A Cartesian Closed Extension of the Category of Locales (Preliminary Version)

Reinhold Heckmann[*]
AbsInt Angewandte Informatik GmbH,
Stuhlsatzenhausweg 69, D-66123 Saarbrücken, Germany
e-mail: heckmann@absint.com

Abstract

We present a Cartesian closed category ELoc of *equilocales*, which contains the category Loc of locales as a reflective full subcategory. The embedding of Loc into ELoc preserves products and all exponentials of exponentiable locales.

1 Introduction

The category Top of topological spaces and continuous functions is not itself Cartesian closed, but embeds into several different Cartesian closed supercategories. This fact allows for using lambda-calculus in topological proofs as e.g., in Escardó's work [2]. So far, no Cartesian-closed supercategory of the category Loc of locales was known. This note fills this gap by presenting one such supercategory, called the category ELoc of *equilocales*.

The new category has some similarity with the category of *equilogical spaces*, which is one of the Cartesian closed supercategories of $\mathcal{T}_0$-Top [1]. In fact, there are several equivalent categories of equilogical spaces of different kinds:

- A category officially called EQU. An object of EQU is a $\mathcal{T}_0$-topological space carrying an equivalence relation.
- A category officially called PEQU. An object of PEQU is an algebraic lattice carrying a *partial* equivalence relation (PER). Note that algebraic lattices, when endowed with the Scott topology, are a special case of injective topological spaces. Taking all injective spaces, i.e., generalizing to continuous lattices, leads to a larger, but equivalent category.

In a similar way, we shall present two different but equivalent categories: the objects of IELoc involve an *injective* locale and a family of PERs, while the objects of SELoc involve an arbitrary locale and a family of PERs satisfying a joint surjectivity condition. The name ELoc without a distinctive initial will be used as a generic name for both of these categories when speaking about properties invariant under equivalence, such as the property of being Cartesian closed.

As both EQU and PEQU obviously embed into the nameless category of $\mathcal{T}_0$-topological spaces carrying a PER, both IELoc and SELoc embed into the category ELoc* whose objects involve

[*]Almost all results in this note were found during a visit of the author at the School of Computer Science of the University of Birmingham, UK.

Dagstuhl Seminar Proceedings 04351
Spatial Representation: Discrete vs. Continuous Computational Models
http://drops.dagstuhl.de/opus/volltexte/2005/133

an arbitrary locale and a family of PERs. For matters of economy, we introduce ELoc^* first together with some machinery that then applies to both IELoc and SELoc.

Section 2 contains some general facts about PERs, which play a major role in the entire development. Section 3 lists those properties of the category Loc of locales that are needed for the development of ELoc. Then Section 4 introduces the auxiliary category ELoc^* and studies some of its properties. In Section 5, ELoc^* is restricted to its full subcategory IELoc, which is shown to be Cartesian closed. Section 6 then defines SELoc as another full subcategory of ELoc^* and proves the equivalence between IELoc and SELoc. In Section 7, we show that Loc embeds into SELoc as a full subcategory, and prove that this embedding preserves products and the exponentials of exponentiable locales, which already exist in Loc. Section 8 then establishes a reflection of SELoc back into Loc.

In showing these results, we never need to delve into the details of the internal structure of locales. In particular, we never work with their frames of opens. We only need some general properties of these objects, which are listed in Section 3. Thus, the results in this note hold in fact for categories different from Loc if only the required general properties are guaranteed.

2 Partial Equivalence Relations

The results in this section are standard and included for later reference.

A *partial equivalence relation* (PER) on a set P is a binary relation '$\sim$' (i.e., subset of $P \times P$), which is symmetric ($a \sim b \Rightarrow b \sim a$) and transitive ($a \sim b,\ b \sim c \Rightarrow\ a \sim c$), but not necessarily reflexive ($a \sim a$ need not hold for all a). Yet elements related to anything are self-related:

2.1 For any PER '$\sim$': If $a \sim b$, then $a \sim a$ and $b \sim b$.

Proof: By symmetry, $a \sim b$ implies $b \sim a$. From $a \sim b$ and $b \sim a$, $a \sim a$ and $b \sim b$ follow by transitivity. □

A subset S of a set P equipped with a PER '$\sim$' is *consistent* if $a \sim b$ holds for all $a, b \in S$ (this includes $a \sim a$ for all a in S). It is *saturated* if $a \in S$ and $a \sim b$ implies $b \in S$. A *PER class* of '$\sim$' is a non-empty saturated consistent subset of P. These PER classes share many properties of equivalence classes.

2.2 If two classes C_1 and C_2 have non-empty intersection, then they are equal.

2.3 If $a \sim a$ holds for some a in P, then $[a] := \{b \in P \mid a \sim b\}$ is a class. It is the unique class containing a. For any class C, $C = [a]$ holds for any a in C. Two classes $[a]$ and $[b]$ are equal ($[a] = [b]$) iff $a \sim b$ holds.

Thus, the PER classes form a partition—not of the entire set P, but of the set $\{a \in P \mid a \sim a\}$ of self-related elements of P.

The set of classes of '$\sim$' is denoted by $P/{\sim}$. We now consider a family of sets $(P_i)_{i \in I}$ and a set Q equipped with PERs '$\sim_{P_i}$' and '$\sim_Q$', respectively. We say a function $F : \prod_{i \in I}(P_i/{\sim_{P_i}}) \to Q/{\sim_Q}$ is induced by a function $f : \prod_{i \in I} P_i \to Q$ if $F([a_i])_{i \in I} = [f(a_i)_{i \in I}]$ holds for all $(a_i)_{i \in I}$ in $\prod_{i \in I} P_i$.

2.4 A function $f : \prod_{i \in I} P_i \to Q$ induces some $F : \prod_{i \in I}(P_i/{\sim_{P_i}}) \to Q/{\sim_Q}$ if and only if f satisfies $(\forall\, i \in I : a_i \sim_{P_i} b_i) \Rightarrow f(a_i)_{i \in I} \sim_Q f(b_i)_{i \in I}$. If it exists, the induced function F is unique.

3 Required Properties of the Basic Category

In this note, we never need to refer to the concrete description of locales and locale maps. Instead, the entire development is based on some abstract properties of the category Loc of locales. These properties are listed in the following. To avoid speaking of unspecific "objects" and "arrows", we keep on talking about Loc, locales, and locale maps, but the reader should keep in mind that the theory not only applies to Loc itself, but to any category satisfying the necessary properties.

3.1 Category

Requirement 1: Loc is a locally small category.

So there is a class of objects, called locales, and for any two locales X and Y, a set $\mathsf{Loc}\,(X, Y)$ of (locale) *maps* from X to Y. (The meaning of "locally small" is that $\mathsf{Loc}\,(X, Y)$ is a *set*, in contrast to a class.) For $f \in \mathsf{Loc}\,(X, Y)$, we also write $f : X \to Y$. The identity on X is written $i_X : X \to X$, and composition of $f : X \to Y$ and $g : Y \to Z$ is denoted by $g\,f : X \to Z$.

The maps $x : S \to X$ from arbitrary locales S are usually considered as generalized points of the locale X. Composition then takes over the role of application: a map $f : X \to Y$ maps a generalized point $x : S \to X$ of X to a generalized point $f\,x : S \to Y$ of Y.

Concretely, a locale X is given by a frame $\mathcal{O}X$, and a locale map $f : X \to Y$ by a frame homomorphism $f^* : \mathcal{O}Y \to \mathcal{O}X$. Thus it is quite obvious that Loc is locally small.

3.2 Products

Requirement 2: The category Loc has products.

This means that for any family $(X_i)_{i \in I}$ of locales, there is a locale $X = \prod_{i \in I} X_i$ and a family $(p_i : X \to X_i)_{i \in I}$ of maps such that for any locale Y and family $(f_i : Y \to X_i)_{i \in I}$ of maps, there is a unique map $f = \langle f_i \rangle_{i \in I} : Y \to X$ with the property $p_i\,f = f_i$ for all i in I.

We also use products $\prod_{i \in I} f_i$ of maps, and the standard notation for the binary case involving $X \times Y$, $\langle x,\, y \rangle$, and $f \times g$.

Concretely, products of locales are obtained by a well-known standard construction [3, II 2.12].

3.3 Monos and Subobjects

A map $e : X \to Y$ is *mono* if for all $x, x' : S \to X$, $e\,x = e\,x'$ implies $x = x'$. Considering the maps $y : S \to Y$ as generalized points of Y, we say that such a y is *contained* in the mono $e : X \to Y$ (and write $y \le e$) if there is a map $x : S \to X$ such that $y = e\,x$. Here, x is uniquely determined because of e's mono property. This unique x is denoted by $e^<\,y$. (This is a non-standard notation; in fact, we do not know of *any* notation for this purpose, but feel that it is useful to have one.)

The following properties are easily proved:

3.1 For any mono $e : X \to Y$, $y : S \to Y$, and $s : R \to S$: If $y \le e$, then $y\,s \le e$ and $e^<\,(y\,s) = (e^<\,y)\,s$. Thus, we may write $e^<\,y\,s$ without risk of ambiguity.

3.2 For any monos $e_1 : X_1 \to Y$ and $e_2 : X_2 \to Y$: $e_1 \leq e_2$ if and only if for all $y : S \to Y$, $y \leq e_1 \Rightarrow y \leq e_2$.

Two monos $e_1 : X_1 \to Y$ and $e_2 : X_2 \to Y$ are *equivalent* ($e_1 \cong e_2$) if $e_1 \leq e_2$ and $e_2 \leq e_1$. Equivalence $e_1 \cong e_2$ implies $X_1 \cong X_2$ and can be characterized in terms of generalized points: $e_1 \cong e_2$ if and only if for all $y : S \to Y$, $y \leq e_1 \Leftrightarrow y \leq e_2$. The equivalence classes of monos to a common target Y are called *subobjects* of Y by categorists. We shall usually not distinguish notationally between a mono and the subobject given by its equivalence class.

There are no requirements here; this section only serves to fix notation.

3.4 Equalizers and Sublocales

Requirement 3: Loc has equalizers.

This means that for all $f, f' : Y \to Z$, there is a mono $e : X \to Y$ such that for all $y : S \to Y$, $y \leq e$ iff $fy = f'y$. All monos arising in this way are called regular in categorists' parlance. Regularity is preserved by equivalence, and thus one can speak of regular subobjects. We shall however follow the custom of localists (similar to that of topologists) and speak of *embeddings* instead of regular monos, and *sublocales* instead of regular subobjects.

Requirement 4: The sublocales of a locale Y form a set (in contrast to a class). Each sublocale (= equivalence class of embeddings) has a canonical representative. The representative of the class of the identity i_Y is i_Y itself.

Concretely, Loc *has equalizers by a standard construction. The sublocales of a fixed locale Y form a set since they correspond to certain subsets of* $\mathcal{O}Y$ *[4, Prop. 6.2.8], which give also rise to canonical representatives. In contrast to this, the subobjects of a locale (in the categorists' sense) do not form a set.*

The requirements set up so far suffice to establish inverse images and meets of sublocales.

3.3 Let $f : X \to Y$ be a map and $e : Y' \to Y$ a sublocale of Y. Then there is a sublocale $e_f : X' \to X$ of X such that for any $x : S \to X$, $x \leq e_f$ iff $fx \leq e$. The map $f_e = e^{<} f\, e_f : X' \to Y'$ is well-defined and satifies $e\, f_e = f\, e_f : X' \to Y$.

Intuitively, e_f is the inverse image of e under f, and f_e is the restriction of f to this inverse image, which maps into e.

Proof: Let $g,\, g' : Y \to Z$ be maps with equalizer e. Define e_f to be an equalizer of $g\, f,\, g'\, f : X \to Z$. Then $x \leq e_f$ iff $g\, f\, x = g'\, f\, x$, iff $fx \leq e$. From $e_f \leq e_f$, $f\, e_f \leq e$ follows so that $f_e = e^{<} f\, e_f$ is well-defined. It satifies $e\, f_e = f\, e_f$ by construction. □

3.4 For any (set-indexed) family $(e_i : X_i \to Y)_{i \in I}$ of sublocales of a fixed locale Y, there is a sublocale $e = \bigwedge_{i \in I} e_i : X \to Y$ with the property that for any $y : S \to Y$, $y \leq e$ iff $y \leq e_i$ for all i in I. This e is the meet of the e_i in the poset of sublocales of Y ordered by '$\leq$'.

Proof: Let $f_i, f'_i : Y \to Z_i$ be maps with equalizer e_i. Define $Z = \prod_{i \in I} Z_i$, $f = \langle f_i \rangle_{i \in I} : Y \to Z$ and $f' = \langle f'_i \rangle_{i \in I} : Y \to Z$, and let $e : X \to Y$ be an equalizer of f and f'. Then $y \leq e$ iff $fy = f'y$, iff $f_i\, y = f'_i\, y$ for all i in I, iff $y \leq e_i$ for all i in I. By 3.2, e is the meet of the e_i. □

3.5 Coequalizers and Quotients

The requirements listed now are only needed to establish a reflection from SELoc to Loc in Section 8.

Requirement 5: The category Loc has coequalizers.

Concretely, the frames describing coequalizers of locales can be easily constructed as equalizers in the category of frames.

The regular epis, i.e., those that arise as coequalizers, are called *quotient maps*, and their equivalence classes are called *quotients*.

Requirement 6: Each quotient (= equivalence class of quotient maps) has a canonical representative. The representative of the class of the identity i_X is i_X itself.

Concretely, the frames of the quotients of X correspond to certain subframes of $\mathcal{O}X$, namely those that can be obtained as equalizers of frame homomorphisms. These subframes give rise to canonical representatives.

3.6 Injective Locales

Categorists often define injective objects w.r.t. monos, while topologists and locale theorists prefer to define them w.r.t. regular monos (embeddings). We adopt the locale theorists' view.

3.5 Definition A locale A is *injective* if for all embeddings $e : X \to Y$ and all $f : X \to A$, there is a (not necessarily unique) extension of f to Y, i.e., a map $\overline{f} : Y \to A$ satisfying $\overline{f}\, e = f$.

We require a rich supply of injective locales.

Requirement 7: Every locale X can be embedded into an injective locale AX by $a_X : X \hookrightarrow AX$. There is a canonical way to construct AX and a_X from X.

Concretely, a locale is injective iff its frame of opens is the Scott topology of a continuous lattice [3, VII Cor. 4.9]). Every locale X can be embedded into the injective locale AX whose frame of opens is the frame of lower sets of $\mathcal{O}X$. The global points of the locale AX form an algebraic lattice consisting of the filters of $\mathcal{O}X$, ordered by inclusion. The frame of lower sets of $\mathcal{O}X$ is isomorphic to the Scott topology of this algebraic lattice. The embedding $a_X : X \to AX$ can be described by the frame homomorphism mapping each lower set of $\mathcal{O}X$ to its join in $\mathcal{O}X$.

3.6 The class of injective locales is closed under products.

This can be shown by categorical reasoning. Thus it is not restricted to concrete locales, but holds on the abstract level of this paper. Additionally we require the following:

Requirement 8: The category Loc_I of injective locales is Cartesian closed.

Concretely, this requirement is satisfied since Loc_I is equivalent to the category of continuous lattices and Scott continuous functions.

4 Generalized Equilocales

Using that Loc is a category, we now introduce the category ELoc^* of generalized equilocales. Using that Loc has products, we show that ELoc^* has products, too.

4.1 Definitions

The equilogical analogue of ELoc^* is the category of PERs on $\mathcal{T}_0$-topological spaces. Note that a PER on a space X in Top, i.e., on the set of points of X, corresponds to a PER on the set $\mathsf{Top}\,(\mathbf{1}, X)$ of continuous functions from the terminal space (one-point space) $\mathbf{1}$ to X. Here, we replace the $\mathcal{T}_0$-topological space X by a locale X, but we also need to get away from considering $\mathbf{1}$ since there are non-trivial locales X with no points ($\mathsf{Loc}\,(\mathbf{1}, X) = \emptyset$), and there is no use of considering a PER on the empty set. The solution is to consider not only a PER on the single set $\mathsf{Loc}\,(\mathbf{1}, X)$, but a family of PERs consisting of one PER on each set $\mathsf{Loc}\,(S, X)$, for any locale S. Here, the elements of $\mathsf{Loc}\,(S, X)$, i.e., the locale maps from S to X, are considered as the generalized points of the locale X at stage S.

4.1 Definition A generalized equilocale (object of ELoc^*) $\mathcal{X}$ is a pair $(X, \sim_{\mathcal{X}})$ consisting of a locale $X = |\mathcal{X}|$ (the *target locale* of $\mathcal{X}$) and a family $\sim_{\mathcal{X}} = (\sim^S_{\mathcal{X}})_{S \in \mathsf{Loc}}$ where $\sim^S_{\mathcal{X}}$ is a PER on the set $\mathsf{Loc}\,(S, X)$ of locale maps from S to X, subject to the following compatibility condition:

$$\forall\, s : R \to S : \;\; x \sim^S_{\mathcal{X}} x' \;\Rightarrow\; x\, s \sim^R_{\mathcal{X}} x' s\,.$$

Thus, a generalized equilocale involves a locale X and an entire class of PERs, one for any locale S. Admittedly, these objects are quite heavy entities, but the homsets of ELoc^* will turn out to be sets in the proper sense. The compatibility condition ensures that equivalence is preserved by composition: if $x, x' : S \to X$ are equivalent at stage S, then the compositions $x\, s$, $x' s : R \to S \to X$ are equivalent at stage R for all locale maps $s : R \to S$.

4.2 Definition Given two generalized equilocales $\mathcal{X} = (X, \sim_{\mathcal{X}})$ and $\mathcal{Y} = (Y, \sim_{\mathcal{Y}})$, we define a relation '$\approx$' on the set $\mathsf{Loc}\,(X, Y)$ of locale maps from X to Y as follows:

$$f \approx f' \iff (\forall\, S \in \mathsf{Loc} : \; x \sim^S_{\mathcal{X}} x' \;\Rightarrow\; fx \sim^S_{\mathcal{Y}} f'x')\,.$$

Although the type of the maps f and f' is $f, f' : X \to Y$, we shall write $f \approx f' : \mathcal{X} \to \mathcal{Y}$ since the definition of '$\approx$' depends on the PERs of $\mathcal{X}$ and $\mathcal{Y}$.

4.3 The relation '$\approx$' of Def. 4.2 is a PER.

Proof: For symmetry, assume $f \approx f'$ and $x \sim^S_{\mathcal{X}} x'$. Then $x' \sim^S_{\mathcal{X}} x$ by symmetry of $\sim^S_{\mathcal{X}}$, whence $fx' \sim^S_{\mathcal{Y}} f'x$ because of $f \approx f'$. Symmetry of $\sim^S_{\mathcal{Y}}$ then yields the relation $f'x \sim^S_{\mathcal{Y}} fx'$ required for $f' \approx f$.

For transitivity, assume $f_1 \approx f_2$ and $f_2 \approx f_3$, and $x \sim^S_{\mathcal{X}} x'$. Then $x \sim^S_{\mathcal{X}} x$ by 2.1, whence $f_1\, x \sim^S_{\mathcal{Y}} f_2\, x \sim^S_{\mathcal{Y}} f_3\, x'$ as required for $f_1 \approx f_3$. □

ELoc^* maps are then defined as $\approx$-classes of locale maps:

4.4 Definition The set $\mathsf{ELoc}^*\,(\mathcal{X}, \mathcal{Y})$ of ELoc^* maps from $\mathcal{X} = (X, \sim_{\mathcal{X}})$ to $\mathcal{Y} = (Y, \sim_{\mathcal{Y}})$ is defined as $\mathsf{Loc}\,(X, Y)/\!\approx$ where '$\approx$' is the relation of Def. 4.2.

Since Loc is a locally small category by Requirement 1, ELoc^* is locally small, too.

4.2 Category

We now specify identity and composition for ELoc^* maps and verify that ELoc^* forms a category.

4.5 For any generalized equilocale $(X, \sim_{\mathcal{X}})$, the identity map $i_X : X \to X$ satisfies $i_X \approx i_X$.

Thus, $[i_X]$ is a well-defined $\approx$-class. We take this class as the identity map ι of $(X, \sim_{\mathcal{X}})$ in ELoc^*.

4.6 If $f \approx f' : \mathcal{X} \to \mathcal{Y}$ and $g \approx g' : \mathcal{Y} \to \mathcal{Z}$, then $g\, f \approx g' f' : \mathcal{X} \to \mathcal{Z}$.

By 2.4, composition $\mathsf{Loc}\,(Y, Z) \times \mathsf{Loc}\,(X, Y) \to \mathsf{Loc}\,(X, Z)$ induces a function $\circ : \mathsf{ELoc}^*\,(\mathcal{Y}, \mathcal{Z}) \times \mathsf{ELoc}^*\,(\mathcal{X}, \mathcal{Y}) \to \mathsf{ELoc}^*\,(\mathcal{X}, \mathcal{Z})$ satisfying $[g] \circ [f] = [g\, f]$, which we take as composition in ELoc^*. Associativity of composition and neutrality of the identities follow directly from this characteristic equation.

Note that isomorphic objects $(X, \sim_{\mathcal{X}})$ and $(Y, \sim_{\mathcal{Y}})$ need not have isomorphic target locales X and Y. In fact, these two locales may be wildly different. The reason is that $[g] \circ [f] = [i_X]$ means $g\, f \approx i_X$, not $g\, f = i_X$.

4.3 Initial Construction

Before we construct products in ELoc^*, we first introduce an initial construction analogous to the construction of the initial topology, but the role of the sets is taken over by locales, and the role of the topology by the PERs. The initial construction is used in Section 4.4 to construct products; in Section 6.3, it will be used for a different purpose.

4.7 Let $(\mathcal{Z}_i)_{i \in I}$ be a family of generalized equilocales, Y a locale, and $(g_i : Y \to Z_i)_{i \in I}$ be a family of locale maps from the locale Y to the target locales $Z_i = |\mathcal{Z}_i|$. Define $y \sim^S_{\mathcal{Y}} y'$ iff $g_i\, y \sim^S_{\mathcal{Z}_i} g_i\, y'$ for all i. Then $\mathcal{Y} = (Y, \sim_{\mathcal{Y}})$ is a generalized equilocale with the following properties:

(1) $g_i \approx g_i : \mathcal{Y} \to \mathcal{Z}_i$ holds for all i in I, leading to ELoc^* maps $\psi_i = [g_i] : \mathcal{Y} \to \mathcal{Z}_i$.

(2) For any generalized equilocale $\mathcal{X} = (X, \sim_{\mathcal{X}})$ and locale maps $f, f' : X \to Y$, $g_i\, f \approx g_i\, f' : \mathcal{X} \to \mathcal{Z}_i$ for all i implies $f \approx f' : \mathcal{X} \to \mathcal{Y}$.

(3) The ELoc^* maps $\psi_i : \mathcal{Y} \to \mathcal{Z}_i$ are jointly mono, i.e., if $\varphi, \varphi' : \mathcal{X} \to \mathcal{Y}$ are two ELoc^* maps such that $\psi_i\, \varphi = \psi_i\, \varphi'$ for all i, then $\varphi = \varphi'$ follows.

PROOF: Symmetry, transitivity, and compatibility for $\mathcal{Y}$ follow directly from the corresponding properties of the objects $\mathcal{Z}_i$. Property (1) is obvious from the definition of $\mathcal{Y}$. For (2), $x \sim^S_{\mathcal{X}} x'$ implies $g_i\, f\, x \sim^S_{\mathcal{Z}_i} g_i\, f' x'$ for all i by hypothesis, whence $f\, x \sim^S_{\mathcal{Y}} f' x'$ by definition of $\sim^S_{\mathcal{Y}}$. For (3), let $\varphi = [f]$ and $\varphi' = [f']$. Then $\psi_i\, \varphi = \psi_i\, \varphi'$ means $g_i\, f \approx g_i\, f'$, whence by (2), $f \approx f'$ follows, i.e., $\varphi = \varphi'$. □

4.4 Products

Now we construct products in ELoc^* from products in Loc as follows: Given a family $(\mathcal{Z}_i)_{i \in I}$ of generalized equilocales with $\mathcal{Z}_i = (Z_i, \sim_{\mathcal{Z}_i})$, we define $Y = \prod_{i \in I} Z_i$ to be the product formed in Loc, and apply 4.7 to the projections $p_i : Y \to Z_i$. This gives a generalized equilocale $\mathcal{Y} = (Y, \sim_{\mathcal{Y}})$ with $y \sim^S_{\mathcal{Y}} y'$ iff $p_i\, y \sim^S_{\mathcal{Z}_i} p_i\, y'$ for all i in I and the following properties:

- By 4.7 (1), $p_i \approx p_i : \mathcal{Y} \to \mathcal{Z}_i$ holds for all i in I, leading to ELoc^* maps $\pi_i = [p_i] : \mathcal{Y} \to \mathcal{Z}_i$.
- For locale maps $f_i : X \to Z_i$, there is a unique map $f = \langle f_i \rangle_{i \in I} : X \to Y$ such that $p_i\, f = f_i$ for all i in I because $Y = \prod_{i \in I} Z_i$. Given $\mathcal{X} = (X, \sim_{\mathcal{X}})$, 4.7 (2) shows that $f_i \approx f'_i : \mathcal{X} \to \mathcal{Z}_i$ for all i in I implies $\langle f_i \rangle_{i \in I} \approx \langle f'_i \rangle_{i \in I} : \mathcal{X} \to \mathcal{Y}$.
- Now consider a family of ELoc^* maps $(\varphi_i)_{i \in I}$ with $\varphi_i : \mathcal{X} \to \mathcal{Z}_i$. If $\varphi_i = [f_i]$, then $f_i \approx f_i : \mathcal{X} \to \mathcal{Z}_i$, whence $\langle f_i \rangle_{i \in I} \approx \langle f_i \rangle_{i \in I} : \mathcal{X} \to \mathcal{Y}$ follows. Thus $\langle \varphi_i \rangle_{i \in I} := [\langle f_i \rangle_{i \in I}] : \mathcal{X} \to \mathcal{Y}$ is a well-defined ELoc^* map. By construction, $\pi_i\, \langle \varphi_i \rangle_{i \in I} = [p_i\, \langle f_i \rangle_{i \in I}] = [f_i] = \varphi_i$ holds.

- It remains to show that $\langle\varphi_i\rangle_{i\in I}$ is the only function with this property. But if $\pi_i\,\varphi = \pi_i\,\varphi'$ for all i holds for two ELoc^* maps $\varphi, \varphi' : \mathcal{X} \to \mathcal{Y}$, then $\varphi = \varphi'$ follows from 4.7 (3).

Altogether we have shown that for any family $(\varphi_i : \mathcal{X} \to \mathcal{Z}_i)_{i\in I}$, there is a unique $\varphi : \mathcal{X} \to \mathcal{Y}$ such that $\pi_i\,\varphi = \varphi_i$ for all i. This proves that $\mathcal{Y}$ is the product of $(\mathcal{Z}_i)_{i\in I}$ in ELoc^*. For later reference, we note with some renaming:

4.8 Products in ELoc^* are given by $\prod_{i\in I}(X_i, \sim_{\mathcal{X}_i}) = (X, \sim_{\mathcal{X}})$ where $X = \prod_{i\in I} X_i$ is the product in Loc and $x \sim^S_{\mathcal{X}} x'$ iff $p_i\,x \sim^S_{\mathcal{X}_i} p_i\,x'$ for all i in I.

4.9 If $(X, \sim_{\mathcal{X}}) = \prod_{i\in I}(X_i, \sim_{\mathcal{X}_i})$, then

(1) For $x_i, x_i' : S \to X_i$: $x_i \sim^S_{\mathcal{X}_i} x_i'$ implies $\langle x_i\rangle_{i\in I} \sim^S_{\mathcal{X}} \langle x_i'\rangle_{i\in I}$.

(2) For $x_i, x_i' : S_i \to X_i$: $x_i \sim^{S_i}_{\mathcal{X}_i} x_i'$ implies $\prod_{i\in I} x_i \sim^S_{\mathcal{X}} \prod_{i\in I} x_i'$ where $S = \prod_{i\in I} S_i$.

Proof: (1) holds by definition of $\sim^S_{\mathcal{X}}$ since $p_i\,\langle x_i\rangle_{i\in I} = x_i$. The hypothesis of (2) implies $x_i\,p_i \sim^S_{\mathcal{X}_i} x_i'\,p_i$ by compatibility of $(X_i, \sim_{\mathcal{X}_i})$. Part (1) then implies $\langle x_i\,p_i\rangle_{i\in I} \sim^S_{\mathcal{X}} \langle x_i'\,p_i\rangle_{i\in I}$, which is the right hand side of (2) since $\langle x_i\,p_i\rangle_{i\in I} = \prod_{i\in I} x_i$. □

4.5 Final Construction

We now introduce the dual of the initial construction—not in the same generality, but only for a single locale map $f : X \to Y$, which is furthermore restricted to be *mono* (see Section 3.3). The reason for these restrictions is twofold: the general case provides additional complications, and the restricted case is the only one needed later (Section 6.1).

4.10 Let $\mathcal{X}$ be a generalized equilocale, Y a locale, and $f : X \to Y$ a *mono* locale map from the target locale X of $\mathcal{X}$ to Y. Define $y \sim^S_{\mathcal{Y}} y'$ iff $y = f\,x$ and $y' = f\,x'$ for some $x, x' : S \to X$ satisfying $x \sim^S_{\mathcal{X}} x'$. Then $\mathcal{Y} = (Y, \sim_{\mathcal{Y}})$ is a generalized equilocale with the following properties:

(1) $f \approx f : \mathcal{X} \to \mathcal{Y}$ holds leading to an ELoc^* map $\varphi = [f] : \mathcal{X} \to \mathcal{Y}$.

(2) For any generalized equilocale $\mathcal{Z} = (Z, \sim_{\mathcal{Z}})$ and locale maps $g, g' : Y \to Z$, $g\,f \approx g'\,f : \mathcal{X} \to \mathcal{Z}$ implies $g \approx g' : \mathcal{Y} \to \mathcal{Z}$.

(3) The ELoc^* map $\varphi : \mathcal{X} \to \mathcal{Y}$ is epi, i.e., if $\psi, \psi' : \mathcal{Y} \to \mathcal{Z}$ are two ELoc^* maps such that $\psi\,\varphi = \psi'\,\varphi$, then $\psi = \psi'$ follows.

(4) $\mathcal{X}$ is the initial generalized equilocale w.r.t. $f : X \to \mathcal{Y}$, and thus $\varphi : \mathcal{X} \to \mathcal{Y}$ is also mono.

Proof: Symmetry and compatibility for $\mathcal{Y}$ follow directly from the corresponding properties of $\mathcal{X}$. For transitivity, let $y_1 \sim^S_{\mathcal{Y}} y_2$ and $y_2 \sim^S_{\mathcal{Y}} y_3$. This means there are $x_1, x_2, x_2', x_3' : S \to X$ such that $y_1 = f\,x_1$, $y_2 = f\,x_2 = f\,x_2'$, $y_3 = f\,x_3'$, $x_1 \sim^S_{\mathcal{X}} x_2$ and $x_2' \sim^S_{\mathcal{X}} x_3'$. Since f is mono, $x_2 = x_2'$ holds, and so transitivity in $\mathcal{X}$ can be applied and yields $x_1 \sim^S_{\mathcal{X}} x_3'$, whence $y_1 \sim^S_{\mathcal{Y}} y_3$.

Property (1) is obvious from the definition of $\mathcal{Y}$. For (2), $y \sim^S_{\mathcal{Y}} y'$ implies $y = f\,x$ and $y' = f\,x'$ for some $x \sim^S_{\mathcal{X}} x'$, whence $g\,y = g\,f\,x \sim^S_{\mathcal{Z}} g'\,f\,x' = g'\,y'$. For (3), let $\psi = [g]$ and $\psi' = [g']$. Then $\psi\,\varphi = \psi'\,\varphi$ means $g\,f \approx g'\,f$, whence by (2), $g \approx g'$ follows, i.e., $\psi = \psi'$.

For (4), $x \sim^S_{\mathcal{X}} x' \iff f\,x \sim^S_{\mathcal{Y}} f\,x'$ must be shown. Direction '⇒' follows from the definition of $\sim^S_{\mathcal{Y}}$. For '⇐', assume $f\,x \sim^S_{\mathcal{Y}} f\,x'$. This means $f\,x = f\,u$ and $f\,x' = f\,u'$ for some $u \sim^S_{\mathcal{X}} u'$. Since f is mono, $x = u \sim^S_{\mathcal{X}} u' = x'$ follows, which completes the proof of the claimed equivalence. The initiality of $\mathcal{X}$ w.r.t. f implies that $\varphi = [f]$ is mono by 4.7 (3). □

5 In-Equilocales

Now we introduce the category IELoc as a full subcategory of ELoc^* and show that it is Cartesian closed. The properties used are closedness under products (3.6) and Cartesian closedness (Requirement 8) of the category Loc_I of injective locales (see also Section 3.6). Cartesian closedness of Loc_I means for any two injective locales B and C there is an injective locale C^B (called *exponential*) equipped with an *evaluation map* $v : C^B \times B \to C$ such that for any injective locale A and any $f : A \times B \to C$, there is a unique *transpose* $f^\sharp : A \to C^B$ satisfying $v\,(f^\sharp \times i_B) = f : A \times B \to C^B \times B \to C$.

5.1 Definition An *in-equilocale* is a generalized equilocale $(A, \sim_\mathcal{A})$ whose target locale A is injective. The full subcategory of ELoc^* whose objects are in-equilocales is called IELoc.

The name in-equilocale was chosen to point to the fact that the target locale is *in*jective. We did not use the name injective equilocale since these equilocales are not injective in the categorical sense.

Considering the product construction in ELoc^*, it is obvious that IELoc is closed under the products of ELoc^* since Loc_I is closed under the products of Loc. To prove that IELoc is Cartesian closed, it remains to show that exponentials exist in it. We now define the exponential candidates and then prove that they really are exponentials.

5.2 Definition Given two in-equilocales $\mathcal{B} = (B, \sim_\mathcal{B})$ and $\mathcal{C} = (C, \sim_\mathcal{C})$, we define $\mathcal{C}^\mathcal{B} = (C^B, \sim_{\mathcal{C}^\mathcal{B}})$ where C^B is the exponential of C and B in Loc_I and '$\sim_{\mathcal{C}^\mathcal{B}}$' is defined by

$$d \sim^S_{\mathcal{C}^\mathcal{B}} d' \iff (\forall R \in \mathsf{Loc} :\ \ b \sim^R_\mathcal{B} b' \ \Rightarrow\ v\,(d \times b) \sim^{S\times R}_\mathcal{C} v\,(d' \times b')\,)\,.$$

A note on types: Since $d : S \to C^B$ and $b : R \to B$, we get $v\,(d \times b) : S \times R \to C^B \times B \to C$ as required.

5.3 $\mathcal{C}^\mathcal{B}$ as defined above is a well-defined in-equilocale.

Proof: The proofs of symmetry and transitivity of the relations $\sim^S_{\mathcal{C}^\mathcal{B}}$ are similar to those for '$\approx$' (4.3). For compatibility, assume $d \sim^S_{\mathcal{C}^\mathcal{B}} d'$ and consider $s : S' \to S$ and $b \sim^R_\mathcal{B} b'$. The required relation $v\,(d\,s \times b) \sim^{S'\times R}_\mathcal{C} v\,(d's \times b')$ then follows from $v\,(d \times b) \sim^{S\times R}_\mathcal{C} v\,(d' \times b')$ by compatibility of $\mathcal{C}$ since $v\,(d\,s \times b) = v\,(d \times b)\,(s \times i_R)$. □

There is an equivalent characterization of $\sim^S_{\mathcal{C}^\mathcal{B}}$:

5.4 $d \sim^S_{\mathcal{C}^\mathcal{B}} d' \iff (\forall s : R \to S :\ \ b \sim^R_\mathcal{B} b' \ \Rightarrow\ v\,\langle d\,s,\, b\rangle \sim^R_\mathcal{C} v\,\langle d's,\, b'\rangle\,)\,.$
Typing: Since $d\,s : R \to S \to C^B$ and $b : R \to B$, we get $v\,\langle d\,s,\, b\rangle : R \to C^B \times B \to C$ as required.

Proof: '$\Rightarrow$': Assume $d \sim^S_{\mathcal{C}^\mathcal{B}} d'$, and consider $s : R \to S$ and $b \sim^R_\mathcal{B} b'$. By Def. 5.2, $v\,(d{\times}b) \sim^{S\times R}_\mathcal{C} v\,(d'{\times}b')$ holds. Composition with $\langle s, i_R\rangle : R \to S{\times}R$ yields $v\,(d{\times}b)\,\langle s, i_R\rangle \sim^R_\mathcal{C} v\,(d' \times b')\,\langle s, i_R\rangle$ by compatibility of $\mathcal{C}$. Since $(d \times b)\,\langle s,\, i_R\rangle = \langle d\,s,\, b\rangle$, the required relation $v\,\langle d\,s,\, b\rangle \sim^R_\mathcal{C} v\,\langle d's,\, b'\rangle$ follows.

'$\Leftarrow$': To show $d \sim^S_{\mathcal{C}^\mathcal{B}} d'$, let $b \sim^R_\mathcal{B} b'$. Using compatibility of $\mathcal{B}$, this relation can be composed with the projection $p_2 : S \times R \to R$ to obtain $b\,p_2 \sim^{S\times R}_\mathcal{B} b'p_2$. Applying the hypothesis to this relation and to $s = p_1 : S \times R \to S$ yields $v\,\langle d\,p_1,\, b\,p_2\rangle \sim^{S\times R}_\mathcal{C} v\,\langle d'p_1,\, b'p_2\rangle$. With $\langle d\,p_1,\, b\,p_2\rangle = (d \times b)$, the relation $v\,(d \times b) \sim^{S\times R}_\mathcal{C} v\,(d' \times b')$ required for $d \sim^S_{\mathcal{C}^\mathcal{B}} d'$ follows. □

We now show that the exponential candidates really satisfy the properties required for exponentials.

5.5 The evaluation function of $\mathsf{Loc_I}$ satisfies $v \approx v : \mathcal{C}^{\mathcal{B}} \times \mathcal{B} \to \mathcal{C}$.

PROOF: Assume $a \sim^S_{\mathcal{C}^{\mathcal{B}}\times\mathcal{B}} a'$. By the construction of products in Section 4.4, $d \sim^S_{\mathcal{C}^{\mathcal{B}}} d'$ and $b \sim^S_{\mathcal{B}} b'$ hold for $d = p_1\, a$ and $b = p_2\, a$. The characterization of $d \sim^S_{\mathcal{C}^{\mathcal{B}}} d'$ given by 5.4 applied to $s = i_S$ yields $v\,\langle d, b\rangle \sim^S_{\mathcal{C}} v\,\langle d', b'\rangle$. Because $\langle d, b\rangle = a$, the required relation $v\,a \sim^S_{\mathcal{C}} v\,a'$ follows. □

Hence $\varepsilon = [v]$ is a well-defined IELoc map $\varepsilon : \mathcal{C}^{\mathcal{B}} \times \mathcal{B} \to \mathcal{C}$.

5.6 If $f \approx f' : \mathcal{A} \times \mathcal{B} \to \mathcal{C}$, then $f^\sharp \approx f'^\sharp : \mathcal{A} \to \mathcal{C}^{\mathcal{B}}$.

PROOF: To prove $f^\sharp \approx f'^\sharp$, consider $a \sim^S_{\mathcal{A}} a'$. Then we have to show $f^\sharp\, a \sim^S_{\mathcal{C}^{\mathcal{B}}} f'^\sharp\, a'$. This is done using 5.2, i.e., for $b \sim^R_{\mathcal{B}} b'$, we have to show $v\,(f^\sharp\, a \times b) \sim^{S\times R}_{\mathcal{C}} v\,(f'^\sharp\, a' \times b')$. Now $v\,(f^\sharp\, a \times b) = v\,(f^\sharp \times i_B)\,(a \times b) = f\,(a \times b)$ by the characteristic property of $f^\sharp$, and likewise for f'. Thus the relation to be shown is $f\,(a \times b) \sim^{S\times R}_{\mathcal{C}} f'(a' \times b')$. But this relation follows from $f \approx f'$ since $a \sim^S_{\mathcal{A}} a'$ and $b \sim^R_{\mathcal{B}} b'$ imply $(a \times b) \sim^{S\times R}_{\mathcal{A}\times\mathcal{B}} (a' \times b')$ by 4.9 (2). □

5.7 For each IELoc map $\varphi : \mathcal{A} \times \mathcal{B} \to \mathcal{C}$, there is a unique IELoc map $\varphi^\sharp : \mathcal{A} \to \mathcal{C}^{\mathcal{B}}$ satisfying $\varepsilon\,(\varphi^\sharp \times \iota_{\mathcal{B}}) = \varphi$.

PROOF: Because of 5.6, the function $(-)^\sharp$ on locale maps induces a function $(-)^\sharp$ on IELoc maps by 2.4. Because all of $(-)^\sharp$, composition, and tupling on IELoc maps are induced from the corresponding operations on locale maps, the equation $\varepsilon\,(\varphi^\sharp \times \iota_{\mathcal{B}}) = \varphi$ holds. We only need to show that $\varphi^\sharp$ is uniquely determined by this equation. So assume $\varepsilon\,(\psi_1 \times \iota_{\mathcal{B}}) = \varepsilon\,(\psi_2 \times \iota_{\mathcal{B}})$ holds for some IELoc maps $\psi_1, \psi_2 : \mathcal{A} \to \mathcal{C}^{\mathcal{B}}$ with $\psi_i = [g_i]$ for $i = 1, 2$. Then $f_1 \approx f_2$ holds where $f_i = v\,(g_i \times i_B)$, whence ${f_1}^\sharp \approx {f_2}^\sharp$ by 5.6. But because of $f_i = v\,(g_i \times i_B)$ and Cartesian closedness of $\mathsf{Loc_I}$, ${f_i}^\sharp = g_i$ follows. Thus we obtain $g_1 \approx g_2$, i.e., $\psi_1 = \psi_2$. □

This concludes the proof that IELoc is Cartesian closed.

6 Sur-Equilocales

The category PEQU of partial equivalence relations on continuous lattices is equivalent to the category EQU of proper equivalence relations on arbitrary $\mathcal{T}_0$-topological spaces. The functor from PEQU to EQU cuts down the continuous lattice to the subspace of self-related elements. A similar cut-down operation can be performed on equilocales, complicated by the fact that there is not a single PER, but a whole class of them. Consequently, the result of the cut-down operation is not a class of proper equivalence relations since it is not possible in general to turn all PERs into equivalence relations at once. Of course, cutting down yields a sublocale (Section 3.4) instead of a subspace.

Section 6.1 introduces the *extension functor* transforming generalized equilocales into in-equilocales. Section 6.3 presents the *core functor* for cutting down generalized equilocales to *sur-equilocales* forming the category SELoc. Then Section 6.4 shows that these two functors form an equivalence of categories when restricted to SELoc and IELoc. Finally, we study the product construction of SELoc in Section 6.5.

6.1 Extension to In-Equilocales

In the following, we need Requirement 7: every locale X can be embedded into an injective locale AX by $a_X : X \hookrightarrow AX$, and there is a canonical way to construct AX and a_X from X (to avoid problems with choice).

Injectivity of the target locale implies a similar property for generalized equilocales:

6.1 Let $\mathcal{X} = (X, \sim_{\mathcal{X}})$, $\mathcal{X}' = (X', \sim_{\mathcal{X}'})$, and $\mathcal{A} = (A, \sim_{\mathcal{A}})$ be generalized equilocales such that A is an injective locale and $\mathcal{X}'$ is final for some embedding $e : X \hookrightarrow X'$. Then for every ELoc^* map $\varphi : \mathcal{X} \to \mathcal{A}$, there is a unique ELoc^* map $\varphi' : \mathcal{X}' \to \mathcal{A}$ such that $\varphi'\,\eta = \varphi$, where $\eta = [e] : \mathcal{X} \to \mathcal{X}'$.

Proof: By 4.10, η is well-defined and epi (part 3), which proves uniqueness. For existence, consider $\varphi = [f] : \mathcal{X} \to \mathcal{A}$. By injectivity of A, the locale map $f : X \to A$ can be extended to a map $f' : X' \to A$ satisfying $f'e = f$. Since $f \approx f$ holds, $f'e \approx f'e$ follows. By 4.10 (2), this relation implies $f' \approx f'$ so that $\varphi' = [f'] : \mathcal{X}' \to \mathcal{A}$ is a well-defined ELoc^* map satisfying $\varphi'\,\eta = \varphi$. □

As a corollary, we get the following isomorphism result:

6.2 Let X be a sublocale of two injective locales A_1 and A_2 via the embeddings $e_1 : X \hookrightarrow A_1$ and $e_2 : X \hookrightarrow A_2$. If $\mathcal{A}_1 = (A_1, \sim_{\mathcal{A}_1})$ and $\mathcal{A}_2 = (A_2, \sim_{\mathcal{A}_2})$ are final for the embeddings e_1 and e_2, respectively, then $\mathcal{A}_1$ are $\mathcal{A}_2$ are isomorphic ELoc^* objects.

Proof: For $i = 1, 2$, $\eta_i = [e_i] : \mathcal{X} \to \mathcal{A}_i$ is well-defined and epi by 4.10 (2). By 6.1, there are ELoc^* maps $\varphi_1 : \mathcal{A}_1 \to \mathcal{A}_2$ and $\varphi_2 : \mathcal{A}_2 \to \mathcal{A}_1$ such that $\varphi_1\,\eta_1 = \eta_2$ and $\varphi_2\,\eta_2 = \eta_1$. Then $\eta_1 = \varphi_2\,\varphi_1\,\eta_1$ and $\eta_2 = \varphi_1\,\varphi_2\,\eta_2$, whence $\varphi_2\,\varphi_1 = \iota_{\mathcal{A}_1}$ and $\varphi_1\,\varphi_2 = \iota_{\mathcal{A}_2}$ since η_1 and η_2 are epi. □

The above isomorphism result means that all ways to extend a generalized equilocale to an in-equilocale by a final construction are isomorphic. We choose a canonical way in the following definition:

6.3 Definition For any generalized equilocale $\mathcal{X} = (X, \sim_{\mathcal{X}})$, define its extension $\mathcal{AX} = (AX, \sim_{\mathcal{AX}})$ to an in-equilocale by giving $\mathcal{AX}$ the final PER structure w.r.t. the embedding a_X, i.e., $y \sim^S_{AX} y'$ iff $y = a_X\,x$ and $y' = a_X\,x'$ for some $x \sim^S_{\mathcal{X}} x'$.

Prop. 4.10 about final generalized equilocales immediately yields the following:

6.4 $\mathcal{AX}$ as in 6.3 is a well-defined in-equilocale, and $a_X \approx a_X : \mathcal{X} \to \mathcal{AX}$ holds leading to an ELoc^* map $\alpha_{\mathcal{X}} = [a_X] : \mathcal{X} \to \mathcal{AX}$, which is both epi and mono.

To get an extension *functor*, $\mathcal{A}$ must also be defined for functions.

6.5 Let $\mathcal{X}$ and $\mathcal{Y}$ be two generalized equilocales. For every ELoc^* map $\varphi : \mathcal{X} \to \mathcal{Y}$, there is a unique ELoc^* map $\mathcal{A}\varphi : \mathcal{AX} \to \mathcal{AY}$ such that $\mathcal{A}\varphi\,\alpha_{\mathcal{X}} = \alpha_{\mathcal{Y}}\,\varphi$. Moreover, the assignment $\varphi \mapsto \mathcal{A}\varphi$ is functorial.

Proof: The target locale AY of $\mathcal{AY}$ is injective, and $\mathcal{AX}$ is final w.r.t. the embedding $a_X : X \hookrightarrow AX$. Thus 6.1 can be applied to $\alpha_{\mathcal{Y}}\,\varphi : \mathcal{X} \to \mathcal{AY}$, giving the claimed function $\mathcal{A}\varphi$. Functoriality of $\mathcal{A}$ follows from the equation $\mathcal{A}\varphi\alpha_{\mathcal{X}} = \alpha_{\mathcal{Y}}\,\varphi$ that characterizes $\mathcal{A}\varphi$ uniquely.□

6.2 Joint Image

The core functor for cutting down generalized equilocales to sur-equilocales relies on the construction of the *joint image*, a general construction for locales not specific to equilocales. As usual, we do not mean concrete locales here; the construction can be performed in any category satisfying the requirements listed in Section 3.

Let Y be a locale. For any class of locale maps y_i with target Y, there is a smallest sublocale of Y containing all the y_i (see Section 3.3 for containment and Section 3.4 for sublocales). This statement is made precise and used later in the following form:

6.6 For any class of maps $y_i : S_i \to Y$ with a fixed target Y, there is a sublocale (= regular subobject) $e : X \to Y$ of Y such that

(1) $y_i \leq e$ for all i;

(2) for all $f, f' : Y \to Z$, $fy_i = f'y_i$ for every i in I implies that $fe = f'e$.

(3) for all $f, f' : Y \to Z$, $fe = f'e$ implies that $fy_i = f'y_i$ for every i in I.

The sublocale e is uniquely determined by (1) and (2), or by (2) and (3). It is called the *joint image* of the class y_i.

Proof: Let E be the class of all sublocales of Y containing all y_i; this class is a set by Requirement 4. Define $e = \bigwedge E$ as in 3.4. Then e contains all y_i by the meet property, and so (1) holds. For (2), let $f, f' : Y \to Z$ such that $fy_i = f'y_i$ for all i in I. This means that all y_i are contained in an equalizer $\tilde{e}$ of f and f', i.e., $\tilde{e}$ is an element of the set E. Since $e = \bigwedge E$, $e \leq \tilde{e}$ follows, whence $fe = f'e$. By (1), $y_i = e\, x_i$ for some x_i, which implies (3).

For uniqueness, assume the embeddings e_1 and e_2 satisfy (2) and in addition (1) or (3). Being an embedding (= regular mono), e_1 must be an equalizer of a pair of maps, say f_1 and f_1'. Then $f_1\, e_1 = f_1'\, e_1$, whence by (1) or (3) $f_1\, y_i = f_1'\, y_i$ for all i. By (2), $f_1\, e_2 = f_1'\, e_2$ follows. Since e_1 is the equalizer of f_1 and f_1', this implies $e_2 \leq e_1$. The opposite containment $e_1 \leq e_2$ also holds because the situation is symmetric. Hence e_1 and e_2 are isomorphic embeddings defining the same sublocale (= equivalence class of isomorphic embeddings). □

A class of maps $x_i : S_i \to X$ with fixed target X is *jointly epi* if for all $g, g' : X \to Z$, $g\, x_i = g' x_i$ for all i implies $g = g'$.

6.7 If $e : X \to Y$ is the joint image of the class of maps $y_i : S_i \to Y$, then the class of maps $x_i = e^{<}\, y_i : S_i \to X$ is jointly epi.

Proof: By 6.6 (1), $y_i \leq e$, and so $x_i = e^{<}\, y_i$ is well-defined as the map with $y_i = e\, x_i$. To show the joint epi property, let $g, g' : X \to Z$ such that $g\, x_i = g' x_i$ for all i. By Requirement 7, there is an embedding $a : Z \to A$ of Z into an injective locale A. Injectivity of A applied to the embedding $e : X \to Y$ and the maps $a\, g,\, a\, g' : X \to A$ yields maps $f, f' : Y \to A$ such that $fe = a\, g$ and $f'e = a\, g'$. Thus, $g\, x_i = g' x_i$ implies $f\, y_i = f\, e\, x_i = a\, g\, x_i = a\, g' x_i = f' y_i$ for all i. By property (2) of 6.6, $f\, e = f'e$ follows, i.e., $a\, g = a\, g'$, whence $g = g'$ since a is an embedding. □

6.8 The joint image of a class of maps $y_i : S_i \to Y$ is (the identity on) Y if and only if the class y_i is jointly epi.

Proof: If the joint image is the identity, then $y_i = i_Y^{<}\, y_i$ is jointly epi by 6.7. Conversely, if the class y_i is jointly epi, then $e = i_Y$ satisfies properties (1) and (2) of 6.6 and thus describes the joint image. □

6.3 Cutting Down In-Equilocales: The Core Functor

The core functor $(-)_0$ will be defined for generalized equilocales and later be restricted to in-equilocales.

6.9 Definition For a generalized equilocale $\mathcal{X} = (X, \sim_{\mathcal{X}})$, its *core* $\mathcal{X}_0$ is $(X_0, \sim_{\mathcal{X}_0})$ where X_0 (or more precisely $e_{X_0} : X_0 \to X$) is the canonical form of the joint image of all self-related $x : S \to X$ (self-related means $x \sim^S_{\mathcal{X}} x$), and the PERs $\sim^S_{\mathcal{X}_0}$ are defined by $x_0 \sim^S_{\mathcal{X}_0} x_0'$ iff $e_{X_0}\, x_0 \sim^S_{\mathcal{X}} e_{X_0}\, x_0'$, i.e., $\mathcal{X}_0$ is initial w.r.t. e_{X_0}.

Proposition 4.7 about initial generalized equilocales immediately yields the following:

6.10 $\mathcal{X}_0$ as in 6.9 is a well-defined generalized equilocale with the following properties:

(1) $e_{X_0} \approx e_{X_0} : \mathcal{X}_0 \to \mathcal{X}$ holds leading to a *mono* ELoc^* map $\eta_{\mathcal{X}} = [e_{X_0}] : \mathcal{X}_0 \to \mathcal{X}$.

(2) For any generalized equilocale $\mathcal{Y} = (Y, \sim_{\mathcal{Y}})$ and locale maps $f, f' : Y \to X_0$, $e_{X_0}\, f \approx e_{X_0}\, f' : \mathcal{Y} \to \mathcal{X}$ implies $f \approx f' : \mathcal{Y} \to \mathcal{X}_0$.

We write $\eta_{\mathcal{X}}$ instead of $\eta_{\mathcal{X}_0}$ since this family of maps will turn out to be natural in $\mathcal{X}$ (see 6.17).

6.11 $\mathcal{X}$ is final w.r.t. e_{X_0}, and so $\eta_{\mathcal{X}} = [e_{X_0}] : \mathcal{X}_0 \to \mathcal{X}$ is mono and epi.

Proof: We need to show that $x \sim^S_{\mathcal{X}} x'$ iff $x = e_{X_0} x_0$ and $x' = e_{X_0} x_0'$ for some $x_0 \sim^S_{\mathcal{X}_0} x_0'$. The direction from right to left follows directly from the definition of $\sim^S_{\mathcal{X}_0}$. For the opposite direction, assume $x \sim^S_{\mathcal{X}} x'$. By 2.1, this implies that x and x' are self-related and so must be contained in X_0, i.e., $x = e_{X_0} x_0$ and $x' = e_{X_0} x_0'$ for some $x_0, x_0' : S \to X_0$. By definition of $\sim^S_{\mathcal{X}_0}$, $x_0 \sim^S_{\mathcal{X}_0} x_0'$ holds. The epi statement comes from 4.10 (3). □

6.12 Definition A *sur-equilocale* is a generalized equilocale $\mathcal{X}$ satisfying $\mathcal{X}_0 = \mathcal{X}$. The full subcategory of ELoc^* consisting of sur-equilocales is called SELoc.

The name sur-equilocale was chosen since the following characterisation, which is an immediate consequence of Property 6.8, expresses a kind of surjectivity condition.

6.13 $\mathcal{X} = (X, \sim_{\mathcal{X}})$ is a sur-equilocale if and only if the class of self-related $x : S \to X$ is jointly epi, i.e., $f\, x = f'\, x$ for all self-related x implies $f = f'$.

6.14 Cores $\mathcal{X}_0$ are always sur-equilocales, i.e., $\mathcal{X}_{00} = \mathcal{X}_0$.

Proof: By Definition 6.9, $x_0 \sim^S_{\mathcal{X}_0} x_0$ iff $e_{X_0}\, x_0 \sim^S_{\mathcal{X}} e_{X_0}\, x_0$ holds for $x_0 : S \to X_0$. By 6.11, $x \sim^S_{\mathcal{X}} x$ iff $x = e_{X_0} x_0$ for some $x_0 \sim^S_{\mathcal{X}_0} x_0$. Together, this means that the self-related maps $x_0 : S \to X_0$ are exactly the maps $e^{<}_{X_0}\, x$ for self-related $x : S \to X$. By 6.7, this class of maps is jointly epi. □

The following lemma will be used to show that $(-)_0$ is a functor, and later in the proof of equivalence of IELoc and SELoc.

6.15 Let $\mathcal{X} = (X, \sim_{\mathcal{X}})$, $\mathcal{Y} = (Y, \sim_{\mathcal{Y}})$, and $\mathcal{Y}' = (Y', \sim_{\mathcal{Y}'})$ be generalized equilocales such that $\mathcal{X}$ is a sur-equilocale and $\mathcal{Y}$ is final for some embedding $e : Y' \hookrightarrow Y$. Then for every ELoc^* map $\varphi : \mathcal{X} \to \mathcal{Y}$, there is a unique ELoc^* map $\varphi' : \mathcal{X} \to \mathcal{Y}'$ such that $\varphi = \eta\, \varphi'$, where $\eta = [e] : \mathcal{Y}' \to \mathcal{Y}$.

Proof: By 4.10, η is well-defined and mono (part 4), which proves uniqueness. For existence, consider $\varphi = [f] : \mathcal{X} \to \mathcal{Y}$, and let $e_f : X' \hookrightarrow X$ be the inverse image of e under f as in 3.3. If $x : S \to X$ is self-related, then so is $f\, x : S \to Y$. Since $\mathcal{Y}$ is final for e, $f\, x$ is contained in e (by the definition of the PERs in the final equilocale). Hence x is contained in e_f. Summarizing, e_f contains all self-related $x : X \to S$, and thus e_{X_0}, which is i_X since $\mathcal{X}$ is a sur-equilocale. Hence, e contains $f\, i_X = f$. This means $f = e\, f'$ for some (unique) locale map $f' : X \to Y'$. By 4.10 (4), $\mathcal{Y}'$ is initial for e. Using property 4.7 (2) of initial equilocales, $e\, f' = f \approx f = e\, f'$ implies $f' \approx f'$, which gives $\varphi' = [f']$ with $\varphi = \eta\, \varphi'$. □

As a corollary, we get the following isomorphism result:

6.16 Let $e_1 : X_1 \hookrightarrow Y$ and $e_2 : X_2 \hookrightarrow Y$ be two sublocales of Y. If $\mathcal{X}_1 = (X_1, \sim_{\mathcal{X}_1})$ and $\mathcal{X}_2 = (X_2, \sim_{\mathcal{X}_2})$ are sur-equilocales, and $\mathcal{Y} = (Y, \sim_{\mathcal{Y}})$ is final for both embeddings e_1 and e_2, then $\mathcal{X}_1$ are $\mathcal{X}_2$ are isomorphic ELoc^* objects.

Proof: For $i = 1, 2$, the maps $\eta_i = [e_i] : \mathcal{X}_i \to \mathcal{Y}$ are well-defined and mono by 4.10 (4). By 6.15, there are ELoc^* maps $\varphi_1 : \mathcal{X}_2 \to \mathcal{X}_1$ and $\varphi_2 : \mathcal{X}_1 \to \mathcal{X}_2$ such that $\eta_1 = \eta_2\,\varphi_2$ and $\eta_2 = \eta_1\varphi_1$. Then $\eta_1 = \eta_1\varphi_1\varphi_2$ and $\eta_2 = \eta_2\,\varphi_2\,\varphi_1$, whence $\varphi_1\varphi_2 = \iota_{\mathcal{X}_1}$ and $\varphi_2\,\varphi_1 = \iota_{\mathcal{X}_2}$ since η_1 and η_2 are mono. □

To get a core *functor*, we must extend the object map $(-)_0$ to functions. Remember the notation $\eta_{\mathcal{X}} : \mathcal{X}_0 \to \mathcal{X}$ for the mono epi ELoc^* map induced by the embedding of the core.

6.17 Let $\mathcal{X}$ and $\mathcal{Y}$ be two generalized equilocales. For every ELoc^* map $\varphi : \mathcal{X} \to \mathcal{Y}$, there is a unique ELoc^* map $\varphi_0 : \mathcal{X}_0 \to \mathcal{Y}_0$ such that $\eta_{\mathcal{Y}}\,\varphi_0 = \varphi\,\eta_{\mathcal{X}}$. The assignment $\varphi \mapsto \varphi_0$ is functorial.

Proof: $\mathcal{X}_0$ is a sur-equilocale by 6.14, and $\mathcal{Y}$ is final for the embedding of $\mathcal{Y}_0$ by 6.11. Thus 6.15 can be applied to $\varphi\,\eta_{\mathcal{X}} : \mathcal{X}_0 \to \mathcal{Y}$, giving the claimed function φ_0. Functoriality of $(-)_0$ follows from the equation $\eta_{\mathcal{Y}}\,\varphi_0 = \varphi\,\eta_{\mathcal{X}}$ characterizing φ_0 uniquely. □

6.4 Equivalence between In-Equilocales and Sur-Equilocales

We now restrict the extension functor $\mathcal{A}$ of Section 6.1 to sur-equilocales, giving $\mathcal{A} : \mathsf{SELoc} \to \mathsf{IELoc}$, and the core functor $(-)_0$ of Section 6.3 to in-equilocales, giving $(-)_0 : \mathsf{IELoc} \to \mathsf{SELoc}$.

6.18 Theorem The functors $\mathcal{A} : \mathsf{SELoc} \to \mathsf{IELoc}$ and $(-)_0 : \mathsf{IELoc} \to \mathsf{SELoc}$ form an equivalence of categories.

The theorem is proved in several steps, given by the following propositions.

6.19 For any sur-equilocale $\mathcal{X}$, $(\mathcal{A}\mathcal{X})_0 \cong \mathcal{X}$ holds.

Proof: We apply 6.16. The equilocales $\mathcal{X}$ and $(\mathcal{A}\mathcal{X})_0$ are sur-equilocales; the first by hypothesis, and the second by construction. The equilocale $\mathcal{A}\mathcal{X}$ is final w.r.t. the embedding of $\mathcal{X}$ by definition, and final w.r.t. the embedding of $(\mathcal{A}\mathcal{X})_0$ by 6.11. From these facts, $(\mathcal{A}\mathcal{X})_0 \cong \mathcal{X}$ follows by 6.16. □

6.20 For any in-equilocale $\mathcal{B}$, $\mathcal{A}(\mathcal{B}_0) \cong \mathcal{B}$ holds.

Proof: We apply 6.2. The target locale of $\mathcal{B}$ is injective by hypothesis, and $\mathcal{A}(\mathcal{B}_0)$ has the same property by construction. The equilocale $\mathcal{A}(\mathcal{B}_0)$ is final w.r.t. the embedding of $\mathcal{B}_0$ by definition, and $\mathcal{B}$ is final w.r.t. the embedding of $\mathcal{B}_0$ by 6.11. From these facts, $\mathcal{A}(\mathcal{B}_0) \cong \mathcal{B}$ follows by 6.2. □

To complete the equivalence proof, one must show that the isomorphisms of 6.19 and 6.20 are natural, or that one of the two functors is full and faithful. We choose the latter and show that $\mathcal{A}$ is full and faithful. Remember that $a_{\mathcal{X}} : \mathcal{X} \to \mathcal{A}\mathcal{X}$ is epi and mono, and $\mathcal{A}\varphi\; a_{\mathcal{X}} = a_{\mathcal{Y}}\,\varphi$ holds.

6.21 $\mathcal{A}$ is faithful.

Proof: For $\varphi, \varphi' : \mathcal{X} \to \mathcal{Y}$, we must show that $\mathcal{A}\varphi = \mathcal{A}\varphi'$ implies $\varphi = \varphi'$. But $\mathcal{A}\varphi = \mathcal{A}\varphi'$ implies $a_{\mathcal{Y}}\,\varphi = \mathcal{A}\varphi\,a_{\mathcal{X}} = \mathcal{A}\varphi'\,a_{\mathcal{X}} = a_{\mathcal{Y}}\,\varphi'$, whence $\varphi = \varphi'$ follows since $a_{\mathcal{Y}}$ is mono. □

6.22 $\mathcal{A} : \mathsf{SELoc} \to \mathsf{IELoc}$ is full.

PROOF: For each $\psi : \mathcal{AX} \to \mathcal{AY}$, we must find a $\varphi : \mathcal{X} \to \mathcal{Y}$ such that $\mathcal{A}\varphi = \psi$. Here, $\mathcal{X}$ is a sur-equilocale since we here consider $\mathcal{A} : \mathsf{SELoc} \to \mathsf{IELoc}$. By definition, $\mathcal{AY}$ is final for the embedding of $|\mathcal{Y}|$ into $|\mathcal{AY}|$. Therefore, we can apply 6.15 to $\psi\, a_{\mathcal{X}} : \mathcal{X} \to \mathcal{AY}$ and thus get $\varphi : \mathcal{X} \to \mathcal{Y}$ such that $a_{\mathcal{Y}}\,\varphi = \psi\, a_{\mathcal{X}}$. On the other hand, $a_{\mathcal{Y}}\,\varphi = \mathcal{A}\varphi\, a_{\mathcal{X}}$ holds, whence $\mathcal{A}\varphi = \psi$ since $a_{\mathcal{X}}$ is epi. □

Note that some generalized equilocales belong to both IELoc and SELoc, namely those which are sur-equilocales *and* have an injective target locale. Such ambiguous objects $\mathcal{X}$ are kept fixed by the core functor ($\mathcal{X}_0 = \mathcal{X}$). On the other hand, $\mathcal{AX}$ and $\mathcal{X}$ usually differ, but 6.20 at least yields that they are isomorphic ($\mathcal{AX} \cong \mathcal{X}$). Thus, no confusion can arise; it is not necessary to distinguish between $\mathcal{X}$ in SELoc and $\mathcal{X}$ in IELoc.

6.5 Products in SELoc

Since SELoc is equivalent to the Cartesian closed category IELoc, it is itself Cartesian closed. The necessary constructions (product and exponential) are obtained by mapping their operands to IELoc by $\mathcal{A}$, performing the construction there, and mapping the result back to SELoc using the core functor $(-)_0$. For products, this yields $(\prod_{i\in I} \mathcal{AX}_i)_0$, but this construction can be simplified.

In Section 4.4, products in ELoc^* have been constructed as $\prod_{i\in I}(X_i, \sim_{\mathcal{X}_i}) = (X, \sim_{\mathcal{X}})$ where $X = \prod_{i\in I} X_i$ is the product in Loc and $x \sim^S_{\mathcal{X}} x'$ iff $p_i\, x \sim^S_{\mathcal{X}_i} p_i\, x'$ for all i in I. While the full subcategory IELoc is closed under these products, there is no reason why SELoc should behave so: it is not obvious that an ELoc^* product of sur-equilocales is again a sur-equilocale (in fact, we believe that it is not a sur-equilocale in general; this may be related to the fact that products of spatial locales are not necessarily spatial again). To avoid confusion, products, projections, tuples etc. are marked in the rest of this note by a star if they are formed in ELoc^* (or IELoc), and by a circle when formed in SELoc.

6.23 A product in SELoc is obtained as the core of the corresponding product in ELoc^*, i.e., $\prod^\circ_{i\in I} \mathcal{X}_i = (\prod^*_{i\in I} \mathcal{X}_i)_0$.

PROOF: Let $\mathcal{X} = \prod^*_{i\in I} \mathcal{X}_i$, and let $\eta : \mathcal{X}_0 \to \mathcal{X}$ be the ELoc^* map induced by the embedding of $|\mathcal{X}_0|$ into $|\mathcal{X}|$. Recall that $\mathcal{X}$ is final w.r.t. the embedding, and η is mono and epi. Using the ELoc^* projections $\pi^*_i : \mathcal{X} \to \mathcal{X}_i$, one easily obtains functions $\pi^\circ_i = \pi^*_i\, \eta : \mathcal{X}_0 \to \mathcal{X}_i$. Given a family of maps $\varphi_i : \mathcal{Y} \to \mathcal{X}_i$ starting from a sur-equilocale $\mathcal{Y}$, one first gets $\varphi^* = \langle \varphi_i \rangle^*_{i\in I} : \mathcal{Y} \to \mathcal{X}$, and then some $\varphi^\circ : \mathcal{Y} \to \mathcal{X}_0$ with $\varphi^* = \eta\, \varphi^\circ$ by 6.15. Then $\pi^\circ_i\, \varphi^\circ = \pi^*_i\, \eta\, \varphi^\circ = \pi^*_i\, \varphi^* = \varphi_i$ as required. If $\psi^\circ : \mathcal{Y} \to \mathcal{X}_0$ is another function with this property, then $\pi^*_i\, \eta\, \varphi^\circ = \pi^*_i\, \eta\, \psi^\circ$, whence $\eta\, \varphi^\circ = \eta\, \psi^\circ$ by the uniqueness property of the ELoc^* tupling operator, whence $\varphi^\circ = \psi^\circ$ since η is mono. □

7 Locales as Equilocales

So far, we have constructed two equivalent Cartesian closed categories, IELoc and SELoc. Now we embed Loc as a full subcategory into SELoc.

7.1 The Embedding of Loc into SELoc

For each locale X, define $\widehat{X} = (X, \sim_{\mathcal{X}})$ where $x \sim^{S}_{\widehat{X}} x'$ iff $x = x'$. This clearly is a well-defined generalized equilocale. It is a sur-equilocale since all $x : S \to X$, including the identity i_X, are self-related, and so the self-related maps are jointly epi (see 6.13).

7.1 For two locale maps $f, f' : X \to Y$, $f \approx f' : \widehat{X} \to \widehat{Y}$ holds if and only if $f = f'$. Hence $X \mapsto \widehat{X} : \mathsf{Loc} \to \mathsf{SELoc}$, $f \mapsto \widehat{f} = \{f\}$ embeds Loc as a full subcategory into SELoc.

Proof: $f \approx f'$ holds iff $x \sim^{S}_{\mathcal{X}} x'$ implies $f\,x \sim^{S}_{\mathcal{Y}} f'x'$, iff $x = x'$ implies $f\,x = f'x'$, iff $f\,x = f'x$ for all $x : S \to X$, iff $f = f'$ (since x may be chosen as i_X). This shows that the SELoc maps from $\widehat{X}$ to $\widehat{Y}$ are singleton classes $\{f\}$ of locale maps $f : X \to Y$. □

7.2 The Preservation of Products

7.2 The embedding of Loc into SELoc preserves products: $\prod^{\circ}_{i\in I} \widehat{X_i} = \prod^{*}_{i\in I} \widehat{X_i} = \widehat{\prod_{i\in I} X_i}$.

Proof: With $X = \prod_{i\in I} X_i$, we need to show $\prod^{*}_{i\in I} \widehat{X_i} = \widehat{X}$. By 4.8, $\prod^{*}_{i\in I} \widehat{X_i}$ is $(X, \sim_{\mathcal{X}})$ where $x \sim^{S}_{\mathcal{X}} x'$ iff $p_i\,x \sim^{S}_{\mathcal{X}_i} p_i\,x'$ for all i in I. By definition of $\widehat{X_i}$, this is equivalent to $p_i\,x = p_i\,x'$ for all i in I, which in turn is equivalent to $x = x'$ by the universal property of the locale product X. But $x = x'$ is the definition of the PERs of $\widehat{X}$, so $\prod^{*}_{i\in I} \widehat{X_i} = \widehat{X}$ follows. By 6.23, $\prod^{\circ}_{i\in I} \widehat{X_i} = (\prod^{*}_{i\in I} \widehat{X_i})_0$ holds. The first part of the proof then implies $\prod^{\circ}_{i\in I} \widehat{X_i} = \widehat{X}_0$. Because $\widehat{X}$ is a sur-equilocale, $\widehat{X}_0 = \widehat{X}$ follows. □

7.3 The Preservation of Exponentials

Although Loc is not Cartesian closed, some exponentials Z^Y exist in Loc. The question now is whether these exponentials are preserved by the embedding of Loc into SELoc, i.e., whether $\widehat{Z}^{\widehat{Y}} \cong \widehat{Z^Y}$ holds. While we do not know the answer to this question in general, we are able to show that exponentials Z^Y of exponentiable locales Y are preserved. Here, a locale Y is *exponentiable* if exponentials Z^Y exist for all locales Z.

Concretely, the exponentiable locales are precisely the core-compact locales, i.e., those locales whose frame of opens is a continuous lattice [3, VII 4.10]. This includes the locale corresponding to the real line, and all locales generated by the Scott topologies of continuous dcpos, in particular all injective locales.

We start with a lemma about exponential transposes. Recall that an exponential Z^Y is characterized by an evaluation map $v : Z^Y \times Y \to Z$ with the property that for any X and any $f : X \times Y \to Z$, there is a unique *transpose* $f^\sharp : X \to Z^Y$ satisfying $v\,(f^\sharp \times i_Y) = f : X \times Y \to Z^Y \times Y \to Z$.

7.3 Let Y and Z be locales with an exponential Z^Y. Then for all locales S and X, all $x, x' : S \to X$ and all $f, f' : X \times Y \to Z$, the equation $f^\sharp\,x = f'^\sharp\,x' : S \to Z^Y$ holds if and only if $f\,(x \times i_Y) = f'(x' \times i_Y) : S \times Y \to Z$.

Proof: $v\,(f^\sharp\,x \times i_Y) = v\,(f^\sharp \times i_Y)\,(x \times i_Y) = f\,(x \times i_Y)$ holds using the characterization of the transpose $f^\sharp$. This equation and the corresponding one for f' and x' prove the implication from left to right. But the equation also shows that $f^\sharp\,x$ is the transpose of $f\,(x \times i_Y)$, and likewise for f' and x'. Thus the implication from right to left follows from the uniqueness of the transpose. □

By 6.23, products in SELoc are given as $\prod^{\circ}_{i\in I}\mathcal{X}_i = (\prod^{*}_{i\in I}\mathcal{X}_i)_0$. By 7.2, the core formation can be avoided in case of locales ($\prod^{\circ}_{i\in I}\widehat{X_i} = \prod^{*}_{i\in I}\widehat{X_i}$), so in particular $\widehat{X}\times^{\circ}\widehat{Y} = \widehat{X}\times^{*}\widehat{Y} (= \widehat{X\times Y})$. A related result holds for arbitrary sur-equilocales $\mathcal{X}$ if Y is restricted to an exponentiable locale.

7.4 If $\mathcal{X} = (X, \sim_{\mathcal{X}})$ is a sur-equilocale and Y is exponentiable, then $\mathcal{X}\times^{*}\widehat{Y}$ is a sur-equilocale and thus equal to $\mathcal{X}\times^{\circ}\widehat{Y}$.

Proof: We apply 6.13, i.e., show that the self-related maps $p : S \to X\times Y$ are jointly epi. To this end, let $f, f' : X\times Y \to Z$ be locale maps such that $f\,p = f'p$ for all $p : S \to X\times Y$ satisfying $p \sim^{S}_{\mathcal{X}\times^{*}\widehat{Y}} p$. We have to show $f = f'$. Note that Z^Y exists since Y is exponentiable, and so we can form transposes $f^{\sharp}, f'^{\sharp} : X \to Z^Y$.

For any self-related $x : S \to X$, $x\times i_Y : S\times Y \to X\times Y$ is self-related by 4.9 (2). By assumption, $f\,(x\times i_Y) = f'(x\times i_Y)$ follows, which by 7.3 implies $f^{\sharp}\,x = f'^{\sharp}\,x$. Since this holds for all self-related x and $\mathcal{X}$ is a sur-equilocale, $f^{\sharp} = f'^{\sharp}$ follows using 6.13. The uniqueness of the transpose then yields $f = f'$ as required. □

In contrast to 7.3, the proof of 7.4 presented above requires exponentiability of Y since the locale Z is not given from the hypotheses, but may be any locale.

7.5 Theorem If Y is an exponentiable locale, then for any locale Z, the exponential Z^Y exists by hypothesis and is preserved by the embedding of Loc into SELoc up to isomorphism, i.e., $\widehat{Z}^{\widehat{Y}} \cong \widehat{Z^Y}$ holds.

Proof: We show that $\widehat{Z^Y}$ satisfies the universal property required for $\widehat{Z}^{\widehat{Y}}$. First, the evaluation map $v : Z^Y\times Y \to Z$ yields $\varepsilon = [v] : \widehat{Z^Y}\times\widehat{Y} \to \widehat{Z}$ by virtue of the embedding (7.1), using the fact that the embedding preserves products (7.2). Second, we have to show that any $\varphi = [f] : \mathcal{X}\times^{\circ}\widehat{Y} \to \widehat{Z}$ (with arbitrary sur-equilocale $\mathcal{X} = (X, \sim_{\mathcal{X}})$) has a transpose $\varphi^{\sharp} : \mathcal{X} \to \widehat{Z^Y}$. Thanks to 7.4, $\times^{\circ}$ may be replaced by $\times^{*}$ so that f has type $X\times Y \to Z$, which can be transposed to $f^{\sharp} : X \to Z^Y$. (Without 7.4, we had $f : S \to Z$ for some sublocale S of $X\times Y$, which could not be transposed in an obvious way.) Of course, our candidate for $\varphi^{\sharp}$ is $[f^{\sharp}]$. In order to verify that this makes sense, we shall now prove that $f \approx f' : \mathcal{X}\times\widehat{Y} \to \widehat{Z}$ implies $f^{\sharp} \approx f'^{\sharp} : \mathcal{X} \to \widehat{Z^Y}$.

If $x \sim^{S}_{\mathcal{X}} x'$, then $(x\times i_Y) \sim^{S\times Y}_{\mathcal{X}\times\widehat{Y}} (x'\times i_Y)$ by 4.9 (2), whence $f\,(x\times i_Y) \sim^{S\times Y}_{\widehat{Z}} f'(x'\times i_Y)$ because of $f \approx f'$. Since the PERs of $\widehat{Z}$ are given by equality, this means $f\,(x\times i_Y) = f'(x'\times i_Y)$, which implies $f^{\sharp}\,x = f'^{\sharp}\,x'$ by 7.3. Since the PERs of $\widehat{Z^Y}$ are also given by equality, the latter means $f^{\sharp}\,x \sim^{S}_{\widehat{Z^Y}} f'^{\sharp}\,x'$ as required.

Because the function $(-)^{\sharp}$ on locale maps preserves '$\approx$' as shown above, it induces a function $(-)^{\sharp}$ on SELoc maps by 2.4. Because all of $(-)^{\sharp}$, composition, and tupling on SELoc maps are in this situation induced from the corresponding operations on locale maps, the equation $\varepsilon\,(\varphi^{\sharp}\times\iota_{\widehat{Y}}) = \varphi$ holds. We only need to show that $\varphi^{\sharp}$ is uniquely determined by this equation. So assume $\varepsilon\,(\psi_1\times\iota_{\widehat{Y}}) = \varepsilon\,(\psi_2\times\iota_{\widehat{Y}})$ holds for some SELoc maps $\psi_1, \psi_2 : \mathcal{X} \to \widehat{Z^Y}$ with $\psi_i = [g_i]$ for $i = 1, 2$. Then $f_1 \approx f_2$ holds where $f_i = v\,(g_i\times i_Y)$, whence ${f_1}^{\sharp} \approx {f_2}^{\sharp}$ as shown above. But because of $f_i = v\,(g_i\times i_Y)$ and the universal property of Z^Y, ${f_i}^{\sharp} = g_i$ follows. Thus we obtain $g_1 \approx g_2$, i.e., $\psi_1 = \psi_2$. □

8 Reflecting Equilocales into Locales

The basic idea for forming the localic reflection of an equilocale $\mathcal{X} = (X, \sim_{\mathcal{X}})$ is to form the quotient of X under all partial equivalence relations at once. More formally, this is the joint coequalizer of all pairs (x, x') such that $x \sim^S_{\mathcal{X}} x'$. We first derive the existence of such joint coequalizers from the required properties of Loc, and then define the reflection and study its properties.

8.1 Joint Coequalizers

The joint coequalizer of a class of pairs (y^1_i, y^2_i) of locale maps $y^1_i, y^2_i : S_i \to Y$ with fixed target Y is constructed using joint image (Section 6.2) and ordinary coequalizer (Requirement 5). First, a class of maps $y_i = \langle y^1_i, y^2_i \rangle : S_i \to Y \times Y$ is constructed from the class of pairs. By 6.6, this class of maps has a joint image $e : X \hookrightarrow Y \times Y$ with the property

8.1 For all f^1, $f^2 : Y \times Y \to Z$, $f^1\, y_i = f^2\, y_i$ for all i in I iff $f^1\, e = f^2\, e$.

The embedding e can be split into components $e^1 = p_1\, e : X \to Y$ and $e^2 = p_2\, e : X \to Y$. Then for any $f : Y \to Z$, 8.1 applied to $f^1 = f\, p_1$ and $f^2 = f\, p_2$ gives the new equivalence

8.2 For all $f : Y \to Z$, $f\, y^1_i = f\, y^2_i$ for all i in I iff $f\, e^1 = f\, e^2$.

Now let $q : Y \to Q$ be the coequalizer of e^1 and e^2. This means that for $f : Y \to Z$, $f\, e^1 = f\, e^2$ iff $f = f'\, q$ for some $f' : Q \to Z$. Combining this with 8.2 yields the final result:

8.3 For any class of pairs $(y^1_i,\, y^2_i)$ of locale maps $y^1_i,\, y^2_i : S_i \to Y$, there is a quotient $q : Y \to Q$ such that for any $f : Y \to Z$, $f\, y^1_i = f\, y^2_i$ for all i in I iff $f = f'\, q$ for some $f' : Q \to Z$.

8.2 The Reflection

We now define a functor $\mathcal{R} : \mathsf{SELoc} \to \mathsf{Loc}$ and show that it is a reflection, i.e., left-adjoint to the embedding of Loc into SELoc. The following defines at once the reflection $\mathcal{RX}$ and a locale map $r_{\mathcal{X}}$ related to it.

8.4 Definition For any $\mathcal{X} = (X, \sim_{\mathcal{X}})$, let $r_{\mathcal{X}} : X \to \mathcal{RX}$ be the canonical form of the joint coequalizer (8.3) of the class of pairs (x, x') such that $x \sim^S_{\mathcal{X}} x'$. This means $r_{\mathcal{X}}$ is epi, $r_{\mathcal{X}}\, x = r_{\mathcal{X}}\, x'$ holds whenever $x \sim^S_{\mathcal{X}} x'$, and for any $f : X \to Y$ such that $fx = fx'$ whenever $x \sim^S_{\mathcal{X}} x'$, $f = f'\, r_{\mathcal{X}}$ holds for some $f' : \mathcal{RX} \to Y$.

The next property is the first step in extending $\mathcal{R}$ to a functor.

8.5 For any $f \approx f : \mathcal{X} \to \mathcal{Y}$, there is a unique locale map $Rf : \mathcal{RX} \to \mathcal{RY}$ satisfying $r_{\mathcal{Y}}\, f = Rf\, r_{\mathcal{X}}$.

Proof: If $x \sim^S_{\mathcal{X}} x'$, then $f\, x \sim^S_{\mathcal{Y}} f\, x'$, whence $r_{\mathcal{Y}}\, f\, x = r_{\mathcal{Y}}\, f\, x'$. Thus, $r_{\mathcal{Y}}\, f$ coequalizes all $x \sim^S_{\mathcal{X}} x'$, whence there is some Rf such that $r_{\mathcal{Y}}\, f = Rf\, r_{\mathcal{X}}$. It is unique since $r_{\mathcal{X}}$ is epi by definition. □

8.6 If $f \approx f' : \mathcal{X} \to \mathcal{Y}$, then $Rf = Rf'$.

Proof: Here we use that $\mathcal{X}$ is a sur-equilocale, i.e., the self-related $x : S \to X$ are jointly epi (6.13). If $x \sim^S_{\mathcal{X}} x$, then $f\, x \sim^S_{\mathcal{Y}} f'x$, whence $r_{\mathcal{Y}}\, f\, x = r_{\mathcal{Y}}\, f'x$. Since these x are jointly epi, $r_{\mathcal{Y}}\, f = r_{\mathcal{Y}}\, f'$ follows. The equation in 8.5 then yields $Rf\, r_{\mathcal{X}} = Rf'\, r_{\mathcal{X}}$, whence $Rf = Rf'$ since $r_{\mathcal{X}}$ is epi. □

8.7 The object map $\mathcal{R}$ of 8.4 becomes a functor $\mathcal{R} : \mathsf{SELoc} \to \mathsf{Loc}$ by defining $\mathcal{R}[f] = Rf$, where Rf is constructed as in 8.5.

PROOF: $\mathcal{R}[f]$ is well-defined because of 8.6. The functoriality follows from the equation $r_\mathcal{Y}\, f = Rf\, r_\mathcal{X}$ characterizing Rf uniquely. □

We now consider the compositions of $\mathcal{R}$ with the embedding $X \mapsto \widehat{X}$.

8.8 $r_{\widehat{X}} = i_X$, $\mathcal{R}\widehat{X} = X$, and $\mathcal{R}\widehat{f} = f$ for all locale maps $f : X \to Y$.

PROOF: $r_{\widehat{X}} : \widehat{X} \to \mathcal{R}\widehat{X}$ is defined as the joint coequalizer of the pairs (x, x') with $x \sim^S_{\widehat{X}} x'$, i.e., $x = x'$. Clearly, the identity of X satisfies $i_X\, x = i_X\, x'$ whenever $x = x'$, and whenever $fx = fx'$ for all $x = x'$, then $f = f\, i_X$. Therefore $r_{\widehat{X}} = i_X$ and thus $\mathcal{R}\widehat{X} = X$. The characteristic equation $r_{\widehat{Y}}\, f = Rf\, r_{\widehat{X}}$ then yields $Rf = f$, whence $\mathcal{R}\widehat{f} = \mathcal{R}\,\{f\} = f$. □

8.9 The maps $r_\mathcal{X}$ induce a natural transformation ρ with $\rho_\mathcal{X} : \mathcal{X} \to \widehat{\mathcal{R}\mathcal{X}}$.

PROOF: Since $r_\mathcal{X}\, x = r_\mathcal{X}\, x'$ whenever $x \sim^S_\mathcal{X} x'$, $r_\mathcal{X} \approx r_\mathcal{X} : \mathcal{X} \to \widehat{\mathcal{R}\mathcal{X}}$ follows so that $\rho_\mathcal{X} = [r_\mathcal{X}] : \mathcal{X} \to \widehat{\mathcal{R}\mathcal{X}}$ is a well-defined SELoc map. Naturality of ρ follows from the equation $r_\mathcal{Y}\, f = Rf\, r_\mathcal{X}$ for $f \approx f : \mathcal{X} \to \mathcal{Y}$. □

8.10 For locales X, $\rho_{\widehat{X}}$ is the identity, i.e., $\rho_{\widehat{X}} = \iota_{\widehat{X}} : \widehat{X} \to \widehat{\mathcal{R}\widehat{X}} = \widehat{X}$ holds. Likewise for equilocales $\mathcal{X}$, $\mathcal{R}\,\rho_\mathcal{X} = i_{\mathcal{R}\mathcal{X}} : \mathcal{R}\mathcal{X} \to \mathcal{R}\,\widehat{\mathcal{R}\mathcal{X}} = \mathcal{R}\mathcal{X}$.

PROOF: The first claim is proved by $\rho_{\widehat{X}} = [r_{\widehat{X}}] = [i_X] = \iota_{\widehat{X}}$ using 8.8. The characteristic equation 8.5 of Rf applied to $f = r_\mathcal{X} \approx r_\mathcal{X} : \mathcal{X} \to \widehat{\mathcal{R}\mathcal{X}}$ yields $r_{\widehat{\mathcal{R}\mathcal{X}}}\, r_\mathcal{X} = Rr_\mathcal{X}\, r_\mathcal{X}$, whence $r_{\widehat{\mathcal{R}\mathcal{X}}} = Rr_\mathcal{X}$ since $r_\mathcal{X}$ is epi. Now $r_{\widehat{\mathcal{R}\mathcal{X}}} = i_{\mathcal{R}\mathcal{X}}$ holds by 8.8, and so $\mathcal{R}\,\rho_\mathcal{X} = Rr_\mathcal{X} = i_{\mathcal{R}\mathcal{X}}$ follows. □

The proof that $\mathcal{R}$ is a reflection is complete once the following property is shown:

8.11 For each SELoc map $\varphi : \mathcal{X} \to \widehat{Y}$, there is a unique locale map $g : \mathcal{R}\mathcal{X} \to Y$ such that $\widehat{g}\,\rho_\mathcal{X} = \varphi : \mathcal{X} \to \widehat{\mathcal{R}\mathcal{X}} \to \widehat{Y}$.

PROOF: The locale map $g = \mathcal{R}\varphi : \mathcal{R}\mathcal{X} \to \mathcal{R}\widehat{Y} = Y$ does the job since $\widehat{\mathcal{R}\varphi}\,\rho_\mathcal{X} = \rho_{\widehat{Y}}\,\varphi = \varphi$ by naturality of ρ and $\rho_{\widehat{Y}} = \iota_{\widehat{Y}}$ (8.10). On the other hand, this is the only choice: if $\widehat{g}\,\rho_\mathcal{X} = \varphi$, then $\mathcal{R}\varphi = \mathcal{R}\widehat{g}\ \mathcal{R}\rho_\mathcal{X} = g\, i_{\mathcal{R}\mathcal{X}} = g$, where the second equality is derived from 8.8 and 8.10.□

A concrete description of the frame of opens of the locale $\mathcal{R}\mathcal{X}$ can be obtained by noting that this frame is isomorphic to $\mathsf{Loc}(\mathcal{R}\mathcal{X}, \$)$ where $\$$ is the Sierpinski locale, and this is isomorphic to $\mathsf{SELoc}(\mathcal{X}, \widehat{\$})$ because of the reflection property 8.11.

9 Related Work and Open Problems

In Section 7.3, we have shown that the embedding of Loc into ELoc preserves exponentials Z^Y of exponentiable locales Y. The preservation of other exponentials existing in Loc is an open problem.

Vickers and Townsend [5] considered the presheaf category $[\mathsf{Loc}^{\mathrm{op}}, \mathsf{Set}]$. This category contains Loc as a full subcategory, but is not Cartesian closed because of size problems. Nevertheless, the exponentials $\$^X$ and $\$^{\$^X}$ exist for any locale X, where $\$$ is the Sierpinski locale. Moreover, $\$^{\$^X}$ is a locale again (although $\$^X$ may be non-localic), which can be obtained directly by applying the double power locale construction to X. An open problem in our context is whether the same result holds for $\$^{\$^X}$ formed in ELoc. More generally, the relationship between ELoc and $[\mathsf{Loc}^{\mathrm{op}}, \mathsf{Set}]$ is not quite clear. A related question is whether the frame of the localic reflection $\mathcal{R}(\$^X)$ is isomorphic to the Scott topology on the frame of X.

Acknowledgement

Martín Escardó posed to me the problem of embedding Loc into a Cartesian closed category on various occasions. He then invited me to visit him at the School of Computer Science of the University of Birmingham, UK. After many discussions with Martín and Steve Vickers about this and related topics, I was able to find the construction presented in this paper during the visit. Certainly, this would not have been possible without Martín's kind invitation, help, and encouragement.

References

[1] A. Bauer, L. Birkedal, and D. S. Scott. Equilogical spaces. URL: `http://www.cs.cmu.edu/Groups/LTC/`. To appear in *Theoretical Computer Science*, 1998, revised 2001.

[2] Martín Escardó. Topology of data types and computability concepts. Submitted for publication, 2003.

[3] P. Johnstone. *Stone Spaces*. Cambridge University Press, 1982.

[4] S. J. Vickers. *Topology via Logic*, volume 5 of *Cambridge Tracts in Theoretical Computer Science*. Cambridge University Press, 1989.

[5] S. J. Vickers and C. F. Townsend. A universal characterization of the double powerlocale. To appear in *Theoretical Computer Science*, special issue on Domain Theory, accepted in 2003.