A geometry of information, II: Sorkin models, and biextensional collapses.

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Abstract. In this second part of our contribution to the workshop, we look in more detail at the Sorkin model, its relationship to constructions in Chu space theory, and then compare it with the Nerve constructions given in the first part. [1]

1 Introduction to part II

In the first part of this article, we saw how the nerve construction from algebraic topology could be applied to 'observational' situations. We also noted how, in theoretical physics, Sorkin had proposed a poset or T_0-space model for an observational context in which a space or spacetime, X, was observed by means of an open cover $\mathcal{F}$. This *Sorkin model* is thus a second means of encoding observational data on X. In fact there is no real need for $\mathcal{F}$ to be a cover of X. Those parts of X which are not covered are just not observed! Therefore we will use the more general term of a *finite family of open sets*, abbreviated to FFOS[2] to describe $\mathcal{F}$.

One of the most intuitive ways of picturing a FFOS is not from physics but from geography, or more exactly geographical information systems (GIS). The surface of the UK theoretically contains an infinite number of positions. These points may be 'observed' in various ways, as belonging to countries, counties, constituencies, postcodes, wards, etc., and this naturally gives something like a FFOS, i.e., by means of the 'attributes' of the points. On the other hand, one may start with the entire UK and base an analysis of some of its 'informational' structure by asking some relevant questions. For example: "which points

[1] This research forms part of a project : *Fractafolds, their geometry and topology*, partially supported by a grant from the Leverhulme Trust. This help is gratefully acknowledged.

[2] *Foss* is Norwegian for a waterfall which gives a nice image of wet patches covering a stone, *Ffos* is Welsh for a ditch which is not so poetic nor so apt, but the Welsh for waterfall is *Rhaeadr* and we challenge someone to come up with a relevent fit for that as an acronym!

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lie within walking distance of a primary school?" or "what are the evacuation zones of the various nuclear power stations?" Again the corresponding predicates correspond to some 'attributes'. The resulting FFOS will then partition the UK into *zones* depending on these questions. For example the above two open sets (predicates) will partition the UK into between one and four zones: one zone, in the case that the entire UK is within walking distance of some primary school and also within the evacuation zone of some nuclear power station, four zones if there exists different points in Britain at which all the four possibilities occur. The Sorkin model reduces this to a set with from 1 to 4 elements together with a partial order which encodes information such as if "all the points within walking distance of a primary school are also within an evacuation zone" is true.

It is often the case that the information we are encoding or recording is not continuously valued, but is itself a finite set of data. For example, the case when the data for each postcode is considered, or when a medical scanner collects data. However in some of these cases, the quantity of data is too large for an effective or quick analysis, and may also contain noise, so we have to approximate, or sample, the data to have any chance of obtaining useful information.

Observing attributes of points or objects is one of the intuitions behind both formal context analysis, [1], and Chu spaces, [2]. In particular for dyadic Chu spaces, which are the only type considered in this article, an object or element either satisfies or does not satisfy an attribute. Our situation thus leads naturally to a view of the space, X, as a union of these zones in which two points are in the same zone if they share exactly the same attributes, i.e., are in exactly the same open sets of the FFOS.

This set of zones seems to be quite a subtle invariant of the pair $(X, \mathcal{F})$ and we have included a brief discussion of some of its less immediate structure, for instance, in the resulting partition of X into zones, there is an intuitive idea of two zones being next to each other. One can easily imagine uses for such information in GIS, for example, where we would like to know whether or not an x is 'next to' a y even if they have been placed in different zones. We would like to encode this sort of information and for it to be preserved in the Sorkin Model. In section 4, we define the concept of two zones being 'next to' or 'close to' each other. This is simply that there is a point in one of the zones which lies in the closure of the other. (It is important to realise that the closure in this case is with respect the underlying topology $\tau(X)$, and not $\tau(\mathcal{F})$, the topology generated from $\mathcal{F}$, since all related zones have this property if the closure is with respect to $\tau(\mathcal{F})$.) This closeness, which we write $x \xrightarrow{c} y$, may also be considered 'nearness' or 'connectedness' between zones. It has the useful property that it is preserved to some extent under refinement and coarsening, that is, if two zones are near to each other, then under refinement, there will be a pair of subzones, one in each, which are near to each other. Whilst under coarsening the two zones will either be coarsened into a single zone or will remain near to each other.

We observe that this is additional information that is placed on the poset. It is not possible, simply given the poset information, to recover the data about which zones are close. A seemingly similar concept, which one can easily calculate for the Sorkin poset, is that of 'oneness'. Two zones are one related if there does not exist an intermediate zone. We give examples to show that in general there is no relationship between which zones are one related and which are close. We also give examples which show that the nice features of closeness under refinement and coarsening are not mirrored for one-related zones.

In the physics, one of the reasons for replacing the continuum of spacetime with a finite spacetime is to handle the problems of infinities that arise because of the point-like nature of particles. However for this finite or discrete model of spacetime to be useful, one requires that we can set up a differential structure on it which has similar properties to the differential geometry of continuous models of spacetimes. We believe that a similar viewpoint will be useful in Computer Science, and so Part III, we look at some 'differential structures' which may be placed on posets.

We are bridging between concepts from mathematical physics, topology and theoretical computer science. and would hope that this "bridging" operation will result in sharing of "technology". By necessity we have to use notation and terminology which combines features of each of the origins. This should ensure easy passage out from this paper to the adjacent regions of theory, but should also lead to maximal notational confusion within the paper!

2 Sorkin models

We will start with the topological situation, but will later generalise to enable more general situations to be handled.

Let X be a topological space and let $\mathcal{F}$ be a finite family of open sets (FFOS) of X. We can define the set $\mathcal{P}_{\mathcal{F}}$ as the quotient

$$\mathcal{P}_{\mathcal{F}} = X/\sim, \qquad \text{where} \qquad x \underset{\mathcal{F}}{\sim} y \text{ if } x \in U \Leftrightarrow y \in U \text{ for all } U \in \mathcal{F} \tag{1}$$

together with the quotient map

$$\pi_{\mathcal{F}} : X \to \mathcal{P}_{\mathcal{F}} . \tag{2}$$

We will drop the subscript $\mathcal{F}$ on both the poset/space $\mathcal{P}_{\mathcal{F}}$ and the equivalence relation $\underset{\mathcal{F}}{\sim}$ when the FFOS $\mathcal{F}$ is obvious. The FFOS $\mathcal{F}$ gives X a second topology, written $\tau(\mathcal{F})$, which is the topology generated by $\mathcal{F}$. To distinguish between the various topologies on X, we shall write $\tau(X)$ for the original topology on X. Since all the sets $U \in \mathcal{F}$ are assumed open in $\tau(X)$ we have $\tau(\mathcal{F}) \subseteq \tau(X)$.

We give $\mathcal{P}$ the quotient topology, written $\tau(\mathcal{P})$, making $\pi_{\mathcal{F}} : (X, \tau(X)) \to (\mathcal{P}, \tau(\mathcal{P}))$ continuous. As a consequence of the definition of $\mathcal{P}$, it is easy to see that $\tau(\mathcal{P})$ is a T_0 topology.

Remark

Here we will use $\mathcal{P}_{\mathcal{F}}$ or $\mathcal{P}$ as a notation for this quotient, rather than $X_{\mathcal{F}}$ as in part 1 of this paper, since we are primarily interested in it as a poset, not as a space. Its 'dual personality' makes a definitive choice of notation problematic!

We observe that given $U \in \tau(\mathcal{F})$, then $\pi_{\mathcal{F}}(U)$ is open in $\tau(\mathcal{P})$, i.e., the map (sometimes called the 'Kolmogorov quotient') $\pi_{\mathcal{F}} : (X, \tau(\mathcal{F})) \to (\mathcal{P}, \tau(\mathcal{P}))$ is continuous and an open mapping.

Definition 1. *Given X and a FFOS $\mathcal{F}$, we say the pair $(\mathcal{P}_{\mathcal{F}}, \pi_{\mathcal{F}})$ is a* Sorkin model *of X relative to $\mathcal{F}$, and call $\mathcal{P}_{\mathcal{F}}$ the* Sorkin poset *for $(X, \mathcal{F})$. For an element $x \in \mathcal{P}$, we will call the corresponding subset $\pi_{\mathcal{F}}^{-1}(x) \subseteq X$ the* zone determined by x*. In general the zones will neither be open nor closed subsets of X.*

As stated in part I, there is a one to one correspondence between finite T_0 spaces and finite posets. The poset structure on $\mathcal{P}$ is given by

$$x \preccurlyeq y \in \mathcal{P} \qquad \text{if and only if} \qquad (\forall V \in \tau(\mathcal{P}),\ \text{then } y \in V \ \Rightarrow\ x \in V). \tag{3}$$

Given $x \in \mathcal{P}$, the downset of x

$$\downarrow\!x = \{z \in \mathcal{P} \,|\, z \preccurlyeq x\} = \bigcap \{V \in \tau(\mathcal{P}) \,|\, x \in V\} \tag{4}$$

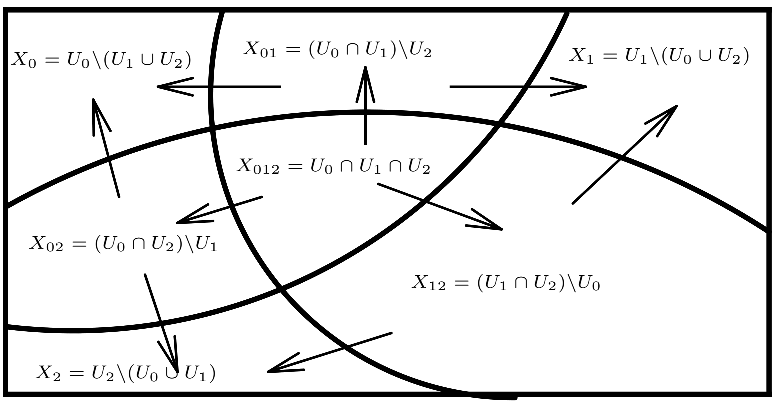
is open. In fact, given a finite poset, $\mathcal{P}$, we can use the downsets to generate the corresponding T_0 topology, $\tau(\mathcal{P})$.

Since posets are also small categories, we will also use the notation $x \to y$ for x. The direction of the arrow makes it consistent with the fact that the sequence $x, x, x, \ldots$ converges to the point y in the T_0 topology on $\mathcal{P}$, since every open set containing y also contains x and this is one reason why we chose the ordering given in (3) and not its dual.

We did not require that $\mathcal{F}$ covers X. If $\mathcal{F}$ does not cover X, then there exists a top element $\top \in \mathcal{P}$, $x \preccurlyeq \top$ for all $x \in \mathcal{P}$. In fact, if points a,b of X are in no set of the FFOS, $\mathcal{F}$, then by default, $a \sim b$, so the elements outside $\bigcup \mathcal{F}$ form a single zone, which is this top element of $\mathcal{P}$.

Example

To illustrate the construction, consider a simple example. Let $X \subset \mathbb{R}^2$, as shown below, and which is covered by three intersecting open sets $\mathcal{F} = \{U_0, U_1, U_2\}$.



(5)

Thus X is partitioned into seven zones: $X = X_0 \cup X_1 \cup X_2 \cup X_{01} \cup X_{12} \cup X_{02} \cup X_{012}$. Let $\pi(X_\alpha) = x_\alpha$, then $\mathcal{P} = \{x_0, x_1, x_2, x_{01}, x_{12}, x_{02}, x_{012}\}$, and the downset basis for the T_0 topology of $\mathcal{P}$ is

$$
\begin{aligned}
&\{x_{012}\}\,, \\
&\{x_{01}, x_{012}\}\,,\ \{x_{01}, x_{012}\}\,,\ \{x_{02}, x_{012}\}\,, \\
&\{x_0, x_{01}, x_{02}, x_{012}\}\,,\ \{x_1, x_{01}, x_{12}, x_{012}\}\,,\ \{x_2, x_{02}, x_{12}, x_{012}\}.
\end{aligned}
$$

Here $\{x_{01}, x_{012}\} = \downarrow x_{01}$, for instance. For each arc of a boundary of an open set in $\mathcal{F}$ in the above diagram, the arrow points from the interior of the open set to the exterior, in other words, from the open side to the closed side. Thus the direction of the arrows is consistent with the arrows in $\mathcal{P}$, so that, for instance, the arrow $X_{02} \to X_0$ in the diagram implies $x_{02} \to x_0 \in \mathcal{P}$. We see that the zone $X_{012} \subset X$ is an open subset and that the zones $X_0, X_1, X_2 \subset X$ are closed, whereas the zones $X_{01}, X_{02}, X_{12} \subset X$ are neither open nor closed. The poset $\mathcal{P}$ in this case is the subdivided 2-simplex and so is the face poset of the nerve of $\mathcal{F}$. This will not always be the case.

3 Sorkin Refinements

The intuition behind the Sorkin model is that observations, corresponding to the open sets of the FFOS $\mathcal{F}$, allow one to distinguish certain points. The zones in the diagram correspond to *clusters* of points that cannot be distinguished by that set of observations. In analysing this sort of situation in the first part of this article, we used nerves and hence Čech refinements were the natural way to consider refining the observations. In the context of the Sorkin model, a different notion of refinement is more natural. We have called it a Sorkin refinement.

Definition 2. *Given two FFOSs $\mathcal{F}$ and $\mathcal{G}$ of a topological space X. We say that $\mathcal{F}$ is a* Sorkin refinement *of $\mathcal{G}$ if $\mathcal{G} \subseteq \tau(\mathcal{F})$.*

This means each open set $U \in \mathcal{G}$ is also open in the topology generated by $\mathcal{F}$.

The effect of a refinement is that we further partition the zones as shown by the following:

Lemma 1. *$\mathcal{F}$ is a refinement of $\mathcal{G}$ if and only if there is a continuous surjective map*

$$\mathcal{P}_\mathcal{F} \xrightarrow{\pi_{\mathcal{F}\mathcal{G}}} \mathcal{P}_\mathcal{G}$$

such that

$$
\begin{array}{ccc}
 & (X, \tau(X)) & \\
 {\scriptstyle \pi_\mathcal{F}} \swarrow & & \searrow {\scriptstyle \pi_\mathcal{G}} \\
(\mathcal{P}_\mathcal{F}, \tau(\mathcal{P}_\mathcal{F})) & \xrightarrow{\quad \pi_{\mathcal{G}\mathcal{F}} \quad} & (\mathcal{P}_\mathcal{G}, \tau(\mathcal{P}_\mathcal{G}))
\end{array}
\tag{6}
$$

commutes.

Note that since $\pi_{\mathcal{G}\mathcal{F}}$ is continuous, it is also order preserving.

Proof. If $\mathcal{G} \subseteq \tau(\mathcal{F})$ and $x \underset{\mathcal{F}}{\sim} x'$, then for $U \in \mathcal{F}$,

$$x \in U \text{ if and only if } x' \in U,$$

so x and x' are in exactly the same open sets of $\mathcal{F}$, but then they are in exactly the same sets of $\tau(\mathcal{F})$ and thus, by restriction, of $\mathcal{G}$, but then $x \underset{\mathcal{G}}{\sim} x'$. If $[x]_{\mathcal{F}} = \pi_{\mathcal{F}}(x) \in \mathcal{P}_{\mathcal{F}}$ is the equivalence class containing x, then we define $\pi_{\mathcal{G}\mathcal{F}}[x]_{\mathcal{F}} = [x]_{\mathcal{G}}$. We have that this is (i) well defined and (ii) satisfies $\pi_{\mathcal{G}\mathcal{F}} \circ \pi_{\mathcal{F}} = \pi_{\mathcal{G}}$, as required. It is clearly a continuous surjective map. The converse is now easy.

If $\mathcal{G} \subseteq \tau(\mathcal{F})$, then we will call the unique natural map $\pi_{\mathcal{G}\mathcal{F}}$ a *coarsening map* and will write it as $\mathcal{F} \to \mathcal{G}$.

Note that, in general, Sorkin refinement is a distinct concept from Čech refinement. The easiest method of satisfying $\mathcal{G} \subseteq \tau(\mathcal{F})$ is simply to have $\mathcal{G} \subseteq \mathcal{F}$. This has a very natural interpretation, namely that we increase the number of observations in going from $\mathcal{G}$ to $\mathcal{F}$. The informational content of the notion of a Sorkin refinement is thus quite intuitive. From the observational viewpoint, the results from analysis of a FFOS $\mathcal{F}$ can be processed, by the allowed operations of geometric logic, to get the topology $\tau(\mathcal{F})$. The condition that $\tau(\mathcal{G}) \subseteq \tau(\mathcal{F})$ then says that observations in $\mathcal{F}$ distinguish at least as many points as those in $\mathcal{G}$. The operations of geometric logic are finite meets / intersections / conjunctions and arbitrary joins / unions / disjunctions, so here we are allowing ourselves to do more logical preprocessing before comparing the distinguishing power of the FFOSs, than in the Čech refinement.

In section 8, we will look at what happens when we have an infinite set of attributes which thus potentially may enable us to distinguish all points, but by increasing the fineness of the FFOS *ad infinitum*, try to interpret the resulting limiting poset.

4 Closeness and Oneness

As mentioned in the introduction, there seem to be additional relations that one can place on the Sorkin model. As we will see, they are not all intrinsic to the poset $\mathcal{P}$ itself, but to it together with the quotient map $\pi_{\mathcal{F}}$.

Definition 3. *Given $x \neq y \in \mathcal{P}$ then we will say that x is* one related *to y, $x \xrightarrow{1} y$ if $x \to y$ and given z such that $x \to z \to y$ then either $z = x$ or $z = y$.*

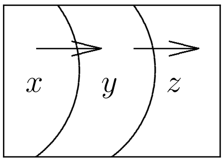
Note that for any finite poset we can establish which pairs of elements are one related. Therefore 'oneness' does not require $\mathcal{P}$ is a poset derived as some $\mathcal{P}_{\mathcal{F}}$ for some FFOS, $\mathcal{F}$. Oneness, which is well known in poset theory under various names [3], is not transitive and the one related pairs form a directed graph, essentially the Hasse diagram, which generates the poset $\mathcal{P}$.

[3] These includes 'covering', which is fairly standard, but would be very confusing in this context!

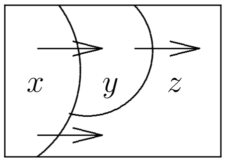
By looking at the nature of the zones in the underlying topology, $\tau(X)$, we can deduce certain information about the poset $\mathcal{P}_{\mathcal{F}}$. In some circumstances, we can gain extra information about the overall model, $(\mathcal{P}_{\mathcal{F}}, \pi_{\mathcal{F}} : X \to \mathcal{P}_{\mathcal{F}})$.

Lemma 2. *If $\overline{\pi^{-1}(x)} \cap \pi^{-1}(y) \neq \emptyset$, where closure is with respect to $\tau(X)$, then $x \to y$.*

However the converse of this statement is not true as the following examples show:



(a),



(b) (7)

In both (7a) and (7b) we have the Sorkin poset is : $x \to y \to z$, so $x \to z$, but in (7a), $\overline{\pi^{-1}(x)} \cap \pi^{-1}(z) = \emptyset$, whilst in (7b), $\overline{\pi^{-1}(x)} \cap \pi^{-1}(z) \neq \emptyset$. To distinguish these types of situation we define the concept of *closeness* or *nearness*:

Definition 4. *Given $x \neq y \in \mathcal{P}$ then we say x is* close *to y, and write $x \stackrel{c}{\to} y$ if*

$$\overline{\pi^{-1}(x)} \cap \pi^{-1}(y) \neq \emptyset \ . \tag{8}$$

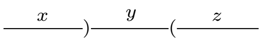
Thus for (7a) $x \stackrel{c}{\not\to} z$, whilst for (7b) $x \stackrel{c}{\to} z$. Thus closeness is a property in addition to $x \to z$.

Note that closure here must be with respect to $\tau(X)$, since if $x \to y$ then y is in the closure of x with respect to $\tau(\mathcal{F})$.

Remark

There is some intuitive link here between 'closeness' and the notion of 'connected to' in certain treatments of the Region Connection Calculus (RCC). Intuitively, both correspond to 'nearness' of the regions or zones. We have not yet had time to investigate this in more detail, but would ask if the other operators of RCC can be appropriately interpreted in the context of a general Sorkin model. We may sometimes use 'near to' or 'connected to' as a synonym for 'close to'.

In general the concepts of oneness and closeness are unrelated. For example in (7b), $x \stackrel{c}{\to} z$ but $x \stackrel{1}{\not\to} y$. For an opposite counter example, consider



$$, \qquad \mathcal{F} = \{\{x\}, \{x, z\}, \{x, y, z\}\} \, , \tag{9}$$

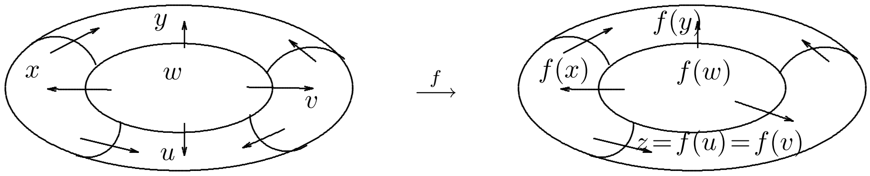
so $x \stackrel{1}{\to} z \stackrel{1}{\to} y, \quad$ but $\quad x \stackrel{c}{\not\to} z$.

We observe that closeness is preserved under refinement and coarsening, more exactly:

Lemma 3. *If $f : \mathcal{F} \to \mathcal{G}$ is a coarsening map and $x \stackrel{c}{\to} y \in \mathcal{P}_{\mathcal{F}}$, then either $f(x) = f(y)$ or $f(x) \stackrel{c}{\to} f(y) \in \mathcal{P}_{\mathcal{F}}$.*

If $f : \mathcal{F} \to \mathcal{G}$ is a coarsening map and $x \stackrel{c}{\to} y \in \mathcal{P}_{\mathcal{F}}$, then there exists $x' \in f^{-1}(x)$ and $y' \in f^{-1}(y)$ such that $x' \stackrel{c}{\to} y'$.

We would like to say the same thing about oneness, but unfortunately we cannot due to the following counter examples:



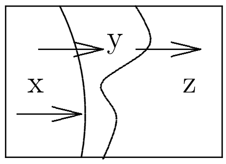
(10)

Here $x \xrightarrow{1} y$, but $f(x) \neq f(y)$ and $f(x) \overset{1}{\not\to} f(y)$, since $f(x) \to z \to f(y)$. The other possibility is countered by simple examples such as:

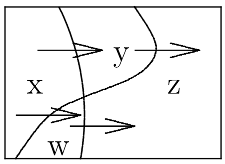
$$\overset{x}{\bullet} \quad \overset{y}{\bullet} \quad \xrightarrow{f} \quad \overset{f(x)}{\bullet} \xrightarrow{1} \overset{f(y)}{\bullet} \tag{11}$$

Towards "nice" or "stable" FFOS

We observe that, to some extent, many of the examples given above, i.e (7b), (9) and (10), are exceptional, that is, they would not arise from a "random" or "generic" FFOS whatever those terms might mean. For example, in (9) we required that the boundary of $\{x, z\}$ coincides with the boundary of $\{x\}$, which "almost never" happens in as much as a small "perturbation" would destroy that property. Many of the other examples are also not stable under "perturbation", as one can see that by slightly altering the FFOS, sometimes will change the Sorkin poset. For example, perturbing (7b) can give

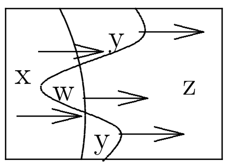


(a),



(b)

or even



(c) (12)

In (12a) the Sorkin poset remains the same but $x \overset{2}{\not\to} z$. In (12b) and (12c) the Sorkin poset becomes $\begin{smallmatrix} x & \to & y & \to & z \\ & \searrow & w & \nearrow & \end{smallmatrix}$.

For some contexts, it would be convenient, if not important, to have precise definitions of 'perturbation', 'generic', 'stable', and so on. Equally it is the case that the 'exceptional', 'non-generic' cases are likely to be significant as being the examples of more singular behaviour, where the geometry of the information changes. In any case it is likely that we will eventually need some idea of 'niceness' in this context. and we offer the following as a provisional list of some possible sufficient requirements for "niceness".

- X is an n dimensional manifold.
- Each $U \in \mathcal{F}$ is a finite union of pathwise connected components.
- If $U_1, \ldots, U_r \in \mathcal{F}$ with $\cap_{i=1}^r U_i \neq \emptyset$, then $\cap_{i=1}^r \partial U_i$ is a finite union of connected submanifolds of dimension $n - r - 1$. This includes the case when $r = 1$.
- If two boundaries intersect then they cross, i.e., they are in general position.

Remarks

(i) Note that a stable FFOS $\mathcal{F}$ could not, in general, be a topologically closed FFOSs, i.e., we must have $\mathcal{F} \neq \tau(\mathcal{F})$, since if $U, V \in \mathcal{F}$ then $U \cap V$ and $U \cup V$ will share boundaries with U and V, which do not intersect.

(ii) The requirements *do* raise questions such as "how might they be observed?" We have already noted, in part 1 of this paper, that 'manifoldness' is problematic from an observational viewpoint, but the others are of similar nature, so we are far from having a complete understanding of this. However this does relate to the *geometry* of the information rather than just its topology and so we have felt the questions should be raised.

The following would be a possible list of 'nice' properties for 'nice' FFOSs, and would look to be feasible given assumptions similar to those listed above.

- If $x \xrightarrow{1} y$, then $x \xrightarrow{c} y$.
- If $x \xrightarrow{c} y$,then $\overline{\pi^{-1}(x)} \cap \overline{\pi^{-1}(y)}$ is a submanifold of dimension r, where $1 \leq r \leq n$ and there is a set $z_1, \ldots, z_{r-1} \in \mathcal{P}$ such that $x \xrightarrow{1} z_1 \xrightarrow{1} \cdots \xrightarrow{1} z_{r-1} \xrightarrow{1} y$.
- If $\mathcal{F}_t$ is a one parameter family of stable FFOSs, which depend continuously on $t \in \mathbb{R}$, then the Sorkin poset is independent of t.
- Let $f : \mathcal{P}_{\mathcal{F}} \to \mathcal{P}_{\mathcal{G}}$ be the corresponding coarsening map, then, if $x \xrightarrow{1} y \in \mathcal{F}$, we should have $f(x) \xrightarrow{1} f(y) \in \mathcal{G}$ and then
- if $x \xrightarrow{1} y \in \mathcal{G}$ then there exists $\hat{x} \in f^{-1}(x)$ and $\hat{y} \in f^{-1}(y)$ such that $\hat{x} \xrightarrow{1} \hat{y} \in \mathcal{F}$.

These ideas are still very speculative and we have left some terms deliberately undefined. Clearly we might think of stable FFOS in a similar way to stable dynamical systems. If a one parameter set of FFOSs $\mathcal{F}_t$ is unstable, say at $t = t_0$ then there may be a change in the topology of $\mathcal{P}_{\mathcal{F}_t}$ as t goes from $t < t_0$ to $t > t_0$. This is analogous to a bifurcation in the theory of dynamical systems. At such a 'bifurcation' the geometry of the information would change. (This idea has been explored in the Physical context.

There are many possible areas of research here. One is to make precise the specification for a stable FFOS of a manifold and then to look at the cases where a FFOS changes topology, but in a "nice" predictable way. Clearly this should be easiest when $\dim(X) = 1$ and the 'space' of FFOSs with a fixed number of open sets is finite dimensional. The second area of research would be to attempt to define a stable FFOS in a more constructive way. Finally we should also look at the connection between dynamical FFOSs and the differential structures on $\mathcal{P}_{\mathcal{F}}$ considered in the later sections.

5 Chu Spaces revisited

Recall from part I, [3], that a Chu space, $\mathcal{C}$, is given as $\mathcal{C} = (C_o, \models_C, C_a)$, where C_o and C_a are sets, respectively, here called the sets of *objects* and of *attributes* and $\models_{\mathcal{C}} \subseteq C_o \times C_a$ is a relation. We write $x \models_C a$ iff $(x, a) \in \models_{\mathcal{C}}$.

Given two Chu spaces, $\mathcal{C} = (C_o, \models_C, C_a)$ and $\mathcal{D} = (D_o, \models_D, D_a)$, then a *Chu morphism* or *Chu transform*, $\mathfrak{f} : \mathcal{C} \to \mathcal{D}$, is given by a pair of maps, f_o or $f_\star : C_o \to D_o$ and f_a or $f^\star : D_a \to C_a$ which satisfy

$$f_\star(x) \models_D a \qquad \Longleftrightarrow \qquad x \models_C f^\star(a). \tag{13}$$

Our basic reference for the theory of Chu spaces is, as before, Pratt's notes, [2].

Example

We can view a space, X, together with a FFOS $\mathcal{F}$ as a Chu space $(X, \in, \mathcal{F})$, with the set of objects being X and $\mathcal{F}$, the set of attributes. An object $x \in X$ has an attribute $U \in \mathcal{F}$ if $x \in U$. This is an example of a special kind of Chu space, called by Pratt a *normal Chu space*, in which the (distinct) attributes are (distinct) subsets of the set of objects and $\models$ is $\in$. Normal Chu spaces are examples of *extensional* Chu spaces.

Definition 5. *Given a Chu space, $\mathcal{C} = (C_o, \models_C, C_a)$, we define two mappings:*

$$\begin{aligned} \breve{\alpha}_C : C_o \to \mathbf{2}^{C_a} \ ; &\qquad \breve{\alpha}_C(x) = \{a \in C_a \mid x \models_C a\} \\ \breve{\omega}_C : C_a \to \mathbf{2}^{C_o} \ ; &\qquad \breve{\omega}_C(a) = \{x \in C_o \mid x \models_C a\}. \end{aligned}$$

The Chu space $\mathcal{C}$ is said to be extensional *if $\breve{\omega}_C : C_a \to \mathbf{2}^{C_o}$ is injective.*
The Chu space $\mathcal{C}$ is said to be separable *if $\breve{\alpha}_C : C_o \to \mathbf{2}^{C_a}$ is injective.*
A Chu space that is both extensional and separable is called biextensional.

Remarks

(i) In an extensional Chu space, because *no two attributes correspond to exactly the same set of objects*, ($\breve{\omega}_C : C_a \to \mathbf{2}^{C_o}$ is injective), we can replace, C_a by $\breve{\omega}_C(C_a)$ to get a normal Chu space, $(C_o, \in, \breve{\omega}_C(C_a))$, which will be isomorphic to the original one. In a separable Chu space, *no two objects have exactly the same sets of attributes*, so, in general, a Chu space of the form, $(X, \in, \mathcal{F})$, will not be separable.

(ii) The two mappings $\breve{\alpha}_C$ and $\breve{\omega}_C$ extend to give mappings $\alpha_C : \mathbf{2}^{C_o} \to \mathbf{2}^{C_a}$ and $\omega_C : \mathbf{2}^{C_a} \to \mathbf{2}^{C_o}$, defined by

$$\alpha_C(X) = \{a \mid \forall x \in X, x \models_C a\} \tag{14}$$

and

$$\omega_C(A) = \{x \mid \forall a \in A, x \models_C a\}. \tag{15}$$

These two mappings are used in Formal Concept Analysis to obtain closure operators on the two powersets, see, for instance, [4].

Representing Chu spaces

It will often be useful to consider a Chu space as an array or matrix with 0-1 entries. (This viewpoint is explored and well exploited in Pratt's notes already mentioned.) The rows are labelled by the 'objects', the columns by the 'attributes', so if $x \models a$, then in the x^{th} row and a^{th} column one finds a 1, and if $x \not\models a$ then there is a 0.

In this representation, $\mathcal{C}$ is extensional if no two columns are the same and separable if no two rows are the same. As we saw, any extensional Chu space can be replaced with no loss of information by a normal one, and this, simply, by omitting the column labels, i.e., by using the columns as labels for themselves.

A related representation of $\mathcal{C}$ can given by the characteristic function of the relation $\models_C$ (as a subset of $C_o \times C_a$). This gives a function

$$r_{\mathcal{C}} = r : C_o \times C_a \to \mathbf{2} = \{0, 1\},$$

$$r(x,a) = \begin{cases} 1 & \text{if } x \models_C a \\ 0 & \text{otherwise.} \end{cases}$$

The two alternative 'Curried' forms of this give the $\check{\alpha}$ and $\check{\omega}$ maps, defined earlier,

$$\check{\alpha}_C : C_o \to \mathbf{2}^{C_a}, \qquad \check{\omega}_C : C_a \to \mathbf{2}^{C_o}.$$

Identifying a subset with its characteristic function, we have

$$\check{\alpha}_C(x) = r_C(x, -) : C_a \to \mathbf{2}, \qquad \check{\omega}_C(a) = r_C(-, a) : C_o \to \mathbf{2}.$$

There is an obvious way to change any Chu space, $\mathcal{C}$, into a biextensional one, namely by 'killing off', or quotienting out, any lack of injectivity of the two maps α_C and ω_C. More formally:

Definition 6. *The biextensional collapse of a Chu space* $\mathcal{C} = (C_o, \models_C, C_a)$ *is the Chu space*

$$\widehat{\mathcal{C}} = (\widehat{C}_o, \models_{\widehat{C}}, \widehat{C}_a) = (\check{\alpha}_C(C_o), \models_{\widehat{C}}, \check{\omega}_C(C_a)), \tag{16}$$

where

$$\check{\alpha}_C(x) \models_{\widehat{C}} \check{\omega}_C(a) \qquad \textit{if and only if} \qquad x \models_C a \tag{17}$$

In the topological context, the Kolmogorov quotient $\pi : (X, \tau(\mathcal{F})) \to (\mathcal{P}, \tau(\mathcal{P}))$ is simply the universal map to the biextensional collapse of $(X, \in, \mathcal{F})$. The Chu space $(\mathcal{P}, \in, \tau(\mathcal{P}))$ is, of course, biextensional, i.e., extensional and separable. The poset structure on $\mathcal{P}$ is given by

$$\check{\alpha}(x) \preccurlyeq \check{\alpha}(y) \iff \forall a \in C_a \text{ then } (y \models_C a \Rightarrow x \models_C a) \iff \check{\alpha}(x) \supseteq \check{\alpha}(y). \tag{18}$$

For a general Chu space, $\mathcal{C}$, there are some problems about this quotienting operation since, although $\check{\alpha}_C : C_o \to \mathbf{2}^{C_a}$ gives an object map in the right direction for a Chu transform from $\mathcal{C}$ to $\widehat{\mathcal{C}}$, the corresponding $\check{\omega}_C : C_a \to \mathbf{2}^{C_o}$ goes in the wrong direction to be its 'adjoint'. If $\mathcal{C}$ is extensional, as was the case in the topological case above, then $\check{\omega}_C : C_a \to \mathbf{2}^{C_o}$ is a bijection onto its image, $\widehat{C}_a$, and its inverse has the right properties to be the adjoint of $\check{\alpha}$. Luckily in our context, although we may have a non-extensional $\mathcal{C}$, it is reasonable to suppose that the 'sample' $\mathcal{F}$ 'is extensional', since if two columns in $\mathcal{C}$ are the same we can include one and not the other in such a sample ... in a sense, there would be no point in keeping repeat columns in $\mathcal{F}$! We will therefore often assume that $\mathcal{F}$ has no repeat columns.

The poset structure on this biextensional collapse shows an example of a general phenomenon discussed by Pratt in his notes, [2]. He considers a general 'alphabet', Σ as the codomain of the characteristic function $r : C_o \times C_a \to \Sigma$. For us, at present, $\Sigma = \mathbf{2} = \{0, 1\}$. The point Pratt makes is that structure on this alphabet induces properties / structure on the Σ-Chu spaces. In the dyadic Chu spaces that we are using, the available structures on $\mathbf{2}$ include those of a Boolean algebra or of a poset / lattice, and thus the basic logical structures of propositional calculus.

For example, the poset structure on $\mathbf{2}$ gives the poset structure on the powerset $\mathbf{2}^{C_o}$, so we can set

$$x \preccurlyeq y \text{ in } C_o, \tag{19}$$

to mean

$$\check{\alpha}_C(x) \supseteq \check{\alpha}_C(y) \text{ in } \mathbf{2}^{C_a}, \tag{20}$$

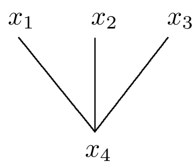
i.e., the value in row x column a is always greater than that in row y column a, whatever attribute a is considered and we get back exactly the formula (18) that we had earlier on.

Here various important points need some comment. Firstly the reversal of order, we are considering attributes as open sets so '$x \preccurlyeq y$' corresponds to 'x is in more open sets of $\mathcal{F}$ than is y', (cf., equation (3) above, '$x \preccurlyeq y$ in $\mathcal{P}$' corresponds to 'if $y \in V \in \mathcal{F}$, then $x \in V$ also'). The convention with the reversal order can also be use and is, of course, completely equivalent.

Example: the trident

Consider the following Chu space, which we will refer to as the 'trident',

$\mathcal{C}$	a_1	a_2	a_3
x_1	1	0	0
x_2	0	1	0
x_3	0	0	1
x_4	1	1	1



(21)

It is biextensional, so is its own biextensional collapse. All the first three rows are less than the fourth one. In fact, this example was constructed from the poset having $\{x_1, x_2, x_3, x_4\}$ as its underlying set with $x_i \geq x_4$ for $i = 1, 2, 3$, and with set of attributes corresponding to the downsets of the first three elements. (Alternatively we can think of this as a 'Chu FFOS' with underlying Chu space constructed as below from this poset (cf. (23), and with $\mathcal{F}$ being the first three non-zero columns / attributes.) Note, once again, that because of our interpretation in terms of FFOSs, the order apparent in the table is the opposite of that in the usual relational representation of the poset. (This reversal of the order is sometimes an annoyance, but, as was said above, *is* natural in our main examples.)

This relation on C_o will not be a partial order unless α_C is injective, that is, unless $\mathcal{C}$ is separable (rows are distinct), but this is independent of extensionality.

The objects we are studying here have already been met in the first part of this work. They consist of a Chu space and the analogue of a FFOS in this more general setting. More formally, we have a Chu space $\mathcal{C} = (C_o, \models_C, C_a)$ giving us a certain model for our 'global information' and then a *finite* sample of the attributes $\mathcal{F}$, giving the pair $(\mathcal{C}, \mathcal{F})$, which is the desired 'Chu FFOS', although here the 'O' for 'open' is not really an appropriate letter to use [4]. Any Chu FFOS $(\mathcal{C}, \mathcal{F})$ yields a Chu space, $\mathcal{C}_{|\mathcal{F}}$, by 'corestricting' the attributes:

Definition 7. *Given a Chu space $\mathcal{C}$, and a finite sample of its attributes, $\mathcal{F}$, making up a Chu FFOS, we call the Chu space,*

$$\mathcal{C}_{|\mathcal{F}} = (C_o, \models_C, \mathcal{F}),$$

the corestriction *of $(\mathcal{C}, \mathcal{F})$.*

The biextensional collapse of $\mathcal{C}_{|\mathcal{F}}$ is analogous to the Sorkin model and so we would expect it to 'be a poset'. The structures arising here need clarification, so we indulge in a short detour to look at the relationship between posets and Chu spaces in more detail.

Posets as Chu spaces

Any poset $\mathcal{P} = (P, \leq)$ gives rise to a Chu space in several equivalent ways. For instance we can take $C_o = C_a = P$ with $x \models_C y$ taken to mean $x \leq y$, or we can take $C_o = P$, $C_a \subseteq \mathbf{2}^P$ given by the upsets

$$\Uparrow\! x = \{z \mid z \geq x\}$$

and

$$x \models_C y \text{ if and only if } \Uparrow\! x \supseteq \Uparrow\! y \text{ if and only if } x \geq y. \tag{22}$$

The upset $\Uparrow\! x = \omega_C(x)$ for $\mathcal{C}$, the Chu space given by the first construction. Of course, we can also take $C_o = P$, $C_a =$ the set of downsets of P (cf. (4) and

[4] FSA = 'finite sample of attributes' would seem a good acronym, but with 'FCA' standing for 'Formal Concept Analysis', the two acronyms might be too close together and confusion would almost certainly result! We have therefore avoided them in this paper.

page 4) and $\models$ to be '$\in$', but note that in this dual construction $x \models \downarrow y$ implies that $x \in \downarrow y$, so $x \leq y$, as one would expect in a dual.

In these constructions, the set of objects of the resulting Chu space is the set of elements of the poset, whilst the set of attributes is derived from it and *is the same size*, so the matrix is square. We will loosely call these 'square' matrix representations, *the standard Chu representations* of the poset, sometimes adding 'square' for emphasis. Clearly if we start with a pair $(X, \mathcal{F})$ and form its Sorkin model, or more generally with a biextensional Chu space $\mathcal{C}$ with, for simplicity, finitely many objects, then we need not have that the Chu space is square, so it may not be clear what poset it represents. In the example, ((21) above), if we took the 'meet' of the columns, then we get a new column, $a_1 \wedge a_2 \wedge a_3$, which corresponds to the downset of the fourth object. If we add it in to $\mathcal{C}$ as a new column, we get a Chu space that is a standard representation of the trident poset, but this is too *ad hoc* for extensive use. To solve this problem and to develop the idea of representing posets by Chu spaces further we need the following ideas adapted from Pratt's notes.

We first make the assumption that $\mathcal{C}$ is a normal Chu space, so $C_a \subseteq \mathbf{2}^{C_o}$ is exactly a set of distinct columns (which thus can be used to label themselves).

Definition 8. *A* property *of $\mathcal{C}$ is any subset $Y \subseteq \mathbf{2}^{C_o}$ containing C_a, so $C_a \subseteq Y \subseteq \mathbf{2}^{C_o}$.*

Example : the trident, continued

For instance, in the 'trident' poset, above, we have a property '$x_1 \geq x_4$' in the commonsense meaning of the word. This corresponds to the subset of $\mathbf{2}^{C_o}$ in which in each a^{th} column we have $r(x_1, a) \leq r(x_4, a)$, i.e., a property in Pratt's sense. (The reversal of order is, once again, a quirk coming from our earlier conventions.)

Pratt's normal realisation of a poset is given by taking a set of defining 'atomic' properties in the poset and carving out from $\mathbf{2}^{C_o}$ the property that is the intersection of all these properties considered as subsets of $\mathbf{2}^{C_o}$. The result for our trident example is easy to construct. The obvious atomic defining properties are $x_i \geq x_4$, for $i = 1, 2, 3$, so we have the normal realisation

$\mathcal{C}$	
x_1	0 0 1 0 0 1 1 0 1
x_2	0 0 0 1 0 1 0 1 1
x_3	0 0 0 0 1 0 1 1 1
x_4	0 1 1 1 1 1 1 1 1

(23)

In general, given a poset $(P, \leq)$, which we will assume finite, fix a set Γ of atomic implications $x \to y$ or properties defining the given poset (e.g., by drawing the Hasse diagram of $(P, \leq)$). For each $\gamma = (x \to y)$ in Γ, interpreted here as, and encoding, $x \geq y$, let X_γ be the set of columns $a \in \mathbf{2}^{C_o}$, such that

$$r(x, a) \leq r(y, a).$$

Then we have:

Definition 9. *The* normal realisation *of* $(P, \leq)$ *is* $(P, \in, X_\Gamma)$*, where* $X_\Gamma = \bigcap X_\gamma$*.*

This normal realisation is maximal as far as the poset structure is concerned. We will write $NR(P, \leq)$ for the normal realisation of a poset $(P, \leq)$. Different choices of Γ yield the same X_Γ, essentially because of the corollary to the following result:

Proposition 1 (Pratt, [2], Proposition 2.1). *A normal Chu space realises a preorder if and only if the set of its columns is closed under arbitrary joins and meets.*

Corollary 1. *A separable normal Chu space realises a poset if and only if the set of its columns is closed under arbitrary joins and meets.*

Here the point is that separability means that different objects satisfy different sets of attributes, so one never has $x \leq y$ and $y \leq x$ unless $x = y$.

The determination of a suitable Γ corresponds in a general sense to specifying a set of generating relations $x \geq y$. It therefore relates to problems of presentation of the poset as an 'algebraic' object.

The importance of this for us is that to form the normal realisation of a given biextensional Chu space, we need only close up the set of its columns under arbitrary meets and joins *within* $\mathbf{2}^{C_o}$. This will never destroy separability and as everything is done *within* $\mathbf{2}^{C_o}$, the result is still extensional. It also avoids a choice of Γ.

To illustrate the usefulness of this further, consider the following variant of the trident example:

Example: trident variant.

Let $\mathcal{C}$ be the Chu space

$\mathcal{C}$	a_{12}	a_{13}	a_{23}
x_1	1	1	0
x_2	1	0	1
x_3	0	1	1
x_4	1	1	1

(24)

This was obtained from the standard trident example (see (21) and (23)) by taking a different sample of observations, namely $a_{12} = a_1 \vee a_2$, etc. Of course, the attributes are 'compound', not being given by the 'atomic' downsets. The poset being represented is the same one, as both (21) and this (24) are biextensional, and that poset is the Sorkin model of both, but in two different forms.

As we said earlier, if we are handed a general Chu space together with a finite sample of its attributes, $(\mathcal{C}, \mathcal{F})$, there is, thus, no reason to suppose that its biextensional collapse will be in standard form. One can, if the size is small, easily check the atomic implications and, for instance, draw its Hasse diagram, but if there are many objects, pairwise checking will be tedious and time consuming at best. Two algorithms suggest themselves for taking such a non-standard Chu poset and returning either its normal realisation or its square standard representation. For the former, we take the Chu space and close up its columns under arbitrary meets and joins, then we can compare any two such Chu spaces on the same set of objects by matching columns. (If the object sets are of equal size, but not 'the same' then, of course, one can extend the pattern matching over permutations of the rows as well, but beware of the automorphism group of the poset. If this group is small but the set of objects is large, searching over permutations needs doing carefully. The worst case may be very bad here.)

The second algorithm extracts the standard representation by constructing the down set of each object and expressing it in terms of the already existing columns. (This can be thought of as extracting the downsets from the normal realisation, but there is no need to build that larger model explicitly.) The explicit algorithm is to order the object set; for each object, x, in turn, check if there is a column representing the downset of that object; if the downset is not there already, create a new column labelled $\downarrow x$, and put a 1 in row y of that new column if all elements in row y are greater than the corresponding entry in row x (remember our reversal of order convention); repeat until there are no more rows to process, and finally delete any original columns that were not identified as downsets and reorder the remaining ones in the same order of the objects. The maximal 'flow formula' for each row (see below, page 19) can be used to express the 'new' column as a meet of the original ones.

Remarks

(i) These two algorithms are not optimised, and we have not attempted to make them efficient in any way. There are clearly many well known ways of doing essentially these processes.

(ii) The normal realisation is better at the theoretical level, as this construction is functorial. Any map of posets $f : (P, \leq) \to (Q, \leq)$ will induce a reverse map $f^{-1} : \mathbf{2}^Q \to \mathbf{2}^P$ by inverse image, and any property of $NR(Q, \leq)$ is mapped by this to a property of $NR(P, \leq)$ in Pratt's sense. It is easy to check that it thus restricts to a Chu transform on the normal realisations. On the other hand, the standard representation is not functorial if we use Chu transforms as the morphisms. For example, consider the folding map from the poset, $\{x_1, x_2, x_3\}$ (with $x_i \geq x_3$ for all i), to $\{y_1, y_2\}$ (with $y_1 \geq y_2$), that maps x_1 and x_2 to y_1 and the bottom elements to each other. It is easy to write down the corresponding map in the normal realisations, but the reverse map sends the downset of y_1 to $\downarrow x_1 \vee \downarrow x_2$ and so does not restrict to a Chu transform on the standard representations.

The biextensional collapse and the Sorkin model

From all this it is clear that the Sorkin model construction and the biextensional collapse are really the same when one considers a FFOS, $(X, \mathcal{F})$, as a Chu space and that for the more general situation of a Chu space together with a finite sample of attributes, similar structures are present. It is worth noting that this gives the Sorkin model construction a universal property that was not that in evidence to start with. The proper setting for this universal property would seem to be these latter objects that we have called '*Chu FFOSs*' for want of a better term. We have loosely referred, earlier, to the biextensional collapse of $\mathcal{C}_{|\mathcal{F}}$ as being the biextensional collapse of the Chu FFOS. We can now put this on a firm basis.

This really works best if $\mathcal{F}$ has 'no repeats' so that $\mathcal{C}_{|\mathcal{F}}$ is extensional. From this perspective, the Kolmogorov quotient

$$\pi_{\mathcal{F}} : X \to X_{\mathcal{F}}$$

is a universal Chu map, so in more generality, we might attempt to construct

$$\pi_{\mathcal{F}} : \mathcal{C}_{|\mathcal{F}} \to \mathcal{C}_{\mathcal{F}}$$

defined, perhaps, by

$$(\pi_{\mathcal{F}})_* = \check{\alpha}_C : C_o \to \check{\alpha}_C(C_o),$$

and if $\mathcal{C}_{|\mathcal{F}}$ is extensional, $\check{\omega}_C : \mathcal{F} \to \check{\omega}_C(\mathcal{F})$ is a bijection and we can use $\check{\omega}_C^{-1}$ as the reverse map $(\pi_{\mathcal{F}})^*$ to get a Chu transform.

If we allow repeats in $\mathcal{F}$, then $\check{\omega}_C : \mathcal{F} \to \check{\omega}_C(\mathcal{F})$ will be a surjection and one might try to use some generalised weaker form of Chu transform (with the reverse map some form of relation, for instance), however for the context of $\mathcal{F}$ being a sample from C_a, the assumption that it is repeat-free seems anodyne enough, so we have not looked at the more general case in any detail.

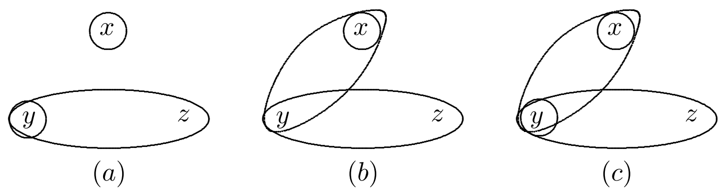
Remarks

(i) It is worth recalling that, in the topological setting, if $\mathcal{F}$ does not cover X, then $\mathcal{P}_{\mathcal{F}}$ has a maximal or top element, namely the equivalence class of any element that is not in the union of the sets in $\mathcal{F}$, likewise if a Chu space $\mathcal{C}$ has an object x with $\check{\alpha}_C(x)$ the empty set, then again in the biextensional collapse the class of this object gives a least element in the poset.

(ii) We further observe that several FFOS on a particular space, X, can give rise to the same Sorkin poset. We already have our trident examples, but here is another in which we can be more exhaustive in listing the possibilities. There are 12 FFOSs on the three point set which all give the Sorkin poset

$$x \qquad y \longrightarrow z. \tag{25}$$

These are



(26)

plus those given by (26) including the empty set or the total set, $\{x, y, z\}$, or both. In other words there are 12 biextensional Chu spaces which have the poset (25). These do not give equivalent Chu spaces, although, of course, some will generate others via the Boolean algebra operations or, more exactly, via 'topological closure', see below.

The essential difference between the Chu space approach and that of the Sorkin Model is that, with the Chu spaces, we are also concerned with exactly which sets are in the FFOS and not just how they separate points. This means that we can easily discuss the nerve of Chu space, as in the first part of this article, see also below, but we have more of a problem defining the Sorkin refinement.

We shall call the Chu space $\mathcal{C}$, *topologically closed* if C_a is a topology on C_o, i.e., C is normal and C_a includes all unions and finite intersections. These are also known as *localic topological systems*, (cf. [5] and p.26 below). We will usually restrict to the case in which $\mathcal{C}$ is also biextensional, but other cases may be useful.

Given a normal Chu space $\mathcal{C}$, there is a topologically closed Chu space denoted $\tau(\mathcal{C}) = (C_o, \in, \tau(C_a))$, which is a topological closure of $\mathcal{C}$ in an obvious sense. There is a universal Chu morphism $\tau : \tau(\mathcal{C}) \to \mathcal{C}$ with $\tau_\star : C_o \to C_o$ as the identity, and $\tau^\star : C_a \to \tau(C_a)$ as the inclusion. For example all the Chu spaces in (26) have the same topological closure, which is given by (26c) plus the empty set and total set.

Remark

From the 'informational / observational' point of view, $\tau(\mathcal{C})$ contains essentially the same information as the original $\mathcal{C}$. In $\tau(\mathcal{C})$, that information has been preprocessed via the propositional operations of geometric logic. Of course, if all the Chu spaces being studied are finite ones, and represent posets, there is no difference between the topological closure and the 'normal realisation' of that poset, however the difference between arbitrary meets, required for the normal realisation, and the finite meets, allowed here, could be important in situations where comparison of arbitrary Chu spaces and their finitary approximations is involved.

Given a finite poset $\mathcal{P}$ then the Chu space, $(\mathcal{P}, \in, \tau(\mathcal{P}))$, is a topological closed Chu space isomorphic to $NR(\mathcal{P})$and we have the following evident lemma which relates the three different concepts: finite posets, finite T_0-spaces and a corresponding class of Chu spaces, and which slightly extends the well known classical result.

Lemma 4. *There is a natural equivalence between the category of finite T_0 spaces (morphisms given by continuous maps), the category of finite posets (morphisms being order preserving maps) and the category of finite biextensional topologically closed Chu spaces (morphisms being Chu transforms).*

Logical interpretations of the Sorkin model

A general Chu space, $\mathcal{C}$, is being thought of, here, as a model for 'informational structure'. In a Chu FFOS, the information is sampled via $\mathcal{F} \subset C_a$, usually finitely. From this viewpoint, there is a good 'logical' interpretation of the biextensional collapse / Sorkin model of $\mathcal{C}_{|\mathcal{F}} = (C_o, \models_C, \mathcal{F})$, the corresponding Chu space.

We will assume $\mathcal{F}$ is chosen so that $\mathcal{C}_{|\mathcal{F}}$ is extensional. (Any two 'observations' in $\mathcal{F}$ give different results on at least one object; the columns of $\mathcal{F}$ are distinct; $\mathcal{F}$ is considered as a subset of $\mathbf{2}^{C_o}$.)

In [6], van Benthem defines a flow formula as being any first order formula produced by the schema

$$x \models a \,\big|\, \neg(x \models a) \,\big|\, \wedge \,\big|\, \vee \,\big|\, \exists x \,\big|\, \forall a.$$

(In [6], the Chu space notation is $(A, \in, X)$, so our attributes are the elements of his X, not of A, and $\models$ is $\in$. The & in his notation is written $\wedge$ here.)

Given any row x in the biextensional collapse / Sorkin poset, $\mathcal{C}_{\mathcal{F}}$, it consists of n-entries 0 or 1, and hence to a flow formula

$$(x \models a_{i_1}) \wedge \ldots \wedge (x \models a_{i_k}) \wedge \neg(x \models a_{i_{k+1}}) \wedge \ldots \wedge \neg(x \models a_{i_n}), \qquad (27)$$

where, of course, the first k a_is are the attributes with $r(x, a_i) = 1$ and the others are those giving 0. This statement, (27), is true as well in $\mathcal{C}_{|\mathcal{F}}$ itself, but now in $\mathcal{C}_{\mathcal{F}}$, the formula *uniquely* determines the row and *vice versa*, this corresponding to the separability of $\mathcal{C}_{\mathcal{F}}$. In other words the rows of the biextensional collapse, $\mathcal{C}_{\mathcal{F}}$ *are* the elementary flow formulae

$$\exists x\Big(\bigwedge_{i \in \mathcal{F}_1} (x \models a_i) \wedge \bigwedge_{i \in \mathcal{F}_2} \neg(x \models a_i)\Big)$$

for some partition $(\mathcal{F}_1, \mathcal{F}_2)$ of $\mathcal{F}$. The question of refinement is then to modify the attribute sample $\mathcal{F}$, redefining some of the 'atomic' statements $x \models a$, replacing them with other flow formulae of this same form. (It may be useful to compare the above, in detail, with parts of Situation Theory, especially ideas on information flow found, for instance, in [7]. Other useful references are [8–10])

6 Sorkin refinement in the language of Chu spaces

Given Chu spaces $\mathcal{C}$ and $\mathcal{D}$, corresponding to FFOSs, $(X, \mathcal{F})$ and $(X, \mathcal{G})$, respectively and where $\mathcal{F}$ is a Sorkin refinement of $\mathcal{G}$, then, in general, there will be no Chu morphism between $\mathcal{C}$ and $\mathcal{D}$. For example, consider (26a) and (26b). Since these represent the same Sorkin poset, each can be considered a refinement of the other, yet there is no Chu morphism between them such that the carrier function is the identity on the three objects.

The problem is that Sorkin refinement uses the topology generated by a FFOS, so we have to mimic that in general for Chu spaces. Of course, that is exactly what is given by the topological closure operation we have just introduced.

Definition 10. *Given Chu spaces $\mathcal{C} = (C_o, \models_C, C_a)$, $\mathcal{D} = (C_o, \models_D, D_a)$ with the same set of objects. We say that $\mathcal{C}$ is a* Sorkin refinement *of $\mathcal{D}$ if there is a Chu transform*

$$\phi : \tau(C) \to \mathcal{D},$$

which is the identity on objects, i.e., $\phi_*(x) = x$*.*

If both $\mathcal{C}$ and $\mathcal{D}$ are normal Chu spaces, then ϕ^* will be an inclusion, $D_a \subseteq \tau(\mathcal{C})$, but, in general, ϕ^* need not be an inclusion, nor even an injection.

Lemma 5. *(i) Any Chu space is a Sorkin refinement of itself.*
(ii) Sorkin refinement is transitive.
(iii) If $\mathcal{C}$ *is extensional and* $\mathcal{C}$ *is a Sorkin refinement of* $\mathcal{D}$*, then the map* ϕ^* *is uniquely determined.*

Each of these is easy consequence of the definitions.

The first part of this lemma suggests that a Sorkin refinement may best be viewed as a pair of morphisms

$$\mathcal{C} \overset{can}{\leftarrow} \tau(\mathcal{C}) \overset{\phi}{\rightarrow} \mathcal{D}$$

with their object parts the identity on C_o, and where *can* is the canonical Sorkin refinement given by (i) of the above lemma.

7 Nerve of a cover and Sorkin models

As mentioned in the first part of this article, we can associate two simplicial complexes with every (dyadic) Chu space, $\mathcal{C}$, one being its Čech nerve, $N(\mathcal{C})$, and the other its Vietoris nerve, $V(\mathcal{C})$. Any simplicial complex, K, gives rise to a poset, namely the poset of its faces having the simplices, σ, τ, etc., of K as elements with $\sigma \leq \tau$ if σ is a face of τ, i.e., $\sigma \subseteq \tau$ as subsets of the set, K_0, of vertices of K.

For a Chu space $\mathcal{C} = (C_o, \models, C_a)$ and a subset $\mathcal{F} \subseteq C_a$, i.e., a Chu FFOS $(\mathcal{C}, \mathcal{F})$, we have the 'corestricted' Chu space, $\mathcal{C}_{|\mathcal{F}} = (C_o, \models, \mathcal{F})$ and thus several 'objects' that encode some of the geometric relationships in $\mathcal{C}$ that can be observed using $\mathcal{F}$. There are the two simplicial complexes $N(\mathcal{C}_{|\mathcal{F}})$ and $V(\mathcal{C}_{|\mathcal{F}})$, and thus the associated partially ordered sets of their faces, and there is also the biextensional collapse / Sorkin poset of $\mathcal{C}_{|\mathcal{F}}$, that will be denoted, as before, by $\mathcal{C}_{\mathcal{F}}$. It would be good to be able to compare these, since they encode 'geometric' information in slightly different ways. The purpose of this section is to reveal some of the relationships between these methods of analysing Chu FFOSs.

Remark

The context for this comparison is already partially in the literature. Given a Chu space $\mathcal{C}$, considered as a Formal Context within FCA theory, see [1, 4, 11], the informational content and structure, (formal concepts, for example), is extracted via the α and ω mappings mentioned earlier, so as to give a closure operation on $\mathcal{P}(C_a)$ and thence a 'concept lattice'. The exact relationship between this and the Sorkin model construction is not yet clear, but we note that if $(D, \leq)$ is any complete lattice, then the corresponding Chu space, $\mathcal{D} = (D, \leq^{op}, D)$, has a concept lattice that is order isomorphic to $(D, \leq)$ itself by the Representation Theorem of Zhang and Shen (see Theorem 4.1. of [4]), but $\mathcal{D}$ is biextensional and so corresponds to its own Sorkin model.

Another mode of analysis of a Chu space, $\mathcal{C}$ is via the nerve constructions. Here there is a variant in which simplices are labelled with subsets of objects that 'witness' to the non-emptiness of the corresponding intersection. This is one of the basic operations of the so-called 'Q-analysis' used within AI and some of the Social Sciences to display or visualise relationships between entities. Thus our initial steps here may lead, to some extent, to a clarification of the gap between these two models for extraction of informational structure from a relational source.

7.1 Comparison Results.

In order to compare nerves with Sorkin models, we clearly need to have a common setting. The above suggested that posets were such a setting, but we must formalise some of the links slightly more. Simplicial complexes are also clearly related to Chu spaces in the manner of their definition. In fact any simplicial complex is naturally a normal Chu space.

Recall if K is a simplicial complex, then we have a set K_0 of vertices and a set $S_K \subset \mathbf{2}^{K_0}$ of 'simplices'. The distinctive properties of S_K are: $\emptyset \notin S_K$ and, if $\tau \neq \emptyset$, $\tau \subset \sigma$ and $\sigma \in S_K$, then $\tau \in S_K$, i.e. S_K is closed under 'non-empty inclusion'. Of course, from a Chu perspective, we can take $\mathcal{S}_K = (K_0, \in, S_K)$ to get a normal Chu space.

Remark

There are some natural questions within this context for which replies would be useful. For example, is it possible to perform the various nerve constructions completely within the Chu context? The fact that these constructions are not quite functorial (see the discussion in the first part of this paper, [3]), suggests that the answer is probably negative perhaps for a relatively trivial reason, but that slightly amending the contexts involved may say more about the extent to which the constructions are 'internal'. If they can be performed internally then there would be some interest in seeing what was the exact structure on $\mathbf{2}$ that they used so that variants of the nerve constructions might be performed in a similar way for other 'alphabets' Σ, and hence, for other 'flavours' of Chu space.

The simplices of a simplicial complex form a poset $(S_K, \subseteq)$ with $\subseteq$ inherited from the power set $\mathbf{2}^{K_0}$, so we have a second Chu space associated with K, namely $NR(S_K, \subseteq)$, the normal realisation of this *face poset*. We will formally define this Chu space by

$$\text{face}(K) := NR(S_K, \subseteq).$$

We can now turn to analysing the relationship between the Sorkin model and the nerves. We will work in the setting of normal Chu spaces and in particular, within that of 'posets as Chu spaces'. We will first reduce the problem to one purely in that 'poset' setting:

Lemma 6. *Let $(\mathcal{C}, \mathcal{F})$ be a Chu FFOS, and $N(\mathcal{C}, \mathcal{F})$ its Čech nerve. Further let $\mathcal{C}_{\mathcal{F}}$ be its biextensional collapse and denote by $\widehat{\mathcal{F}}$ the corresponding family of* *attributes. Assume that there are no repeated columns in $\mathcal{F}$, so $\mathcal{C}_{|\mathcal{F}}$ is extensional, then the quotient map*

$$\pi_{\mathcal{F}} : \mathcal{C}_{|\mathcal{F}} \to \mathcal{C}_{\mathcal{F}}$$

exists and induces an isomorphism

$$N(\mathcal{C}, \mathcal{F}) \to N(\mathcal{C}_{\mathcal{F}}, \widehat{\mathcal{F}})$$

of simplicial complexes.

Proof. The assumption of extensionality for $\mathcal{C}_{|\mathcal{F}}$, as we saw earlier, implies that $\pi^*_{\mathcal{F}}$ is a bijection, so, by results in part 1, we have an induced map. That map is a bijection on vertices, so we only need to check what it does to simplices. As, in a nerve $N(\mathcal{C}, \mathcal{F})$, a n-simplex $\sigma = \langle a_0, \ldots, a_n \rangle$ is a set $\{a_0, \ldots, a_n\}$ such that

$$\exists x \ ((x \models a_0) \wedge \ldots \wedge (x \models a_n)) \tag{28}$$

holds and

$$(x \models a_0) \text{ if and only if } \check{\alpha}_C(x) \models \check{\omega}_C(a) \tag{29}$$

(cf. (17)), we have that $\langle a_0, \ldots, a_n \rangle \in N(\mathcal{C}, \mathcal{F})$ if and only if

$$\langle \check{\omega}_C(a_0), \ldots, \check{\omega}_C(a_n) \rangle \in N(\mathcal{C}_{\mathcal{F}}, \widehat{\mathcal{F}}),$$

which completes the proof.

We can thus assume that $\mathcal{C}_{|\mathcal{F}}$ is itself biextensional and thus essentially is a poset. One has, however, to remember that we have a sample of the columns of the corresponding normal realisation. Of course, the original sample $\mathcal{F}$ may not 'cover' the original $\mathcal{C}$. If that is the case $C_{\mathcal{F}}$ has a top element, but $\widehat{\mathcal{F}}$ misses it out so the lemma still holds. We will shortly produce a comparison map from $\mathcal{C}_{\mathcal{F}}$ to its nerve, but if there is a top element in $\mathcal{C}_{\mathcal{F}}$, there will be no corresponding simplex in the nerve as the families which are simplices in $N(\mathcal{C}_{\mathcal{F}}, \widehat{\mathcal{F}})$ are assumed 'non-empty'. We will therefore assume that $\mathcal{F}$ does 'cover' $\mathcal{C}$, so there is no zero row in $\mathcal{C}_{|\mathcal{F}}$. The more general case is left to the reader!

There are various properties that a Chu FFOS may have that are clearly relevant to our comparison and which also correspond, intuitively, to good 'informational' properties. Our discusssion will continue to assume $\mathcal{F}$ covers $\mathcal{C}$. First a useful technical definition.

Definition 11. *Let $(\mathcal{C}, \mathcal{F})$ be a Chu FFOS with $\mathcal{C}_{|\mathcal{F}}$ extensional. The* free $\wedge$-attribute completion *of $\mathcal{C}$ is the Chu space, denoted $\bigwedge_{fin} \mathcal{C}$, obtained by freely adding new attributes corresponding to all finite conjunctions of columns of $\mathcal{C}$:*
For each non-singleton finite subset, $A \subseteq C_a$, we form a new column, labelled $\bigwedge A$ or $a_{i_1} \wedge \ldots \wedge a_{i_n}$ if $A = \{a_{i_1}, \ldots, a_{i_n}\}$, with

$$r(x, \bigwedge A) = 1 \textit{ if and only if } r(x, a) = 1 \quad \forall a \in A. \tag{30}$$

Example

If $\mathcal{C}$ is the standard trident, (21), then we have new columns $a_1 \wedge a_2$, $a_1 \wedge a_3$, $a_2 \wedge a_3$ and $a_1 \wedge a_2 \wedge a_3$ in $\bigwedge_{\text{fin}} \mathcal{C}$. Each new column is 0 0 0 1.

Definition 12. *Let $(\mathcal{C}, \mathcal{F})$ be a Chu FFOS with $\mathcal{C}_{|\mathcal{F}}$ extensional.*

(i) We say $\mathcal{F}$ is minimal *if, for any $a \in \mathcal{F}$, there is an object x such that $r(x, b) = 0$ for all $b \in \mathcal{F}$ with $b \neq a$, (i.e., you cannot omit any attributes from the 'sample' without destroying the 'covering' property).*

(ii) We say $(\mathcal{C}, \mathcal{F})$ is generic *if $\bigwedge_{\mathit{fin}} \mathcal{C}_{|\mathcal{F}}$ is extensional, in other words, if $A = \{a_0, \ldots, a_m\}$ and $B = \{b_0, \ldots, b_n\}$ are finite subsets of $\mathcal{F}$ such that $\omega_C(A) = \omega_C(B)$, then $A = B$.*

Remarks

(i) We note that 'generic' clearly implies 'extensional', since the columns of $\bigwedge_{\text{fin}} \mathcal{C}_{|\mathcal{F}}$ include those of $\mathcal{C}_{|\mathcal{F}}$. The converse is not true however, as the standard trident example shows. Using the notation already introduced for that example, we have $a_1 \wedge a_2$ and $a_1 \wedge a_3$ in $\bigwedge_{\text{fin}} \mathcal{C}_{|\mathcal{F}}$ are identical columns. In other words we have a relation:

$$a_1 \wedge a_2 = a_1 \wedge a_3.$$

Of course, there is a maximal representation for this column of values, namely $a_1 \wedge a_2 \wedge a_3$.

(ii) The condition of 'genericity' compares the free wedge completion with its biextensional collapse and thus with that part of $\mathbf{2}^{C_o}$ generated from the given Chu space by intersection. (If $\mathcal{C}$ is separable, then this will form part of the normal realisation.) It thus compares the values of 'formal conjunctions' of attributes with the actual values. The 'formal' aspect is captured by the nerve, the 'actual' one by the Sorkin poset.

(iii) We suggest that 'generic' is another aspect of 'stability', as small 'perturbations' of the structure of a non-generic cover often seem to return to 'genericity', but, as we have said, we cannot as yet define these other terms precisely!

We can now examine an 'obvious' map from a $(\mathcal{C}, \mathcal{F})$ to the corresponding $N(\mathcal{C}, \mathcal{F})$. We can use lemma 6 to reduce to the case where $\mathcal{C}_{|\mathcal{F}}$ is biextensional and our discussion to impose the condition that $\mathcal{F}$ covers $\mathcal{C}$. We can think of $\mathcal{C}$ as being a normal Chu space in its normal realisation and $\mathcal{F}$ as a finite sample of the columns, so that $\mathcal{C}_{|\mathcal{F}}$ has no zero rows and no repeat rows.

Suppose $x \in C_o$ is an object of $\mathcal{C}$. As $\mathcal{C}_{|\mathcal{F}}$ is extensional, x corresponds to a flow formula,

$$((x \models a_{i_0}) \wedge \ldots \wedge (x \models a_{i_n})),$$

where $\{a_{i_0}, \ldots, a_{i_n}\} \subseteq \mathcal{F}$ is the set of columns, a of $\mathcal{C}_{|\mathcal{F}}$ for which $r(x, a) = 1$. We clearly have that

$$\psi(x) = \langle a_{i_0}, \ldots, a_{i_n} \rangle \in N(\mathcal{C}, \mathcal{F})_n,$$

since x satisfies them all. Note that $\psi(x) = \langle \alpha_{\mathcal{F}}(x) \rangle$, where $\alpha_{\mathcal{F}}$ is the 'α-map' for $\mathcal{C}_{|\mathcal{F}}$. We will think of this as being an object of the face poset, face$(N(\mathcal{C}, \mathcal{F}))$

of the nerve of $(\mathcal{C},\mathcal{F})$. We thus have a map between the elements of two posets. Is that map order preserving?

Suppose that $x \leq y$ in $\mathcal{C}$, this means that, for all $a \in C_a$,

$$r(x,a) \geq r(y,a),$$

so $\psi(y)$ is a subset of $\psi(x)$, i.e., it is an (iterated) face of it. (Note that we need that $\mathcal{C}_{|\mathcal{F}}$ has no zero rows, otherwise $r(y,a)$ might be a zero row and $\psi(y)$ the empty set, which is not a simplex.) As the order in the face poset of $N(\mathcal{C},\mathcal{F})$ is inclusion, ψ gives us an order preserving map / Chu transform:

$$\psi : NR(\mathcal{C}_{|\mathcal{F}}) \to \text{face}(N(\mathcal{C},\mathcal{F})).$$

It is useful to note that if $x \in C_o$, so $\psi(x) = \langle \alpha_{\mathcal{F}}(x)\rangle$, then $\downarrow\!x = \bigwedge \alpha_{\mathcal{F}}(x)$ within $NR(\mathcal{C}_{|\mathcal{F}})$.

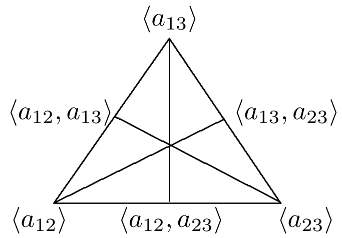
For particularly nicely behaved 'covers' $\mathcal{F}$, this order preserving map will be an isomorphism of posets. If this is the case we will say that $\mathcal{F}$ is *simplicial.* The properties identified above go some way to analysing this notion, for instance, if $\mathcal{F}$ is minimal, then for any $a \in \mathcal{F}$, there is an x such that

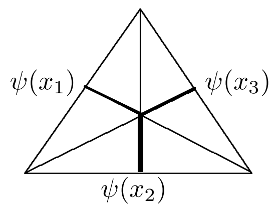
$$\{b \in \mathcal{F} \mid x \models b\} = \{a\},$$

so $\psi(x) = \langle a\rangle$, and conversely if all of the original vertices are to be in the image of ψ, then $\mathcal{F}$ must be minimal.

Example

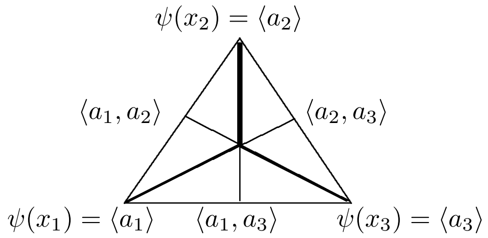
The trident variant example, (24), is not minimal, since $\mathcal{F}' = \{a_{12}, a_{13}\}$ is also a cover: there are no zero rows in $\mathcal{C}_{|\mathcal{F}'}$, which is also no longer separable as x_1 and x_4 now have the same row. The map ψ sends the trident to the nerve as in the following pictures, showing face$(N(\mathcal{C},\mathcal{F}))$ on the left,





with the image of ψ shown by thicker lines in the righthand one.

By contrast, the original trident, (21), which is *minimal* but not generic, gives



Here the images of the x_i are the corresponding downsets.

Neither of these covers is 'simplicial'. It is clear what prevents it in each case, and shows the close link between 'minimality' and 'genericity'.

If $\mathcal{F}$ is not minimal, we noted that some of the original vertices may be missed out. Intuitively, if $\mathcal{F}$ is not generic, then the 'poset' does not see the difference between two 'simplices', whilst the nerve records their labels as corresponding to distinct simplices.

Proposition 2. *If $\mathcal{F}$ is a simplicial cover, then it is minimal and generic.*

Proof. 'Minimality' has been proved above, but we give a more detailed analysis of 'genericity'.

Suppose $A, B \subseteq \mathcal{F}$ are subsets of attributes for which

$$\omega_C(A) = \omega_C(B), \tag{31}$$

so the set of objects satisfying the attributes in A, and in B are the same. We can thus assume $A \subseteq B$ without loss of generality, so

$$B = \{a_0, \ldots, a_n, b_0, \ldots, b_m\}$$

with $A = \{a_0, \ldots, a_n\}$ and we write $B' = B \setminus A = \{b_0, \ldots, b_m\}$. We want to show that B' is empty. We have

$$\omega_C(A) = \omega_C(B) \subseteq \omega_C(B'),$$

as $B' \subseteq B$.

If $\omega_C(A) = \emptyset$, then A does not form a simplex, but on the other hand, there is no object that satisfies all of A, so this does not cause a problem and we can assume $\omega_C(A) \neq \emptyset$, and as a consequence, $\langle A \rangle$ is a simplex of $N(\mathcal{C}, \mathcal{F})$ with $x \in \omega_A$ being proof of that fact. Also $\langle B \rangle$ and $\langle B' \rangle$ are simplices, so, as $\mathcal{F}$ is simplicial, there are objects $x, y, z \in C_o$ with

$$\alpha_C(x) = A, \alpha_C(y) = B, \alpha_C(z) = B'.$$

In the partially ordered set face$(N(\mathcal{C}, \mathcal{F}))$,

$$\langle A \rangle \vee \langle B' \rangle = \langle B \rangle.$$

(The poset face$(N(\mathcal{C}, \mathcal{F}))$ need not have all joins, but does have these.) As ψ is assumed to be an isomorphism,

$$x \vee z = y,$$

but then, as $B' \cap A$ is empty, x and z agree on no attribute:

$\mathcal{C}$	A B'
x	$1 \ldots 1\, 0 \ldots 0\, 0 \ldots$
z	$0 \ldots 0\, 1 \ldots 1\, 0 \ldots$
y	$1 \ldots 1\, 1 \ldots 1\, 0 \ldots$

(32)

so $y \in \omega_C(B)$, but is not in $\omega_C(A)$, contradicting (31).

7.2 Sorkin Models in the Language of Frames

Recall [5] that a poset $\mathcal{A}$ is a frame if, every subset has a join ($\bigvee$) and every finite subset has a meet ($\bigwedge$) and binary meets distribute over joins. In our language we may consider frames as topologies but without having the underlying set of points. The poset structure mirrors subset inclusion, meets and joins intersections and unions and the bottom $\perp$ and top $\top$ thus represent the empty set and the entire set.

If $\mathcal{A}$ and $\mathcal{B}$ are frames, a function from $\mathcal{A}$ to $\mathcal{B}$ is a *frame homomorphism* if and only if it preserves all joins and finite meets. It is important to remember the typical example of $\mathcal{A} = \tau(Y)$ and $\mathcal{B} = \tau(X)$, being topologies on spaces Y and X, and $f : X \to Y$ being a continuous map from X to Y. The frame homomorphism $f^{-1} : \tau(Y) \to \tau(X)$ goes in the reverse direction.

Throughout the discussion, above, of Chu spaces, we have assumed that there are some 'objects' that are 'observed' via their attributes. This does seem a bit strange if we merely have the observations and have no way of grabbing hold of some 'points' or 'objects', i.e., if we just have a frame. There is, however, a way to extract a Chu space (or more exactly a 'topological system' in the terminology of Vickers [5] p.52) from a given frame. We will briefly recall this to see how it fits with our overall 'philosophy' (cf, [5], p.60-61). We first give a more detailed formal definition of a topological system.

Definition 13. *Let $\mathcal{A}$ be a frame, X a set and $\models \subset X \times \mathcal{A}$, written, as usual, as $x \models a$, then $(X, \mathcal{A}, \models)$ is a* topological system *if and only if*

- *if S is a finite subset of $\mathcal{A}$, then*

$$x \models \bigwedge S \text{ if and only if } \forall a \in S (x \models a);$$

- *if S is any subset of $\mathcal{A}$,*

$$x \models \bigvee S \text{ if and only if } \exists a \in S (x \models a).$$

Given a frame $\mathcal{A}$, we can construct a topological system by taking X to be the set of frame homomorphisms

$$x : \mathcal{A} \to \mathbf{2}$$

with $x \models a$ if and only if $x(a) = 1$. This, of course, just identifies a point as a row in a possible Chu space matrix. The requirement that x is a frame homomorphism imposes conditions on the row x, relative to the meets and joins in $\mathcal{A}$. We will sometimes refer to the frame $\mathcal{A}$ as a *localic Chu space* and this will mean that we are using this associated topological system.

We shall use $\mathcal{T}$ for a"large" frame that we wish to model or sample. It may be infinite, e.g., if it comes from some topological space such as a manifold, or it may be finite, but just very large! We shall used $\mathcal{A}$, $\mathcal{B}$, etc., for the small finite frames, which may be considered as samples of $\mathcal{T}$.

[5] For example from [5]

Suppose A is a finite family of elements of $\mathcal{T}$, then we can form $\tau(A)$, its topological closure (within $\mathcal{T}$) as being the subframe of $\mathcal{T}$ generated by A. As a point of $\mathcal{T}$ is a $x : \mathcal{T} \to \mathbf{2}$, by restriction, we get $x|_{\tau(A)} : \tau(A) \to \mathbf{2}$, but different $\mathcal{T}$-points can restrict to the same $\tau(A)$-point. In fact, as $x|_{\tau(A)}$ is a frame homomorphism, it is completely determined by its values on the elements of A and we can put an equivalence relation on the points of $\mathcal{T}$ by

$$x \sim_A y \text{ if and only if } x|_A = y|_A.$$

Of course, this is exactly the analogue of Sorkin's original construction in this context. This is constructed solely from the pair $(\mathcal{T}, A)$. As $\tau(A)$ is closed under meets and joins, this Chu space is the normal realisation of a poset. Thus a sample or FFOS corresponds to a finite subframe $\tau(A)$ and here the corresponding locale *is* the Sorkin poset of that FFOS. (Note that there is no reason to expect a point of $\tau(A)$ to extend to one of $\mathcal{T}$ in general.)

In this context, given finite subframes, $\mathcal{A}$ and $\mathcal{B}$ of $\mathcal{T}$, $\mathcal{A}$ will be a *Sorkin refinement* of $\mathcal{B}$ if and only if $\mathcal{B} \subseteq \mathcal{A}$.

We next turn to the study of zones here. Since we do not directly have the set X, we cannot construct the quotient map $\pi : X \to \mathcal{P}$, nor can we consider the zones $\pi^{-1}(x)$ for $x \in \mathcal{P}$, since, in general, these are not elements of $\tau(X)$. In other words, when dealing with frames, we must express all our statements in terms of the 'open sets', i.e., elements of $\mathcal{A}$ or $\mathcal{T}$. We cannot so easily talk about general subsets of X.

Closeness.

Can we establish if two zones x, y are next to each other, i.e., $x \xrightarrow{c} y$, solely in the language of frames? The answer is yes. Clearly the formula (8) cannot be used, since we cannot define $\pi^{-1}(x)$ or talk about its closure in $\tau(X)$.

First note that in a frame there is the concept of the complement given by

$$\bullet^c : \mathcal{T} \to \mathcal{T} \; ; \qquad x \to x^c = \bigvee \{y \in \mathcal{T} \,|y \wedge x \neq \bot\} \tag{33}$$

If $\mathcal{T}$ is the topology $\tau(X)$, then X^c is the interior of the compliment of X, i.e. $X^c = X \backslash \overline{X}$. We can define the map

$$\text{intc} : \mathcal{P} \to X \; ; \quad \text{intc}(x) = \iota_{\mathcal{A}}(\downarrow\!x)^c \vee \left(\bigvee \{\iota_{\mathcal{A}}(\downarrow\!z) | z < x\}\right)^c \tag{34}$$

Again if $\mathcal{T}$ is the topology $\tau(X)$ then $\text{intc}(x)$ is the interior of the compliment of the zone $\pi^{-1}(x)$, i.e. $\text{intc}(x) = X \backslash \overline{\pi^{-1}(x)}$. We say $x \xrightarrow{c} y$ if

$$\text{intc}(x) \vee \text{intc}(y) \neq \top \quad \text{and} \quad \text{intc}(x) \vee \text{intc}(y) \vee \iota_{\mathcal{A}}(\downarrow\!y) \neq \text{intc}(x) \vee \text{intc}(y) \tag{35}$$

To see this we observe that if $\mathcal{T} = \tau(X)$, then $\text{intc}(x) \vee \text{intc}(y) \neq \top$ implies that $\overline{\pi^{-1}(x)} \cap \overline{\pi^{-1}(y)} \neq \emptyset$. This is not equivalent to $x \xrightarrow{c} y$ since it does not guarantee that there exists $z \in \overline{\pi^{-1}(x)} \cap \overline{\pi^{-1}(y)}$ such that $z \in \pi^{-1}(y)$. Cosider, for example, the case when $\pi^{-1}(x)$ and $\pi^{-1}(y)$ correspond to two open sets whose boundaries intersect. Thus the statement $\text{intc}(x) \vee \text{intc}(y) \vee \iota_{\mathcal{A}}(\downarrow\!y) \neq \text{intc}(x) \vee \text{intc}(y)$ implies there exists $z \in \overline{\pi^{-1}(x)} \cap \overline{\pi^{-1}(y)}$ such that $z \in \pi^{-1}(\downarrow\!y)$. From this we can prove that $\pi(z) = y$.

8 Limits of Sorkin models.

The informational content of a large set of data, such as that given by numerical simulations of a physical process or by medical scans, can only be analysed by sampling and that usually means finite sampling. Increasing the density of the sampling corresponds to increasing the refinement of the observations in the sense we have been using that term. It is thus useful to model the extent to which increasing the level of observational refinement 'ad infinitum' can retrieve the (abstract) object being observed. To do this we wish to take the limit of the Sorkin models of a space X with respect to increasing refinements in the FFOSs $\mathcal{F}$ and will generally limit ourselves to this spatial case, as this is the case that we understand best.

Each FFOS will generate a poset and the sequence of posets has a limit which is also a poset. We wish to interpret this limit.

Let $\{\mathcal{F}_i\}$, $i \in \mathbb{N}$ be a sequence of FFOS with each $\mathcal{F}_{i+1}$ being a Sorkin refinement of the previous one, $\mathcal{F}_i$. Since the posets depend only on the topology $\tau(\mathcal{F}_i)$, we shall assume that each $\mathcal{F}_i$ is topologically closed, i.e. $\mathcal{F}_i = \tau(\mathcal{F}_i)$. Thus

$$\cdots \hookleftarrow \mathcal{F}_{i+2} \hookleftarrow \mathcal{F}_{i+1} \hookleftarrow \mathcal{F}_i \hookleftarrow \cdots \tag{36}$$

For each $\mathcal{F}_{i+1} \hookrightarrow \mathcal{F}_i$, there is a corresponding order preserving surjection $\mathcal{P}_{i+1} \stackrel{\pi_{i+1,i}}{\longrightarrow} \mathcal{P}_i$, thus (36) gives an inverse sequence of posets:

$$\cdots \stackrel{\pi_{i+3,i+2}}{\longrightarrow} \mathcal{P}_{i+2} \stackrel{\pi_{i+2,i+1}}{\longrightarrow} \mathcal{P}_{i+1} \stackrel{\pi_{i+1,i}}{\longrightarrow} \mathcal{P}_i \stackrel{\pi_{i,i-1}}{\longrightarrow} \cdots \quad . \tag{37}$$

The nature of the limit of these sequences depends not only on the structure to X and the choice of FFOSs $\{\mathcal{F}_i\}$ but may also depend on the category in which one chooses to take the limit.

We shall first look at the limit of (37) in the category of posets / T_0-spaces. The limit is given by the poset $\mathcal{P}_\infty$, which as a set is given by

$$\mathcal{P}_\infty = \{\underline{x} = (x_0, x_1, x_2, \ldots) \mid x_i \in \mathcal{P}_i\,,\ x_i = \pi_{i+1,i}(x_{i+1})\}$$

and

$$\pi_{i\infty} : \mathcal{P}_\infty \to \mathcal{P}_i\ ; \qquad \pi_{i\infty}((x_0, x_1, x_2, \ldots)) = x_i; \tag{38}$$

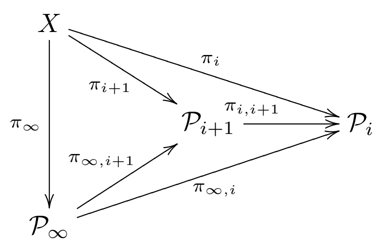
the order on $\mathcal{P}_\infty$ is given by

$$(x_0, x_1, x_2, \ldots) \preccurlyeq (y_0, y_1, y_2, \ldots) \qquad \Longleftrightarrow \qquad x_i \preccurlyeq y_i\ \forall i \in \mathbb{N}, \tag{39}$$

There is the natural map

$$\pi_\infty : X \to \mathcal{P}_\infty\ ; \qquad \pi_\infty(x) = (\pi_0(x), \pi_1(x), \pi_2(x), \ldots), \tag{40}$$

so that the following commutes



(41)

Let us assume, for simplicity, that we wish to approximate a manifold or similar, so we will assume X is a complete metric space. We also assume that $\{\mathcal{F}_i\}$ separates X, that is, for any two points $x, y \in X$, there exist $\mathcal{F}_i \in \{\mathcal{F}_i\}$ and $U \in \mathcal{F}_i$ such that U distinguishes x from y. Given these two assumptions we can define the map

$$\sigma : \mathcal{P}_\infty \to X \ ; \qquad \bigcap_{i=0}^{\infty} \overline{\pi_i^{-1}(x_i)} = \{\sigma(x_0, x_1, x_2, \ldots)\} \tag{42}$$

which is a left inverse of π_∞

$$\sigma(\pi_\infty(x)) = x \tag{43}$$

and

$$\underline{y} \preccurlyeq \pi_\infty(\sigma(\underline{y})) \tag{44}$$

What is the nature of $\mathcal{P}_\infty$? In fact, $\mathcal{P}_\infty$ does not seem to be very interesting! All the activity happens on the subset $\partial\mathcal{F}_\bullet \in X$ given by

$$\partial\mathcal{F}_\bullet = \bigcup_{i=0}^{\infty} \bigcup_{U \in \mathcal{F}_i} \partial U \tag{45}$$

This set is dense in X.

Lemma 7.

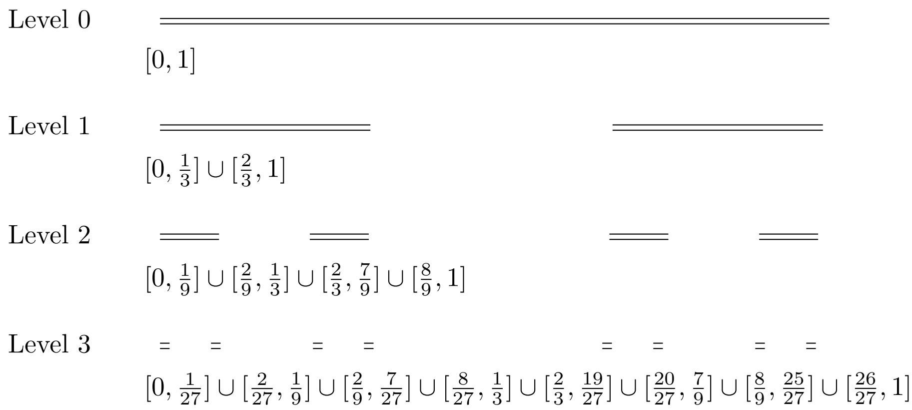
- *Given $\underline{x}, \underline{y} \in \mathcal{P}_\infty$ such that $\underline{x} \preccurlyeq \underline{y}$ and $\underline{x} \neq \underline{y}$, then $\sigma(\underline{x}) = \sigma(\underline{y}) \in \partial\mathcal{F}$.*
- *Given $x \in X \backslash \partial\mathcal{F}_\bullet$, then $\sigma^{-1}(X)$ is a singleton.*
- *Given $x \in \partial\mathcal{F}_\bullet$ then there exists $\underline{x}, \underline{y} \in \mathcal{P}_\infty$ such that $\underline{x} \preccurlyeq \underline{y}$, $\underline{x} \neq \underline{y}$ and $\sigma(\underline{x}) = \sigma(\underline{y}) = x$.*

We can also look at the sequence (37) in the category of topological spaces, where each $\pi_{i,i+1}$ is a continuous map. The limit in this category is given by the topological space $(\mathcal{P}_\infty, \mathcal{F}_\infty)$ where the points are given by $\mathcal{P}_\infty$ and a basis for the topology is given by $\bigcup_{i=1}^{\infty} \mathcal{F}_i$. (The exact nature of this topological space needs investigating further, see Kopperman, [12, 13]).

We could get other limits by working, say, in the category of Chu spaces and the category of frames, but have not studied their interrelations as yet.

Examples of Sorkin refinements: The Cantor set

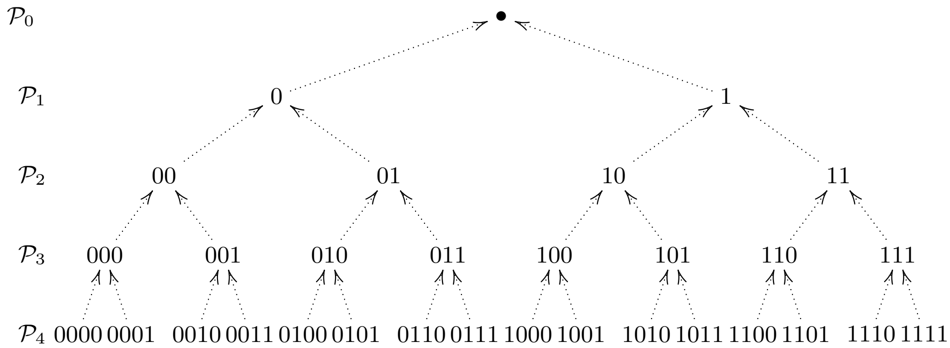
We will look at two Sorkin refinement sequences for the Cantor set, which we construct in the standard way by successively removing the middle third of each interval, i.e., by polyhedral approximations.



First, consider the simplest case where at each level, we give the open set to be the entire interval of the polyhedral approximation, i.e.,

$$
\begin{aligned}
\mathcal{F}_0 &= \{[0,1]\},\\
\mathcal{F}_1 &= \left\{[0,\tfrac{1}{3}]\,,\,[\tfrac{2}{3},1]\right\},\\
\mathcal{F}_2 &= \left\{[0,\tfrac{1}{9}]\,,\,[\tfrac{2}{9},\tfrac{1}{3}]\,,\,[\tfrac{2}{3},\tfrac{7}{9}]\,,\,[\tfrac{8}{9},1]\right\},\ldots
\end{aligned}
$$

This gives rise to the following sequence of discrete posets. (All points are unrelated, the dotted lines represent the mappings coming from the refinement process.)



The corresponding limit $\mathcal{P}_\infty$ is the Cantor set with $\partial\mathcal{F}_\bullet = \emptyset$, i.e., no structure remains. This can be considered in terms of information arriving from a bit stream. However at level n we simply have n disconnected points so it is not very interesting topologically. We can, however, choose a different sequence of covers and refinements. For this second case, at each level we consider the star open cover.

$$\mathcal{F}_0 = \{[0,1)\,,\,(0,1]\}$$
$$\mathcal{F}_1 = \left\{[0,\tfrac{1}{3})\,,\,(0,\tfrac{1}{3}]\,,\,[\tfrac{2}{3},1)\,,\,(\tfrac{2}{3},1]\right\}$$
$$\mathcal{F}_2 = \left\{[0,\tfrac{1}{9})\,,\,(0,\tfrac{1}{9}]\,,\,[\tfrac{2}{9},\tfrac{1}{3})\,,\,(\tfrac{2}{9},\tfrac{1}{3}]\,,\,[\tfrac{2}{3},\tfrac{7}{9})\,,\,(\tfrac{2}{3},\tfrac{7}{9}]\,,\,[\tfrac{8}{9},1)\,,\,(\tfrac{8}{9},1]\right\}$$

This gives rise to the corresponding sequence of posets and the limit $\mathcal{P}_\infty$ is the Cantor set with $\partial\mathcal{F}_\bullet$ being the countable subset of all edges. If we write the elements of the Cantor set as binary expansions, then these are the points whose binary expansions terminate in all 0s or all 1s. Again this can be considered in terms of information arriving from a bit stream. In this case however at each level we are given one of the three following possible pieces of information.

All the remaining digits are 0's	Represented above by a 0 in the last digit
All the remaining digits are 1's	Represented above by a 1 in the last digit
There are 1's and 0's in the remaining digits	Represented above by a $\star$ in the last digit

In this case there is a non trivial poset at each level and there is a corresponding non trivial differential calculus, for which see part III.

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