^d) is symmetric or R_0 and so it generates a (topological) nearness structure $\xi^d=\left\{\mathfrak{A}|\,\mathfrak{A}\subseteq 2^X, \bigcap\left\{\mathrm{cl}^d A|\, A\in\mathfrak{A}\right\}\neq\varnothing\right\}$, where cl^d is the closure in (X,τ^d), such that $\mathrm{cl}_{\xi^d}=\mathrm{cl}^d$.

Example 2.4. Let $(X,A,\vdash)$ be a topological system. As a dual topological system to $(X,A,\vdash)$ we can take any topological system $(X,B,\models)$, such that the topology $\tau_X(B)$ on X induced by the frame B is dual to the topology $\tau_X(A)$ induced by the frame A. Note that for a frame B such a topological system exists iff there is a frame epimorphism $e: B\to(\tau_X(A))^d$. Indeed, if $(\tau_X(A))^d=\tau_X(B)$, then the spatialization mapping $e: B\to(\tau_X(A))^d$ is the desired frame epimorphism. Conversely, if such epimorphism e exists, one can put $x\models b$ iff $x\in e(b)$ for every $b\in B$ and $x\in X$. In this case we say that B represents a dualization of $(X,A,\vdash)$.

In the several following propositions and lemmas, in fact we will follow the author's construction presented already in [5] for topological spaces. However, now we are in a situation which is more complicated and so some steps of the original proof cannot be repeated so straightforwardly. It comes out that it is more easy and convenient to work with down-complete families. If we do not have such a family, we can complete it. However, we have to check that the completion behaves well with respect to the dual.

Proposition 2.2. *Let* $\mathfrak{C}\subseteq 2^X$. *Then* $\mathfrak{C}^d=\mathfrak{C}^{*d}$.

Proof. Firstly, let us notice that $\leq_{\mathfrak{C}^*}$ is the same binary relation, as $\leq_{\mathfrak{C}}$. Indeed, since $\mathfrak{C}\subseteq\mathfrak{C}^*$, it follows that $\leq_{\mathfrak{C}^*}\subseteq\leq_{\mathfrak{C}}$. Conversely, suppose that for some $x,y\in X$ we have $x\leq_{\mathfrak{C}} y$. Suppose that $y\in C$ for some $C\in\mathfrak{C}^*$. If $C\in\mathfrak{C}$, we immediately have $x\in C$. Let $C\in\mathfrak{C}^*\smallsetminus\mathfrak{C}$. There exists some $z\in X$ such that $C=\downarrow_{\mathfrak{C}}\{z\}$. Then $y\leq_{\mathfrak{C}} z$, so $x\leq_{\mathfrak{C}} z$ and again $x\in C$, which gives $x\leq_{\mathfrak{C}^*} y$. Therefore, $\leq_{\mathfrak{C}}$ and $\leq_{\mathfrak{C}^*}$ are the same relations. Since $\mathfrak{C}\subseteq\mathfrak{C}^*$, any set that is compact with respect to $\mathfrak{C}^*$, obviously is compact with respect to $\mathfrak{C}$. Involving the previous observation, we get $\mathfrak{C}^{*d}\subseteq\mathfrak{C}^d$. Suppose that $K\in\mathfrak{C}^d$ and take any $\zeta\subseteq\mathfrak{C}^*$ such that the family $\{K\}\cup\zeta$ has f.i.p. There exist some $\xi\subseteq\mathfrak{C}$ and $S\subseteq X$ such that $\zeta=\xi\cup\{\downarrow_{\mathfrak{C}}\{s\}\,|\, s\in S\}$. Put $\psi=\xi\cup\{C|\, C\in\mathfrak{C}, C\cap S\neq\varnothing\}\subseteq\mathfrak{C}$. We will show that $\{K\}\cup\psi$ has f.i.p. Let $\kappa\subseteq\psi$ be finite. Then $\kappa=\kappa_1\cup\kappa_2$, where $\kappa_1\subseteq\xi$ and $C\cap S\neq\varnothing$ for every $C\in\kappa_2$. Take some $s_C\in C\cap S$ for every $C\in\kappa_2$. Observe that $\downarrow_{\mathfrak{C}}\{s_C\}\subseteq C$. It follows $\eta=\kappa_1\cup\{\downarrow_{\mathfrak{C}}\{s_C\}\,|\, C\in\kappa_2\}$ is a finite subset of ζ and so $\varnothing\neq K\cap(\bigcap\eta)\subseteq K\cap(\bigcap\kappa)$. Hence, $\{K\}\cup\psi$ has f.i.p. Since K is compact with respect to $\mathfrak{C}$, it follows that $\varnothing\neq K\cap(\bigcap\psi)=K\cap(\bigcap\xi)\cap(\bigcap\{C|\, C\in\mathfrak{C}, C\cap S\neq\varnothing\})=K\cap(\bigcap\xi)\cap(\bigcap_{s\in S}\bigcap\{C|\, C\in\mathfrak{C}, s\in C\})=K\cap(\bigcap\xi)\cap(\bigcap_{s\in S}\downarrow_{\mathfrak{C}}\{s\})=K\cap(\bigcap\zeta)$. It follows that $K\in\mathfrak{C}^{*d}$ and hence $\mathfrak{C}^d=\mathfrak{C}^{*d}$.

The dual already is down-complete, as the next proposition shows.

Proposition 2.3. *Let $\mathfrak{C} \subseteq 2^X$. Then $\mathfrak{C}^d$ is down-complete and up-compact.*

Proof. One can easily check that the set $\uparrow_{\mathfrak{C}} \{x\}$ is compact with respect to $\mathfrak{C}$ for every $x \in X$. Then $\{\uparrow_{\mathfrak{C}} \{x\} \,|\, x \in X\} \subseteq \mathfrak{C}^d$. Hence, $\mathfrak{C}^d$ is down-complete. Suppose firstly, that $\mathfrak{C}$ is down-complete. Then every element of $\mathfrak{C}^d$ is compact with respect to $\{\downarrow_{\mathfrak{C}} \{x\} \,|\, x \in X\}$, which gives that $\mathfrak{C}^d$ is an up-compact family. But $\mathfrak{C}^d = \mathfrak{C}^{*d}$ by Proposition 2.2 and $\mathfrak{C}^*$ is down-complete. Hence, $\mathfrak{C}^d$ is up-compact in any case.

Another important point of our construction it is that the compactness of a set is, as we can expect, preserved by saturation.

Proposition 2.4. *Let $K \subseteq X$ be compact with respect to $\mathfrak{C} \subseteq 2^X$. Then $\uparrow_{\mathfrak{C}} K$ is compact with respect to $\mathfrak{C}$.*

Proof. Let $\xi \subseteq \mathfrak{C}$ be a family such that the family $\xi \cup \{\uparrow_{\mathfrak{C}} K\}$ has f.i.p. Then, for every $C_1, C_2, \ldots C_k \in \xi$, it follows $(\uparrow_{\mathfrak{C}} K) \cap C_1 \cap C_2 \cap \cdots \cap C_k \neq \varnothing$. There exists an element $t \in (\uparrow_{\mathfrak{C}} K) \cap C_1 \cap C_2 \cap \cdots \cap C_k$ (depending on the choice of $C_1, C_2, \ldots C_k$). Hence, there is some $s \in K$, such that $s \leq_{\mathfrak{C}} t$. But, for every $i = 1, 2, \ldots, k$ it follows $C_i = \downarrow_{\mathfrak{C}} C_i$ and so $s \in K \cap C_1 \cap C_2 \cap \cdots \cap C_k$. Then the family $\xi \cup \{K\}$ has f.i.p. Since K is compact with respect to $\mathfrak{C}$, we have $\varnothing \neq K \cap (\bigcap \xi) \subseteq (\uparrow_{\mathfrak{C}} K) \cap (\bigcap \xi)$. It follows that $\uparrow_{\mathfrak{C}} K$ is compact with respect to $\mathfrak{C}$.

We also need to verify some intersection properties of compact sets with respect to the original family $\mathfrak{C}$ as well as with its second dual.

Proposition 2.5. *Let $C \in \mathfrak{C} \subseteq 2^X$ and $K \subseteq X$ be compact with respect to $\mathfrak{C}$. Then $C \cap K$ is compact with respect to $\mathfrak{C}$.*

Proof. Let $\xi \subseteq \mathfrak{C}$ be a family such that the family $\xi \cup \{C \cap K\}$ has f.i.p. Then also family $\xi \cup \{C\} \cup \{K\}$ has f.i.p. But $\zeta = \xi \cup \{C\} \subseteq \mathfrak{C}$ and K is compact with respect to $\mathfrak{C}$, so $(C \cap K) \cap (\bigcap \xi) = K \cap (\bigcap \zeta) \neq \varnothing$. Hence, $C \cap K$ is compact with respect to $\mathfrak{C}$.

The following result is a modification of a lemma due to B. Burdick [1].

Proposition 2.6. *Let $\mathfrak{C} \subseteq 2^X$ be down-complete, K be compact with respect to $\mathfrak{C}$ and $P \in \mathfrak{C}^{dd}$. Then $K \cap P$ is compact with respect to $\mathfrak{C}$.*

Proof. Let $\zeta \subseteq \mathfrak{C}$ be a family such that the family $\zeta \cup \{K \cap P\}$ has f.i.p. Let $\eta = \{\uparrow_{\mathfrak{C}} (K \cap D_1 \cap D_2 \cap \cdots \cap D_m) |\, D_1, D_2, \ldots, D_m \in \zeta\}$. It follows from Proposition 2.4 and Proposition 2.5 that $\eta \subseteq \mathfrak{C}^d$ and $\{P\} \cup \eta$ has f.i.p. Since $P \in \mathfrak{C}^{dd}$, it follows that there exists some $x \in P \cap (\bigcap \eta) \neq \varnothing$. Hence, $x \in P \cap \uparrow_{\mathfrak{C}} (K \cap D_1 \cap D_2 \cap \cdots \cap D_m)$ for every $D_1, D_2, \ldots, D_m \in \zeta$. It follows that there is some $t \in K \cap D_1 \cap D_2 \cap \cdots \cap D_m$ (depending on the choice of $D_1, D_2, \ldots, D_m$) such that $t \leq_{\mathfrak{C}} x$. Then $\downarrow_{\mathfrak{C}} \{x\} \cap K \cap D_1 \cap D_2 \cap \cdots \cap D_m \neq \varnothing$ for every $D_1, D_2, \ldots, D_m \in \zeta$, so the collection $\{K\} \cup (\{\downarrow_{\mathfrak{C}} \{x\}\} \cup \zeta)$ has f.i.p. Since $\mathfrak{C}$ is down-complete, it follows that $\{\downarrow_{\mathfrak{C}} \{x\}\} \cup \zeta \subseteq \mathfrak{C}$. Since K is compact with respect to $\mathfrak{C}$ and $\mathfrak{C}^{dd}$ is down-complete by Proposition 2.3, it follows $\varnothing \neq K \cap \downarrow_{\mathfrak{C}} \{x\} \cap (\bigcap \zeta) \subseteq K \cap P \cap (\bigcap \zeta)$. Hence, $K \cap P$ is compact with respect to $\mathfrak{C}$.

Now, we can start the main part of the proof. We need the following two lemmas for the proof of the main theorem.

Lemma 2.1. *Let $\mathfrak{C} \subseteq 2^X$ be down-complete, $K \in \mathfrak{C}^d$, $\psi \subseteq \mathfrak{C} \cup \mathfrak{C}^{dd}$, $M \in \psi \cap \mathfrak{C}^{dd}$ and let $\{K\} \cup \psi$ has f.i.p. Then there exists $\xi(M) \in M$ such that $\{K\} \cup \psi \cup \{\downarrow_{\mathfrak{C}} \{\xi(M)\}\}$ has f.i.p.*

Proof. Let $\varphi = \{\uparrow_{\mathfrak{C}} (K \cap P_1 \cap P_2 \cap \cdots \cap P_k) | \, P_1, P_2, \ldots, P_k \in \psi\}$. From Proposition 2.4, Proposition 2.5 and Proposition 2.6 it follows that $\varphi \subseteq \mathfrak{C}^d$ and $\{M\} \cup \varphi$ has f.i.p. Hence, since $M \in \mathfrak{C}^{dd}$ is compact with respect to $\mathfrak{C}^d$, there exists $\xi(M) \in M \cap (\bigcap \varphi)$. Then for every $P_1, P_2, \ldots, P_k \in \psi$ it follows that $\xi(M) \in \uparrow_{\mathfrak{C}} (K \cap P_1 \cap P_2 \cap \cdots \cap P_k)$ so there exists $t \in K \cap P_1 \cap P_2 \cap \cdots \cap P_k$ (depending on the choice of $P_1, P_2, \ldots, P_k$) with $t \leq_{\mathfrak{C}} \xi(M)$. Then $t \in \downarrow_{\mathfrak{C}} \{\xi(M)\}$ which implies that $\downarrow_{\mathfrak{C}} \{\xi(M)\} \cap K \cap P_1 \cap P_2 \cap \cdots \cap P_k \neq \varnothing$. It follows that $\{K\} \cup \psi \cup \{\downarrow_{\mathfrak{C}} \{\xi(M)\}\}$ has f.i.p.

Lemma 2.2. *Let $\mathfrak{C} \subseteq 2^X$ be down-complete, $K \in \mathfrak{C}^d$, $\psi \subseteq \mathfrak{C} \cup \mathfrak{C}^{dd}$ and let $\{K\} \cup \psi$ has f.i.p. Then for every $M \in \psi \cap \mathfrak{C}^{dd}$ there exists $\xi(M) \in M$ such that $\{K\} \cup \{\psi \cap \mathfrak{C}\} \cup \left\{\downarrow_{\mathfrak{C}} \{\xi(M)\} \,|\, M \in \psi \cap \mathfrak{C}^{dd}\right\}$ has f.i.p.*

Proof. We put $\psi_1 = \psi \cap \mathfrak{C}$, $\psi_2 = \psi \cap \mathfrak{C}^{dd}$. Let $\psi_2 = \{M_\alpha | \, \alpha < \mu\}$ where μ is an ordinal number. From Lemma 2.1 it follows that there exists $\xi(M_0) \in M_0$ such that $\{K\} \cup \psi \cup \{\downarrow_{\mathfrak{C}} \{\xi(M_0)\}\}$ has f.i.p. Suppose that for some $\beta < \mu$ and every $\alpha < \beta$ there exists $\xi(M_\alpha) \in M_\alpha$ such that, in the notation $\chi_\alpha = \psi \cup \{\downarrow_{\mathfrak{C}} \{\xi(M_\gamma)\} \,|\, \gamma \leq \alpha\}$, the family $\{K\} \cup \chi_\alpha$ has f.i.p. Let $\chi = \bigcup_{\alpha<\beta} \chi_\alpha$. Obviously, the family $\{K\} \cup \chi$ has f.i.p. and since $\mathfrak{C}^{dd}$ is down-complete by Proposition 2.3, it follows that $\chi \subseteq \mathfrak{C} \cup \mathfrak{C}^{dd}$ and $M_\beta \in \psi_2 \subseteq \chi \cap \mathfrak{C}^{dd}$. Then, by Lemma 2.1, there exists $\xi(M_\beta) \in M_\beta$ such that $\{K\} \cup \chi \cup \{\downarrow_{\mathfrak{C}} \{\xi(M_\beta)\}\}$ has f.i.p. But $\chi_\beta = \chi \cup \{\downarrow_{\mathfrak{C}} \{\xi(M_\beta)\}\}$, which implies that the family $\{K\} \cup \chi_\beta$ has f.i.p. By induction, we have defined $\xi(M_\beta) \in M_\beta$ for every $\beta < \mu$. Obviously, the family $\{K\} \cup (\bigcup_{\beta<\mu} \chi_\beta) = \{K\} \cup \psi \cup \{\downarrow_{\mathfrak{C}} \{\xi(M_\beta)\} \,|\, \beta < \mu\}$ has f.i.p which implies that also its subfamily $\{K\} \cup \psi_1 \cup \{\downarrow_{\mathfrak{C}} \{\xi(M)\} \,|\, M \in \psi_2\}$ has f.i.p.

Theorem 2.1. *Let $\mathfrak{C} \subseteq 2^X$. Then $\mathfrak{C}^d = (\mathfrak{C} \cup \mathfrak{C}^{dd})^d$.*

Proof. Firstly, suppose that $\mathfrak{C}$ is down-complete. Since $\mathfrak{C} \subseteq \mathfrak{C} \cup \mathfrak{C}^{dd}$, it follows that $\mathfrak{C}^d \supseteq (\mathfrak{C} \cup \mathfrak{C}^{dd})^d$. Let $K \in \mathfrak{C}^d$ and let $\psi \subseteq \mathfrak{C} \cup \mathfrak{C}^{dd}$ be a family such that $\{C\} \cup \psi$ has f.i.p. Denote $\psi_1 = \psi \cap \mathfrak{C}$ and $\psi_2 = \psi \cap \mathfrak{C}^{dd}$. From Lemma 2.2. it follows that for every $M \in \psi_2$ there exists $\xi(M) \in M$ such that $\{K\} \cup \psi_1 \cup \{\downarrow_{\mathfrak{C}} \{\xi(M)\} \,|\, M \in \psi_2\}$ has f.i.p. But K is compact with respect to $\mathfrak{C}$ which is down-complete. Since $\psi_1 \cup \{\downarrow_{\mathfrak{C}} \{\xi(M)\} \,|\, M \in \psi_2\} \subseteq \mathfrak{C}$, it follows $\varnothing \neq K \cap (\bigcap \psi_1) \cap (\bigcap \{\downarrow_{\mathfrak{C}} \{\xi(M)\} \,|\, M \in \psi_2\}) \subseteq K \cap (\bigcap \psi)$. Therefore, K is compact with respect to $\mathfrak{C} \cup \mathfrak{C}^{dd}$. But $K = \uparrow_{\mathfrak{C}} K = \uparrow_{\mathfrak{C}}^{dd} K$ by Proposition 2.1 and so $K = \uparrow_{\mathfrak{C} \cup \mathfrak{C}^{dd}} K$. Hence, $K \in (\mathfrak{C} \cup \mathfrak{C}^{dd})^d$ and so $\mathfrak{C}^d = (\mathfrak{C} \cup \mathfrak{C}^{dd})^d$. If $\mathfrak{C}$ is not down-complete, by Proposition 2.2 we have $\mathfrak{C}^d = \mathfrak{C}^{*d} = (\mathfrak{C}^* \cup \mathfrak{C}^{*dd})^d = (\mathfrak{C}^* \cup \mathfrak{C}^{dd})^d = (\mathfrak{C} \cup \mathfrak{C}^{dd})^d$.

Corollary 2.1. *Let $\mathfrak{C} \subseteq 2^X$. Then $\mathfrak{C}^{dd} = \mathfrak{C}^{dddd}$.*

Proof. Since $\mathfrak{C}^{dd} \subseteq \mathfrak{C} \cup \mathfrak{C}^{dd}$, by Theorem 2.1 it follows $\mathfrak{C}^d = (\mathfrak{C} \cup \mathfrak{C}^{dd})^d \subseteq \mathfrak{C}^{ddd}$. Applying Theorem 2.1 again we have $\mathfrak{C}^{dd} = (\mathfrak{C}^d \cup \mathfrak{C}^{ddd})^d = \mathfrak{C}^{dddd}$.

Corollary 2.2. *Let (X, τ) be a topological space. Then $\tau^d = (\tau \vee \tau^{dd})^d$.*

Proof. Let $\mathfrak{C}$ be the family of all closed sets in (X, τ). Then $\mathfrak{C}^d$ is a closed subbase for τ^d and $(\mathfrak{C} \cup \mathfrak{C}^{dd})^d$ is a closed subbase for $(\tau \vee \tau^{dd})^d$, which gives the assertion.

Corollary 2.3. *Let (X, τ) be a topological space. Then $\tau^{dd} = \tau^{dddd}$.*

Proof. One can use formally a similar approach as in the proof of Corollary 2.1.

Corollary 2.4. *Let (X, cl) be a closure space. Then $\mathrm{cl}^{dd} = \mathrm{cl}^{dddd}$.*

Proof. Let $\mathfrak{C} = \{\mathrm{cl}\, A |\, A \subseteq X\}$. Then $\mathfrak{C}^{dd} = \mathfrak{C}^{dddd}$ by Corollary 2.1. But the closures cl^{dd} and cl^{dddd} are topological and their topologies coincide since they are generated by the same closed subbase.

Corollary 2.5. *Let (X, ξ) be a nearness space. Then $\xi^{dd} = \xi^{dddd}$.*

Proof. Let τ be the topology generated by the nearness ξ. Then $\tau^{dd} = \tau^{dddd}$ and so $\mathrm{cl}^{dd} = \mathrm{cl}^{dddd}$ for the corresponding closures. As we mentioned in Example 2.3, both of the topological spaces (X, τ^{dd}) and (X, τ^{dddd}) are R_0 and so both of them generate the same nearness structure $\xi^{dd} = \xi^{dddd}$.

Corollary 2.6. *A frame B represents the 2-nd iterated dualization of a topological system $(X, A, \vdash)$ iff B represents its 4-th iterated dualization.*

Proof. The assertion follows directly from the definition of a dual of a topological system and from Corollary 2.3.

In connection with the previous results it should be noted that there exist spaces whose four iterated de Groot duals, including the original space, are different. One simple example of such a space was given by B. Burdick [1]. We repeat it briefly because of completeness.

Example 2.5. Let (X, τ) be the first uncountable ordinal $X = \omega_1$ with the topology $\tau = \{[0, \alpha) \smallsetminus F |\, 0 \leq \alpha \leq \omega_1, \mathrm{F} \text{ is finite}\}$. Then τ^d is cocountable, τ^{dd} is cofinite and τ^{ddd} is discrete. Hence $\tau \vee \tau^{dd} = \tau$ and one can easily check that in consistency with Corollary 2.2 it follows $(\tau \vee \tau^{dd})^d = \tau^d$.

Let us close the paper by reviving a problem that the author already mentioned several times in some occasions. Taking the space (X, τ) of the previous example, it can be easily observed that it also holds $\tau^{ddd} = (\tau \wedge \tau^{dd})^d$, as well as in other examples that were given during the years by J. de Groot, H. Herrlich, G. E. Strecker and E. Wattel (e.g. see [3]). Hence, the following open question has its own place here.

Problem 2.1. Is it true that $\tau^{ddd} = (\tau \wedge \tau^{dd})^d$ for every topological space (X, τ)?

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