DIHOMOTOPY CLASSES OF DIPATHS IN THE GEOMETRIC REALIZATION OF A CUBICAL SET: FROM DISCRETE TO CONTINUOUS AND BACK AGAIN. EXTENDED ABSTRACT.

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1. Introduction

This extended abstract should be considered as an overview of some results, and the more precise statements and proofs are found in the references. Hence the approach here is less rigorous than in the references. The new results presented at Dagstuhl are in [1] and [3].

The subject is directed topology in spaces which have been subdivided into cubes. Such spaces are used as the geometric model of Higher Dimensional Automata [6] and for Dijkstras PV-models [2], and going back and forth between the algebraic and the geometric representation is a very concrete example of a geometrization/discretization process or a discrete - continuous correspondence.

2. Local partial orders and d-spaces.

In the geometric model of for instance an HDA, there is an underlying time direction, which should be preserved, when manipulating the geometric object. There are different approaches to how one should encode that extra structure. M. Grandis in [5] does it by taking out a subset of the set of continuous paths and specifying, that these are the increasing paths; this defines a d-space. In [4], we define a local partial order on a topological space; this is an open cover of the space by partially ordered open sets such that the partial orders are closed and they are compatible on intersections. A directed path is then a path, which is locally increasing.

Equivalence of executions coming from local commutativity or independence relations modeled by the HDA, has a geometric counterpart, namely dihomotopy or d-homotopy in Grandis' approach.

Two directed paths $\gamma_1, \gamma_2 : \vec{I} \to X$ are dihomotopic, if there is a continuous family $H : I \times \vec{I} \to X$ of dipaths connecting one to another. In the d-space approach, the equivalence relation is the transitive hull of the relation defined by a continuous family $H : \vec{I} \times \vec{I} \to X$ - preserving local partial order in both parameters. It is clear that d-homotopy implies dihomotopy, but the converse is not true in general.

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3. Cubical complexes and cubicalized spaces.

The algebraic version of a 3-cube is a graded set: $M_3 = \{Q\}$ (1 3-dimensional object), $M_2 = \{A, B, C, D, E, F\}$ (Representing the 2- dimensional faces), $M_1 = \{a, b, c, d, e, f, g, h, i, j, k, l\}$ (12 edges) and $M_0 = \{p, q, r, s, t, u, v, x\}$ (8 vertices), and boundary maps $\partial_i^k : M_n \to M_{n-1}$, $k \in \{0,1\}$, $i \in \{1, 2, 3, \dots, n\}$ such that

$$\partial_i^k \partial_j^l \quad = \quad \partial_{j-1}^l \partial_i^k \quad (i < j)$$

The boundary maps reflect the attaching of the $n-1$ dimensional faces onto the n-dimensional objects, and since these are cubes, there is both an upper, $k = 1$, and a lower, $k = 0$, face in each coordinate direction - indexed by i.

The general construction with a graded set and boundary maps satisfying the above relations is called a semi-cubical set, and geometrically one may think of this as several cubes glued along common faces. Indeed, there is a geometric realization of such a semi-cubical set, which glues cubes following the underlying recipe given by the boundary maps. When the semi-cubical complex is *non-selfintersecting*, the geometric realization is a locally partially ordered space, [4]. The local partial order on each cube is induced by the coordinatewise po on $\mathbb{R}^n$: $(x_1, \dots, x_n) \leq (y_1, \dots, y_n)$ if $x_i \leq y_i$ for all i, and we takte the transitive hull of this locally (i.e., in the star of each vertex).

A Higher Dimensional Automaton is an example of a semi-cubical set.

4. Approximation in a directed setting.

In an HDA, the computational paths representing executions are sequences of edges $e_1, \dots, e_m$ such that $\partial_1^1(e_k) = \partial_1^0(e_{k+1})$. If there is a 2-cube F such that $e_k = \partial_2^0(F)$ and $e_{k+1} = \partial_1^1(F)$, then $e_1, \dots, e_m$ is equivalent to the sequence where e_k, e_{k+1} is replaced by $\partial_1^0(F), \partial_2^1(F)$. The equivalence relation on execution paths is the transitive hull of this relation. This equivalence relation is denoted combinatorial equivalence. The geometric realization of such dipaths are called combinatorial dipaths.

In the geometric model of an HDA, the natural paths to study are the directed paths and the equivalence relation should be dihomotopy (or d-homotopy). It is immediate that the combinatorially equivalent execution paths are geometrically realized as dipaths, which are dihomotopic and also d-equivalent. However, it takes some work to see, that there is a converse to this:

Let $\gamma : \vec{I} \to X$, where X is the geometric realization of a *geometric*[1] Then there is a combinatorial dipath $\tilde{\gamma}$ which is dihomotopic to γ.

Suppose γ_1 and γ_2 are combinatorial dipaths, which are dihomotopic. Then they are also combinatorially equivalent.

[1] In a geometric complex, the intersection of two cubes is a unique face in both [1] cubical complex.

5. Cubicalized spaces and directed cubicalizations.

So how complicated can the geometry of a space be, when we demand that it is subdivided into cubes - geometrically- and that there is a partial order on each cube induced by the standard po on $\mathbb{R}^n$ and such that these agree on common faces. Even if there is a cubical subdivision of a space, there may not be a consistent ordering of the edges, as one can see in a standard Möbius band, but in fact, this twodimensional obstruction is the only obstruction there is, so a cubicalized space without immersed cubical Möbius bands has a cubical local partial order. And moreover, If a cubicalization is barycentrically subdivided once, the resulting cubical subdivision has a cubical local partial order. These results are both in [3]

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