On Properly Pareto Optimal Solutions

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Abstract. In this paper we study ε-proper efficiency in multiobjective optimization. We introduce various new definitions of ε-proper efficiency, relate them with existing ones, study various concepts and develop very general necessary optimality conditions for a few of them.

1 Extended Abstract

Approximate solutions are referred to as ε-efficient solutions where ε refers to the precision parameter. Several authors have studied ε-efficiency in multiobjective optimization see for example [6], [2], [10]. The concept of ε-efficiency is practically useful from the fact that to a decision maker good approximate solutions are very practical and helpful in decision making. However like Pareto points or efficient points there are also ε-Pareto points with undesirable preperties. Thus even in the approximate case we need to filter out the bad ones and keep the so called ε-proper Pareto solutions.

Consider the following vector optimization problem (VP):

Minimize $f(x) = (f_1(x), f_2(x), \ldots, f_m(x))$

subject to $x \in X$

where each $f_i : \mathbb{R}^n \to \mathbb{R}$, $X \subseteq \mathbb{R}^n$. In what follows we will consider $\varepsilon \in \mathbb{R}^m_+$, i.e. $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_m)$, $\varepsilon_i \geq 0$ for all i. In some cases we will set $\varepsilon_i = \varepsilon'$, for all i and then $\varepsilon = e\varepsilon'$ where $e = (1, \ldots, 1) \in \mathbb{R}^m_+$.

Definition 1 **ε-Pareto optimality** *Let $\varepsilon \in \mathbb{R}^m_+$ be given then a point $x^* \in X$ is said to be an ε-Pareto optimal of (VP) if there exists no $x \in X$ such that,*

$$f_i(x) \leq f_i(x^*) - \varepsilon_i, \quad \forall i \in \{1, 2, \ldots, m\}. \tag{1}$$

and with strict inequality holding for at least one index.

Let us denote the set of all Pareto optimal solutions as X_{par}. Observe that if $\varepsilon = 0$, the above definition reduces to that of a Pareto optimal solution. Let us denote the set of ε-Pareto points as $X_{\varepsilon-par}$.

Definition 2 **Geoffrion proper Pareto optimality** [3] *$x_0 \in X$ is called Geoffrion proper Pareto optimal if x_0 is Pareto optimal and if there exists a*

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number $M > 0$ such that for all i and $x \in X$ satisfying $f_i(x) < f_i(x_0)$, there exists an index j such that $f_j(x_0) < f_j(x)$ and moreover $(f_i(x_0) - f_i(x))/(f_j(x) - f_j(x_0)) \leq M$.

Let us denote the set of all Geoffrion properly Pareto optimal solutions as X_G.

Lemma 1 *A point $x_0 \in X_G$ if and only if there exists $M > 0$ such that the following system is inconsistent (for all $i = 1, 2, \ldots, m$ and for all $x \in X$).*

$$-f_i(x_0) + f_i(x) < 0$$
$$-f_i(x_0) + f_i(x) < M(f_j(x_0) - f_j(x)) \quad \forall j \neq i.$$

Note that in Geoffrion's definition $x \in X$. However as shown in next lemma, when $Y = f(X)$ is $\mathbb{R}^m_+$ compact (i.e. the sections $(y - \mathbb{R}^m_+) \cap Y$ are compact for all $y \in Y$) then this can be replaced by $x \in X_{par}$.

Lemma 2 *Suppose that $Y = f(X)$ is $\mathbb{R}^m_+$ compact, then $x^0 \in X_G$ if x_0 is Pareto optimal and if there exists a number $M > 0$ such that for all i and $x \in X_{par}$ satisfying $f_i(x) < f_i(x^0)$, there exists an index j such that $f_j(x^0) < f_j(x)$ and moreover $(f_i(x^0) - f_i(x))/(f_j(x) - f_j(x^0)) \leq M$.*

Definition 3 Liu ε-properly Pareto optimality (Liu [7]) *A point, $x^* \in X$ is called ε-proper Pareto optimal in the sense of Liu [7], if x^* is ε-Pareto optimal and there exists a number $M > 0$ such that for all i and $x \in X$ satisfying $f_i(x) < f_i(x^*) - \varepsilon_i$, there exists an index j such that $f_j(x^*) - \varepsilon_j < f_j(x)$ and moreover $(f_i(x^*) - f_i(x) - \varepsilon_i)/(f_j(x) - f_j(x^*) + \varepsilon_j) \leq M$.*

Observe that if $\varepsilon = 0$, the above definition reduces to that of a Geoffrion proper Pareto optimal. Let us denote the set of all Liu properly Pareto optimal solutions as $X_L(\varepsilon)$.

Remark 1 *Let us however observe in the above definition and definition 2.2, M is arbitrary. On the other side M provides a bound on the trade-offs betwen the components of the objective vector. It is more natural to expect in practice the decision maker will provide a bound on such trade offs. Thus we are motivated to define the following.*

Definition 4 Geoffrion M properly Pareto optimality *Given a positive number $M > 0$, $x^0 \in X$ is called Geoffrion M proper Pareto optimal if x^0 is Pareto optimal and if for all i and $x \in X$ satisfying $f_i(x) < f_i(x^0)$, there exists an index j such that $f_j(x^0) < f_j(x)$ and moreover $(f_i(x^0) - f_i(x))/(f_j(x) - f_j(x^0)) \leq M$.*

Let us denote the set of all Geoffrion M properly Pareto optimal solutions as X_M. It is to be noted that a similar modified definition is also possible for Liu ε-proper Pareto optimal solutions. Let us denote the set of all M ε-proper Pareto optimal solutions as $X_M(\varepsilon)$.

Theorem 1 *Let* $\varepsilon = \varepsilon' e$ *where* $\varepsilon' \in \mathbb{R}$, $\varepsilon' > 0$ *and* $e = (1, 1, \ldots, 1)$, *then for any fixed* M,

$$X_M = \cap_{\varepsilon' > 0} X_M(\varepsilon) \tag{2}$$

Proposition 11 *Consider a* (VP) *in which* X *is a finite set. Then there exists an* $\varepsilon > 0$, *such that* $X_M = X_M(\varepsilon)$.

Definition 5 Benson's ε-proper Pareto optimality *A point* $x^0 \in X$ *is called Benson's* ε*-proper Pareto optimal, if*
$cl(cone(f(X) + (C + \varepsilon) - (f(x^0)))) \bigcap (-C) = \{0\}$
where C *is the ordering cone.*

This definition is a modification of Benson's proper efficiency (Benson [1])

Lemma 3 *If a point* x_0 *is Benson's* ε *proper-Pareto optimal then its also* ε*-Pareto optimal.*

Definition 6 Henig ε-efficiency *A point* $x^* \in X$ *is Henig* ε*-Pareto optimal if*

1. $(f(x^*) - \varepsilon - C \setminus \{0\}) \cap f(X) = \emptyset$, *or equivalently*
2. $(f(X) + \varepsilon - f(x^*)) \cap (-C \setminus \{0\}) = \emptyset$, *or*

where C *is the ordering cone, such that* $\mathbb{R}^m_+ \setminus \{0\} \subseteq intC$

Definition 7 Henig ε-weak efficiency *A point* $x^* \in X$ *is Henig* ε*-weak efficient point if*

1. $(f(x^*) - \varepsilon - intC) \cap f(X) = \emptyset$, *or equivalently*
2. $\exists$ *no* $x \in X$, *s.t.* $f(x^*) - f(x) - \varepsilon \in intC$

where as usual C *is the ordering cone, and* $\mathbb{R}^m_+ \setminus \{0\} \subseteq intC$

Thus Henig ε-weak efficient points can be seen as weak points obtained when $intC$ is perturbed by an amount ε.

Theorem 2 *Let us consider the problem* (VP) *where* $f : \mathbb{R}^n \to \mathbb{R}^m$ *is a* C*-convex and* X *be a closed convex set. Let* $\varepsilon = \varepsilon' e$, *where* $\varepsilon' > 0$ $\varepsilon' \in \mathbb{R}$. *Let* $x_0 \in X$ *be Henig* ε*-weak minimum, then there exists* $\mu \in C^*$, *with* $\langle \mu, e \rangle = 1$ *such that* x_0 *is a* ε*-minimum for the following scalar minimum problem* (MP)

$$\min_{x \in X} \langle \mu, f(x) \rangle$$

Definition 8 Henig ε-proper efficiency *The Henig* ε*-proper Pareto optimal set (with respect to cone* C*) is defined as*
$X_{\varepsilon, pH}(f(X)) = \{x \in X \mid (f(x) - \varepsilon - (\Theta \setminus \{0\})) \cap f(X) = \emptyset\}$ *where* C *is the ordering cone with* $C\{0\} \subseteq int\Theta$.

This definition is a modification of Henig's *global* proper efficiency (Henig [4])

Lemma 4 *Let $\varepsilon = \varepsilon' e$ and H denote the set of all Henig weak minimum of the program (VP) and for any given $\varepsilon > 0$, let H_ε denote the set of all Henig ε-weak minimum of (VP). Then,*

$$H = \cap_{\varepsilon' > 0} H_\varepsilon \tag{3}$$

When the ordering cone is $\mathbb{R}^m_+$, the above theorem reduces to

Corollary 1 Lemaire [5]

Let $\varepsilon = \varepsilon' e$ and E denote the set of all weak vector minimum of the program (VP) and for any given $\varepsilon > 0$, let E_ε denote the set of all ε-weak minimum of (VP). Then,

$$E = \cap_{\varepsilon > 0'} E_\varepsilon \tag{4}$$

Let $(f_i)'_\varepsilon(x; d)$ denote the ε-directional derivative of a convex function f_i at x in the direction d.

Lemma 5 *Consider the problem (VP). Let $\varepsilon = \varepsilon' e$. If*

$$((f_1)'_\varepsilon(y; x-y), \ldots, (f_m)'_\varepsilon(y; x-y)) \in W = \mathbb{R}^m \setminus (-intC) \quad \forall x \in X \tag{5}$$

then $y \in H_\varepsilon$. When $\varepsilon = 0$, the converse is also true.

1.1 Kuhn Tucker type optimality conditions for Benson ε-efficiency.

We can derive the necessary and sufficient Kuhn Tucker type optimality conditions for Benson ε-proper Pareto optimal solutions.

Theorem 3 *Consider the problem (VP) and let $f(x) = (f_1(x), f_2(x), \ldots, f_m(x))$ and let the set X be given by inequality constraints $g(x) = (g_1(x), g_2(x), \ldots, g_l(x))$. Suppose that f is a convex function with respect to C and that $g_1, g_2, \ldots, g_m$ are convex functions. Assume that the Slater Constraint Qualification holds. Then $x_0 \in X$ is an ε-properly Pareto optimal in Benson's sense if and only if there exists scalars $\mu_j \in intC^*, j \in T = \{1, 2, \ldots, m\}$, $\lambda_i \geq 0$, $i \in L = \{1, 2, \ldots, l\}$, $\delta_{j^*} \geq 0, j \in T = \{1, 2, .., m\}$ and $\varepsilon_{i^*} \geq 0, i \in L = \{1, 2, \ldots, l\}$ such that*

1. $0 \in \sum_{j=1}^m \partial_{\delta_{j^*}}(\mu_j f_j)(x_0) + \sum_{i=1}^l \partial_{\varepsilon_{i^*}}(\lambda_i g_i)(x_0)$, *and*
2. $\sum_{j=1}^m \delta_{j^*} + \sum_{i=1}^l \varepsilon_{i^*} - \langle \mu, \varepsilon \rangle \leq \sum_{i=1}^l \lambda_i g_i(x_0) \leq 0$

The concept of M ε-proper Pareto optimality is useful among other concepts like ε-Pareto optimality, weak ε-Pareto optimality and proper ε-Pareto optimality. The above lemma shows that if we take the limit of any M ε-proper Pareto solutions as $\varepsilon \to 0$, then it will give only the set of M proper solutions. This cannot be said of any other concepts like ε-Pareto optimality, weak ε-Pareto optimality and proper ε-Pareto optimality, in the limit they get to weak Pareto optimal solutions. In *MOEA's* the concept of ε-Henig efficiency can be thought of as combining an ε-MOEA with Branke's guidance approach.

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