Normal Form Theorem for Logic Programs with Cardinality Constraints

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Abstract

JEFF(New abstract) We study cardinality-constraint (CC) logic programs [NSS99]. A CC-logic program is *body-normal* if for every clause C of P the body of C consists of atoms and negated atoms, that is cardinality constraints of the form $1\{p\}$ or $\{q\}0$. For a class of programs P whose heads are *not* of the form $X0$, we prove that there is a body-normal program $bn(P)$ such that $bn(P)$ is in the same language as P and P and $bn(P)$ have the same stable models. If the heads of the form $X0$ are admitted, then we show that in the language with just one additional atom a similar result can be achieved.

1 Introduction

In this paper we investigate the cardinality-constraint programs. Those are logic programs that admit, besides of usual atoms, generalized atoms called cardinality-constraints atoms of the form kXl where X is a finite set of propositional atoms and k is a non-negative integers, $k \leq |X|$ and l is an integer or ∞ and $k \leq l$. This extension of logic programming has been implemented in the logic-programming solver *smodels*, [NSS99, Syr01, SNS02]. However the roots of cardinality-constraints are in both SAT and in Integer Programming communities. It should be mentioned that cardinality-constraints are naturally represented as pseudo-boolean integer inequalities (i.e. integer inequalities

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where admited solutions must take values in $\{0, 1\}$). We refer the reader to papers such as [DG03, WB96] for the discussion of the developments in these other areas.

The solver *smodels* allows for the use of cardinality-constraints both in the heads and in the bodies of clauses. Niemelä and collaborators [NSS99] introduced the stable semantics for programs admitting cardinality constraints. At the first glance it has not been clear at all that the stable semantics of programs as introduced in [NSS99] correctly generalizes the generally accepted stable semantics of normal logic programs [GL88]. The relationship of the stable semantics of programs admitting cardinality-constraint atoms has been studied by Ferraris and Lifschitz in [FL01] and by the authors in [MR03]. Ferraris and Lifschitz reduced the stable semantics for such programs to answer sets of programs with nested expressions (a natural generalization of logic programs). The present authors reduced the stable semantics of CC-logic programs to the usual semantics of normal programs extended by **hide** operation.

In [MNR90] the authors developed a proof-theoretical technique to study stable models of logic programs. The technique was based on *proof-schemes*, context-dependent proofs of atoms out of programs. The characterization of stable models that one obtains with the proof schemes is based on a fixpoint of anti-monotonic operator. The technique of proof-schemes has been extended by the authors in [MR03] to handle the context of CC-logic programs. This extension provides, as in the case of normal logic programs, a characterization of stable models of CC-logic programs in proof-theoretic terms.

The goal of this paper is to prove a normal form theorem for CC-logic programs. To see this result in perspective, let us look at the simpler case of normal logic programs. For such programs Dung and Kanchansut [DK89] proved a certain normal form theorem. Let us call a normal program P *purely negative* if the clauses of the program P do not contain positive literals. Next, let us call programs P and P' *equivalent* if the families of stable models of P and of P' coincide. Dung and Kanchansut stated the following normal form theorem: for every normal program P there is a purely negative program P' such that P and P' are equivalent.

For CC-logic programs elimination of positive facts from the bodies of clauses (while keeping heads) is not, in general, possible. An example of such program is given below (Example 2.4, Section 2. Yet a weaker normal form theorem can be shown. Let us call a CC-logic program *body normal* if the clauses of C contain in the bodies only the CC-atoms of the form $1\{a\}1$, and $0\{B\}0$[1]. That is only the atoms or negated atoms. We show that for every CC-logic program P there is a strongly equivalent body-normal program P' such that the heads in P' occur in P **JEFF**.

We also show some complexity results for body-normal CC-logic programs. We discuss conclusions in Section 4.

[1]This is equivalent to having in the bodies only expressions $1\{a\}1$ and $0\{b\}0$.

2 Logic programs, CC-logic programs and their stable semantics

Recall that a clause C of a logic program P is a rule of the form

$$p \leftarrow q_1, \ldots, q_m, \neg r_1, \ldots, \neg r_n. \tag{1}$$

where $p, q_1, \ldots, q_m, r_1, \ldots, r_n$ are atoms from the set of atoms At of the program. We shall refer to p as the head of C, $head(C)$, $\{q_1, \ldots, q_m\}$ as the premises of C, $prem(C)$, $\{r_1, \ldots, r_n\}$ as the constraints of C, $cons(C)$, and $q_1 \wedge \ldots \wedge q_m \wedge \neg r_1 \wedge \ldots \wedge \neg r_n$ as the body of p, $body(C)$. A (normal) logic program is a set P of clauses. To distinguish them from other clauses described below for CC-logic programs, we shall refer to clause of the form of (1) as an *ordinary* logic program clause. When $n = 0$, the clause C is called a *Horn clause*. A Horn program is a set of Horn clauses.

A Horn program always has a least model which we denote that model by M_P. This model can be constructed as the closure of the one-step provability operator T_P as follows. Suppose that P is Horn Program. Thus all the clause of P are of the form

$$p \leftarrow a_1, \ldots, a_n. \tag{2}$$

Let H_P denote the Herbrand Base of P. In general, if Q is logic program, the Herbrand base of Q is the set of atoms a such that either a or $\neg a$ occurs in Q. Let 2^{H_P} denote the set of all subsets of H_P. The one step provability operator T_P associated with P is map $T_P : 2^{H_P} \rightarrow 2^{H_P}$ such that

$$T_P(S) = \{p : \exists \text{ clause } C = p \leftarrow a_1, \ldots, a_n \in P \text{ such that } \{a_1, \ldots, a_n\} \subseteq S\}. \tag{3}$$

We can then define $T_P^n(S)$ for $n \geq 1$ by induction on n by defining $T_P^1(S) = T_P(S)$ and $T_P^{n+1}(S) = T_P(T_P^n(S))$. It is easy to see that T_P is monotone operator so that

$$\emptyset \subseteq T_P(\emptyset) \subseteq T_P^2(\emptyset) \subseteq T_P^3(\emptyset) \subseteq \cdots.$$

We let $T_P^\omega(\emptyset) = \bigcup_{n=1}^{\infty} T_P^n(\emptyset)$. Then the minimal model of P, M_P, is defined to be $T_P^\omega(\emptyset)$

There is a natural extension of the one step provability operator to ordinary logic programs. That is, suppose P is program which consists of clauses of the form of (1) and $M \subseteq H_P$. Then we can define an operator $T_{P,M} : 2^{H_P} \rightarrow 2^{H_P}$ by

$$\begin{aligned} T_{P,M}(S) \quad = \quad & \{p : \exists C = p \leftarrow a_1, \ldots, a_n,\ \neg b_1, \ldots,\ \neg b_m \in P \\ & \text{such that } \{a_1, \ldots, a_n\} \subseteq S \ \&\ \{b_1, \ldots, b_m\} \cap M = \emptyset\}. \end{aligned} \tag{4}$$

We can then define $T_{P,M}^n(S)$ for $n \geq 1$ by induction on n by defining $T_{P,M}^1(S) = T_{P,M}(S)$ and $T_{P,M}^{n+1}(S) = T_{P,M}(T_{P,M}^n(S))$. Again it easy to see that $T_{P,M}$ is monotone operator so that

$$\emptyset \subseteq T_{P,M}(\emptyset) \subseteq T_{P,M}^2(\emptyset) \subseteq T_{P,M}^3(\emptyset) \subseteq \cdots.$$

We let $T^{\omega}_{P,M}(\emptyset) = \bigcup_{n=1}^{\infty} T^n_P(\emptyset)$. Then we say that M is a stable model of P if $M = T^{\omega}_{P,M}(\emptyset)$.
Alternatively, we can define stable models of logic programs via the Gelfond-Lifschitz operator $GL(\cdot,\cdot)$ [GL88]. Here the operator $GL(\cdot,\cdot)$ assigns to a logic program P and a set of atoms $M \subseteq H_P$, the least model of the Horn program P^M where P^M consists of the set of all Horn clauses C^M obtained from a clause C of P of the form of (1) as follows.

$$C^M = \begin{cases} \textbf{nil} & \text{if for some } i, 1 \leq i \leq n, r_i \in M \\ p \leftarrow q_1, \ldots, q_m & \text{otherwise} \end{cases} \tag{5}$$

We say that M is a stable model of P if $M = GL(P,M)$. This definition is equivalent to the one given above. Gelfond and Lifschitz proved that every stable model M of P is a model of P, in fact, a minimal and supported model of P.
Next we define a natural proof-theoretic construct associated to logic programs called *proof schemes*. Let P be a logic program, then the set of proof schemes of P can be defined inductively as follows.

1. If $C = p \leftarrow \neg r_1, \ldots, \neg r_n$ is a clause of P (the case $n = 0$ is allowed), then $\langle\langle p\rangle, \langle C\rangle, \{r_1, \ldots, r_n\}\rangle$ is a proof scheme for p in P

2. If $\langle\langle s_1, \ldots, s_k\rangle, \langle C_1, \ldots, C_k\rangle, \{t_1, \ldots, t_l\}\rangle$ is a proof scheme and $C = p \leftarrow q_1, \ldots, q_m, \neg r_1, \ldots, \neg r_n$ is a clause in P and $\{q_1, \ldots, q_m\} \subseteq \{s_1, \ldots, s_k\}$, then

$$\langle\langle s_1, \ldots, s_k, p\rangle, \langle C_1, \ldots, C_k, C\rangle, \{t_1, \ldots, t_l, r_1, \ldots, r_n\}\rangle$$

   is a proof scheme for p in P.

If $\mathfrak{S} = \langle\langle s_1, \ldots, s_k\rangle, \langle C_1, \ldots, C_k\rangle, \{t_1, \ldots, t_l\}\rangle$ is a proof scheme in P, we refer to s_k as the conclusion of $\mathfrak{S}$, $concl(\mathfrak{S})$, and say that $\mathfrak{S}$ is a proof scheme of s_k in P. We also refer to $\{t_1, \ldots, t_k\}$ as the constraints of of $\mathfrak{S}$, $const(\mathfrak{S})$. We say that $\mathfrak{S}$ is proof scheme of length k. We say that $\mathfrak{S}$ is *reduced* if $s_1, \ldots, s_k$ are pairwise distinct.
One can think of a proof scheme for a logic program as the analogue of a derivation or proof in classical logic. However, a proof scheme $\mathfrak{S}$ for p not only contains the clauses that can be used to derive p but also keeps track of the set of atoms that must be absent from prospective stable model M, namely $const(\mathfrak{S})$, if p is to be an element of $GL(P,M)$. Thus we say that a proof scheme $\mathfrak{S}$ is *admitted* by M if $M \cap const(\mathfrak{S}) = \emptyset$.

Example 2.1 Let P consist of clauses:

$C_1 = p \leftarrow q, \neg r$
$C_2 = q \leftarrow \neg s$
$C_3 = s \leftarrow \neg q$.

It is easy to check that there are exactly two reduced proof schemes of length 1, namely, $\mathfrak{S}_1 = \langle\langle q\rangle, \langle C_2\rangle, \{s\}\rangle$ is a proof scheme for q and $\mathfrak{S}_2 = \langle\langle s\rangle, \langle C_3\rangle, \{q\}\rangle$ is a proof scheme for s. There are three reduced proof schemes of length 2. The triple

$$\mathfrak{S}_3 = \langle\langle q, p\rangle, \langle C_2, C_1\rangle, \{r, s\}\rangle,$$

is a proof scheme for p in P. In addition, there are two other reduced proofs which come from concatonating $\mathfrak{S}_1$ and $\mathfrak{S}_2$, namely, $\mathfrak{S}_4 = \langle\langle q, s\rangle, \langle C_2, C_3\rangle, \{s, q\}\rangle$ and $\mathfrak{S}_5 = \langle\langle s, q\rangle, \langle C_3, C_2\rangle, \{s, q\}\rangle$. It should be clear that neither of these proof schemes can be used in the construction of stable model. Finally there are 3 more reduced proof schemes of length 3, namely, $\mathfrak{S}_6 = \langle\langle q, p, s\rangle, \langle C_2, C_1, C_3\rangle, \{r, s, q\}\rangle$, $\mathfrak{S}_7 = \langle\langle q, s, p\rangle, \langle C_2, C_3, C_1\rangle, \{r, s, q\}\rangle$, and $\mathfrak{S}_8 = \langle\langle s, q, p\rangle, \langle C_3, C_2, C_1\rangle, \{r, s, q\}\rangle$. Let us observe that r can never be in the stable model since r is not the head of a caluse of P and hence $r \notin T^{\omega}_{P,M}(\emptyset)$ for any M. Thus any stable model M of P must be contained in $\{p, q, s\}$. In this case, it is easy to check that there are exactly two stable models of P, $M_1 = \{s\}$ and $M_2 = \{p, q\}$. Clearly, M_1 admits $\mathfrak{S}_2$ but not $\mathfrak{S}_1$ and $\mathfrak{S}_3$. M_2 admits $\mathfrak{S}_1$ and $\mathfrak{S}_3$, but not $\mathfrak{S}_2$. □

The following result is proven in [MNR90].

Proposition 2.1 *Let M be a set of atoms contained the Herbrand base H_P of the a logic program P. Then M is a stable model of P if and only if*

1. *Every atom p of M possesses a proof scheme $\mathfrak{S}_p$ in P such that M admits $\mathfrak{S}_p$*
2. *No atom p in $At \setminus M$ possesses a proof scheme admitted by M.*

The proposition immediately follows from the definition of stable model and the following lemma.

Lemma 2.2 *Let M be a set of atoms contained the Herbrand base H_P of the a logic program P. Then*

$$T^{\omega}_{P,M}(\emptyset) = \{p : M \text{ admits a proof scheme } \mathfrak{S} \text{ for } p\}.$$

Proposition 2.1 implies the following property of *models* of programs.

Corollary 2.3 *Let P be a logic program and let M be a model of P. Then M is a stable model of P if and only if every element of M possesses a proof scheme in P admitted by M.*

The advantage of proof schemes is that they are entities associated with programs and atoms and not with models. Proof schemes carry within themselves the information about their own applicability. Let us observe that Corollary 2.3 establishes a condition for models of P that is easier to check than the conditions given in Proposition 2.1. Below we will extend the notion of proof scheme to CC-logic programs and prove a result analogous to Corollary 2.3. This is one reason why we believe that the definition of CC-stable models of cardinality-constraint programs is a natural generalization of stable semantics for ordinary logic programs.

There is one other property that we can derive via proof schemes. Namely, we can show that every program P is equivalent to a program Q, in the sense that P and Q have the same stable models, where each clause of Q has no premises. This result due to Dung and Kanchansut [DK89] becomes very natural in the context of proof schemes. To this end consider the set of clauses of the form

$$p \leftarrow \neg b_1, \ldots, \neg b_m$$

where m may be zero. We call such a program, a *purely negative* program. Let us suppose that we start with a logic program P and for each reduced proof scheme

$$\mathfrak{S} = \langle \langle s_1, \ldots, s_n \rangle, \langle C_1, \ldots, C_n \rangle, \{t_1, \ldots, t_l\} \rangle,$$

we construct a clause

$$C_{\mathfrak{S}} = s_n \leftarrow \neg t_1, \ldots, \neg t_m$$

whose body consists entirely of negative atoms. Let $Neg(P)$ consist of the program whose clauses are precisely the set of $C_{\mathfrak{S}}$ such that $\mathfrak{S}$ is reduced proof scheme of P. If P is a finite program, then so is $Neg(P)$. Then we have the following theorem which was implicit in [MNR90].

Theorem 2.4 *For any logic program P, P and $Neg(P)$ have the same stable models.*

We observe that all supported models of $Neg(P)$ are automatically stable models of $Neg(P)$. Thus supported models of P are not necessarily supported models of $Neg(P)$. **JEFF**

Example 2.2 Recall the program P of Example 1 which consist of clauses:

$C_1 = p \leftarrow q, \neg r$
$C_2 = q \leftarrow \neg s$
$C_3 = s \leftarrow \neg q.$

Then it is easy to see by our analysis of the reduced proof schemes of P that $Neg(P)$ consists of the following eight clauses where in each case S_i is derived from $\mathfrak{S}_i$.

$S_1 = q \leftarrow \neg s$
$S_2 = s \leftarrow \neg q$
$S_3 = p \leftarrow \neg r, \neg s$
$S_4 = s \leftarrow \neg q, \neg s$
$S_5 = q \leftarrow \neg q, \neg s$
$S_6 = s \leftarrow \neg r, \neg s, \neg q$
$S_7 = p \leftarrow \neg r, \neg s, \neg q$
$S_8 = p \leftarrow \neg r, \neg s, \neg q$

Let us observe that it is possible to get the same rule from two different proof schemes as in the case of S_7 derived from $\mathfrak{S}_7$ and $\mathfrak{S}_8$. Moreover, we can get clauses C and C', like S_1 and S_3, such that $head(C) = head(C')$ and $const(C) \subseteq const(C')$. In such a situation, there is no loss in dropping clause C' from the program. In our case, if we drop all such instance it is easy to see that $Neg(P)$ is equivalent to clauses S_1, S_2, and S_3. The stable models of $Neg(P)$ are, as expected, $\{s\}$ and $\{p, q\}$. □

We now formally define cardinality-constraint logic programs (CC-logic programs). The syntax of CC-logic programs admits two types of atoms: (i) ordinary atoms from set At and (ii) atoms of the form kXl where X is a finite set of atoms from At, k is a natural number (i.e. $k \in \omega$), $l \in \omega \cup \{\infty\}$ and $k \leq l$. Such new atoms will be called *cardinality constraints*. The intended meaning of an atom kXl is "out of atoms in X at least k but not more than l belong to the intended model."[2] Let us observe that the meaning of the negated atom, $\neg p$ is precisely the same as that of $\{p\}0$. Therefore we shall assume that the bodies of rules of CC-logic programs contain only atoms of the form kXl and atoms from At. That is, a CC-clause is either a clause of the form

$$p \leftarrow q_1, \ldots, q_m, k_1 X_1 l_1, \ldots, k_n X_n l_n \tag{6}$$

or

$$kXl \leftarrow q_1, \ldots, q_m, k_1 X_1 l_1, \ldots, k_n X_n l_n. \tag{7}$$

We note that either m or n can be zero. Thus the head of CC-clauses is either of the form p where p is an atom from At or kXl where k, X, and l satisfy the conventions described above. We say that a set of atoms $M \subseteq At$ satisfies the cardinality constraint kXl, $M \models kXl$ if $k \leq |X \cap M| \leq l$. Similarly we say that $M \models p$ where $p \in At$, if $p \in M$. By treating the commas in the bodies of clauses as conjunctions, we say that $M \models body(C)$ if all atoms occurring in $body(C)$ belong to M and all cardinality constraints occurring in $body(C)$ are satisfied by M. We say that M satisfies a clause C, $M \models C$, if either M does not satisfy the body of C or M satisfies the head of C.
A CC-logic program is a set of CC-clauses of the form (6) or (7). We say that M is a *model* of P, $M \models P$, if M satisfies all CC-clauses $C \in P$.
There is a particular class of programs called Horn constraint programs that play a role similar to that of Horn programs in ordinary logic programming. A *Horn constraint clause* is a CC-clause where the head of the clause is an ordinary atom and all the cardinality-constraint atoms $k_i X_i l_i$ in the body have $l_i = \infty$, i.e., it is of the form

$$H = p \leftarrow q_1, \ldots q_m, k_1 X_1, \ldots, k_n X_n.$$

Niemelä, Simons and Soininen [NSS99] observe that the one step provability operator associated with a such Horn constraint program P is monotone and

[2] Customarily we do not write the lower bound if it is 0 nor the upper bound, if it is ∞ but not always.

hence a Horn constraint program P has a least fixed point, M^P. That is, let $T_P : 2^{At} \to 2^{At}$ be defined so that for each $S \subseteq At$

$$T_P(S) = \{p : \exists H = p \leftarrow q_1, \ldots q_m, k_1X_1, \ldots, k_nX_n \in P \text{ such that } \{q_1, \ldots, q_m\} \subseteq S \text{ and for all } i = 1, \ldots n,\ |X_i \cap S| \geq k_i\}. \quad (8)$$

Again it is easy to see that T_P is monotone compact operator and that

$$\emptyset \subseteq T^1_P(\emptyset) \subseteq T^2_P(\emptyset) \subseteq T^2_P(\emptyset) \subseteq \cdots.$$

Thus

$$T^\omega_P(\emptyset) = \bigcup_{n=1}^{\infty} T^n_P(\emptyset)$$

is the least fixed point of T_P. Niemelä, Simons and Soininen observe that that $M^P = T^\omega_P(\emptyset)$ is the least model of P.
Next we introduce the analogue of the Gelfond-Lifschitz reduct for CC-logic clauses which we call the NSS-reduct. The NSS-reduct of a cardinality-constraint clause C with respect to a set M of ordinary atoms is defined as follows. First, eliminate all clauses C such $M \not\models \mathit{body}(C)$. Next,

1. if $C = p \leftarrow q_1, \ldots, q_m,\ k_1X_1l_1, \ldots, k_nX_nl_n$, then $C^M = p \leftarrow q_1, \ldots, q_m$, $k_1X_1, \ldots, k_nX_n$
2. If $C = kXl \leftarrow q_1, \ldots, q_m, k_1X_1l_1, \ldots, k_nX_nl_n$, then C^M is a collection of Horn constraint clauses of the form $p \leftarrow q_1, \ldots, q_m, k_1X_1, \ldots, k_nX_n$ for each $p \in X \cap M$.

Given a CC-program P, we let P^M denote a Horn constraint program consisting of all NNS-reducts of clauses $C \in P$. Following [NSS99], we say that M is a *CC-stable model* of P if (i) M is a model of P and (ii) M is the least model of the Horn constraint program P^M. It appears that, in the case of ordinary programs, the NSS-reduct prunes more clauses than GL-reduct[3].
We can also introduce a one-step provability operator $T_{P,M} : 2^{At} \to 2^{At}$ for any CC-program P and $M \subseteq At$. That is, for any $S \subseteq At$, we let $T_{P,M}(S)$ equal the set of all $p \in At$ such that either

(1) there is a clause $C = p \leftarrow q_1, \ldots, q_m, k_1X_1l_1, \ldots, k_nX_nl_n$ such that $M \models \mathit{body}(C)$, $\{q_1, \ldots, q_m\} \subseteq S$ and for all $i = 1, \ldots, n$, $|S \cap X_i| \geq k_i$ or

(2) there is a clause $C = kXl \leftarrow q_1, \ldots, q_m, k_1X_1l_1, \ldots, k_nX_nl_n$ such that $M \models \mathit{body}(C)$, $p \in (M \cap X)$, $\{q_1, \ldots, q_m\} \subseteq S$ and for all $i = 1, \ldots, n$, $|S \cap X_i| \geq k_i$.

Note that M affects $T_{P,M}(S)$ in two ways. First M restricts the clauses C

[3] M. Truszczyński (unpublished) proved that for models of P this reduct results in the same notion of stable model.

that can be used to put elements into $T_{P,M}(S)$ to be only those clauses such that $M \models body(C)$. Second, if $C = kXl \leftarrow q_1, \ldots, q_m, k_1X_1l_1, \ldots, k_nX_nl_n$ is such that $M \models body(C)$, then we can only use C to put elements from $M \cap X$ into $T_{P,M}(S)$. Nevertheless, it is easy to see that $T_{P,M}$ is a monotone operator so that

$$\emptyset \subseteq T^1_{P,M}(\emptyset) \subseteq T^2_{P,M}(\emptyset) \subseteq T^2_{P,M}(\emptyset) \subseteq \cdots .$$

Thus

$$T^{\omega}_{P,M}(\emptyset) = \bigcup_{n=1}^{\infty} T^n_P(\emptyset)$$

is the least fixed point of $T_{P,M}$. It is then easy to check that M is a CC-stable model of P if and only if (i) M is a model of P and (ii) $T^{\omega}_{P,M}(\emptyset) = M$.
Next we define the notion of a proof scheme for a CC-logic program and state a result analogous to Corollary 2.3. The basic idea is that a proof scheme should carry along all the information that is needed to see that an element p is in a CC-stable model M. In particular, when we deal with atoms of the form kXl, we need to know the information that $k \leq |M \cap X| \leq l$. Thus our proof schemes will carry along the information about what we expect $M \cap X$ to be. Formally, the notion of CC-proof scheme for a CC-logic program P is defined inductively as follows.

1. Whenever $C = p \leftarrow l'_1X_1l''_1, \ldots, l'_nX_nl''_n$ is a clause in P and for all $1 \leq i \leq n$, $l'_i = 0$ and Y_i is a subset of X_i such that $l'_i \leq |Y_i| \leq l''_i$ then

$$\langle\langle p\rangle, \langle C\rangle, \langle(l'_1X_1l''_1, Y_1), \ldots, (l'_nX_nl''_n, Y_n)\rangle\rangle$$

   is a CC-proof scheme for P. (The case $n = 0$ is allowed.)

2. Whenever $l'Xl'' \leftarrow l'_1X_1l''_1, \ldots, l'_nX_nl''_n$ is a clause in P and for all $1 \leq i \leq n$, $l'_i = 0$ and Y_i is a subset of X_i such that $l'_i \leq |Y_i| \leq l''_i$ and Y is a subset of X such that $k \leq |Y| \leq l$, then for every $p \in Y$

$$\langle\langle p\rangle, \langle C\rangle, \langle(kXl, Y), (l'Xl'', Y), (l'_1X_1l''_1, Y_1), \ldots, (l'_nX_nl''_n, Y_n)\rangle\rangle$$

   is a CC-proof scheme for P. (Again, the case $n = 0$ is allowed.)

3. Whenever

$$\mathfrak{S} = \langle\langle s_1, \ldots, s_w\rangle, \langle C_1, \ldots, C_w\rangle, \langle(k'_1X_1k''_1, Y_1), \ldots, (k'_rX_rk''_r, Y_r)\rangle\rangle$$

   is a CC-proof scheme in P and

$$C = p \leftarrow q_1, \ldots, q_m, l'_1Z_1l''_1, \ldots, l'_nZ_nl''_n$$

   is a clause in P such that $\{q_1, \ldots, q_m\} \subseteq \{s_1, \ldots, s_w\}$ and for all $1 \leq i \leq n$, $|Z_i \cap \{s_1, \ldots, s_w\}| \geq k_i$ and T_i is a subset of Z_i such that $l'_i \leq |T_i| \leq l''_i$, then

$$\langle\langle s_1, \ldots, s_w, p\rangle, \langle C_1, \ldots, C_w, C\rangle, \\ \langle(k'_1X_1k''_1, Y_1), \ldots, (k'_rX_rk''_r, Y_r), (l'_1Z_1l''_1, T_1) \ldots, (l'_nZ_nl''_n, T_n)\rangle\rangle$$

   is a CC-proof scheme for P.

4. Whenever

$$\mathfrak{S} = \langle\langle s_1, \ldots, s_w\rangle, \langle C_1, \ldots, C_w\rangle, \langle(k_1' X_1 k_1'', Y_1), \ldots, (k_r' X_r k_r'', Y_r)\rangle\rangle$$

is a proof scheme in P and

$$C = l' Z l'' \leftarrow q_1, \ldots, q_m, l_1' Z_1 l_1'', \ldots, l_n' Z_n l_n''$$

is a clause in P such that $\{q_1, \ldots, q_m\} \subseteq \{s_1, \ldots, s_w\}$, for all $1 \leq i \leq n$, $|Z_i \cap \{s_1, \ldots, s_w\}| \geq k_i$ and T_i is a subset of Z_i such that $l_i' \leq |T_i| \leq l_i''$ and Y is a subset of Z such that $l' \leq |Y| \leq l''$, then for every $p \in Y$

$$\begin{array}{l}\langle\langle s_1, \ldots, s_w, p\rangle, \langle C_1, \ldots, C_w, C\rangle, \\ \langle(l' Z l'', Y), (k_1' X_1 k_1'', Y_1), \ldots, (k_r' X_r k_r'', Y_r), (l_1' Z_1 l_1'', T_1) \ldots, (l_n' Z_n l_n'', T_n)\rangle\rangle\end{array}$$

is a proof scheme for P.

Now, given a CC-proof scheme

$$\mathfrak{S} = \langle\langle s_1, \ldots, s_k\rangle, \langle C_1, \ldots, C_k\rangle, \langle(k_1' X_1 k_1'', Y_1), \ldots, (k_r' X_r k_r'', Y_r)\rangle\rangle$$

for P, we say that $\mathfrak{S}$ is a CC-proof scheme for s_k in P. We shall refer to the sequence $\langle(k_1' X_1 k_1'', Y_1), \ldots, (k_r' X_r k_r'', Y_r)\rangle$ as the *cardinality constraint sequence* of $\mathfrak{S}$. If $M \subseteq At$, then we say that $\mathfrak{S}$ is *admitted* by M if $M \cap X_i = Y_i$ for $i = 1, \ldots, k$. We say that $\mathfrak{S}$ is *reduced* if $s_1, \ldots, s_k$ are pairwise distinct. We say that $\mathfrak{S}$ is *self-consistent* if for all $i = 1, \ldots, r$, $Y_i = X_i \cap (\bigcup_{j=1}^{r} Y_j)$.
It is easy to see that if M admits a proof scheme

$$\mathfrak{S} = \langle\langle s_1, \ldots, s_k\rangle, \langle C_1, \ldots, C_k\rangle, \langle(k_1' X_1 k_1'', Y_1), \ldots, (k_r' X_r k_r'', Y_r)\rangle\rangle$$

then $\mathfrak{S}$ is self-consistent and $M \models k_1' X_1 k_1'', \ldots, M \models k_r' X_r k_r''$ since the sets Y_i, $i = 1, \ldots, k$ witnesses that the corresponding constraints are satisfied. It is then easy to see by induction that M must satisfy the body of every clause C_i in $\mathfrak{S}$. Thus a proof scheme provides a derivation of an atom and proposes a way of satisfying constraints occurring in bodies of all clauses used in that derivation. Moreover, the proof schemes for ordinary programs can be easily transformed into the CC-proof schemes for the corresponding cardinality-constraint program. That is, instead of having an element r be in set of constraints in the third component of a proof scheme for an ordinary logic program, we simply add a pair $(0\{r\}0, \emptyset)$ to the cardinality constraint sequence of the corresponding CC-proof scheme because a set $M \subseteq At$ will admit such a proof scheme if and only if $r \notin M$.

Example 2.3 Let P be the following CC-logic program:

$C_1 = 1\{p, q\}2 \leftarrow r, 0\{t\}0$
$C_2 = r \leftarrow 0\{s\}0$
$C_3 = s \leftarrow 0\{r\}0$

The CC-program P has four stable models: $M_1 = \{r, p\}$, $M_2 = \{r, q\}$, $M_3 = \{r, p, q\}$ and $M_4 = \{s\}$. M_1 and M_2 are included in M_3.
The triple $\langle\langle r, p\rangle, \langle C_2, C_1\rangle, \langle(1\{p, q\}2, \{p\}), (0\{s\}0, \emptyset), (0\{t\}0, \emptyset)\rangle\rangle$ is admitted by M_1, but not by M_2. Also, the scheme $\mathfrak{S}_3$

$$\langle\langle r, p, q\rangle, \langle C_2, C_1, C_1\rangle, \langle(1\{p, q\}2, \{p, q\}), (0\{s\}0, \emptyset), (0\{t\}0, \emptyset)\rangle\rangle$$

is admitted by M_3 but not by M_1, because atom q does not belong to M_1. Let us observe that clause C_1 is used in $\mathfrak{S}_3$ twice, once to derive p and again to derive q. This phenomenon does *not* occur in case of normal logic programs where where, in a reduced scheme, every clause can be used at most once. □

The following analogue of Corollary 2.3 is proved in [MR03].

Proposition 2.5 *Let P be a CC-logic program, and let $M \subseteq At$, $M \models P$. Then M is a CC-stable model of P if and only if every element p of M possesses a proof scheme $\mathfrak{S}_p$ such that $\mathfrak{S}_p$ is admitted by M.*

Next we want to prove the analogue of Theorem 2.4 for CC-programs. It turns out we need to be careful. To this end, we shall say a CC-program P is *totally negative* if all the clauses of P are of the form

$$p \leftarrow 0T0 \tag{9}$$

for some set finite T or

$$kXl \leftarrow 0T0 \tag{10}$$

for some set finite T. In the case of ordinary logic programs, we were able to show that for every logic program P, there was totally negative program Q such that P and Q have the same stable models and the set of heads of clauses in P contains the set of heads of clauses in Q. Our next example will show that it is not the case that for every CC-program P, there is a totally negative CC-logic program Q such that P and Q have the same CC-stable models and the set of heads of clauses in P contains the set of heads of clauses of Q.

Example 2.4 Consider the CC-logic program P with the following two clauses.

$C_1 : 0\{1, 2\}1 \leftarrow$
$C_2 : 3 \leftarrow 1$

It is easy to check that P has three CC-stable models, $M_1 = \emptyset$, $M_2 = \{2\}$, and $M_3 = \{1, 3\}$. Now if Q is a totally negative program such that the set of heads of P contains the set of heads of Q, then Q must consists of two types of clauses.

$$E_1 : 0\{1, 2, \} \leftarrow 0A0 \text{for some set } A \text{ and}$$
$$E_2 : 3 \leftarrow 0B0 \text{for some set } B.$$

However one can not have any clauses of the type E_2 since NSS-reduct of Q relative to $\emptyset$ would be a clause D of the form

$$D : 3 \leftarrow 0B\infty$$

for some set B. But then E would show that $3 \in T_{Q,\emptyset}(\emptyset)$ so that $\emptyset$ not a CC-stable model of Q. But if Q has no clauses of the form of E_2, then all the clauses of Q must be of the form E_1. But this is impossible since then there would be no way to have $3 \in T_{Q,\{1,3\}}(\emptyset)$ and hence $\{1,3\}$ is not a stable model of Q. Thus there can be no such Q. □

Despite Example 2.4, we can still use CC-proof schemes to show that for every CC-logic program P, there is a CC-logic program Q such that P and Q have the same CC-stable models, the set of heads of clauses of P contains the set of heads of clauses of Q, and every clause of Q is of the form

$$p \leftarrow q_1, \ldots, q_m, \neg b_1, \ldots, \neg b_n \tag{11}$$

or

$$kXl \leftarrow q_1, \ldots, q_m, \neg b_1, \ldots, \neg b_n \tag{12}$$

That is, the bodies of the all the clauses of Q are of the form of bodies for ordinary logic programs. We shall call CC-logic programs all of whose clauses are of the form (11) or (12) *body-normal* CC-logic programs. We note that we can re-write clauses of the form (11) or (12) as follows.

$$p \leftarrow q_1, \ldots, q_m, 0\{b_1, \ldots, b_n\}0 \tag{13}$$

or

$$kXl \leftarrow q_1, \ldots, q_m, 0\{b_1, \ldots, b_n\}0 \tag{14}$$

Thus we shall assume that the clauses of a body-normal CC-logic programs are always of the form (13) or (14).
Now suppose that we are given a CC-logic program P. Our goal is to construct a body normal CC-logic program $BN(P)$ such that P and $BN(P)$ have the sames set of CC-stable models. Suppose that $\mathfrak{S}$ is a reduced proof scheme

$$\mathfrak{S} = \langle \langle s_1, \ldots, s_n \rangle, \langle C_1, \ldots, C_n \rangle, \langle (k_1 X_1 l_1, T_1), \ldots (k_t X_t l_t, T_t) \rangle \rangle$$

of P where

$$C_n = p \leftarrow q_1, \ldots, q_m, l'_1 Y_1 l''_1, \ldots, l'_r Y_r l''_r.$$

Then we construct clause

$$C_{\mathfrak{S}} = s_n \leftarrow s_1, \ldots, s_{n-1},\ 0R_{\mathfrak{S}}0$$

where $R_{\mathfrak{S}} = \bigcup_{i=1}^{t} Z_i$ and, for each $i = 1, \ldots, t$,

$$Z_i = \begin{cases} X_i - T_i & \text{if } |X_i| < l_i, \\ \emptyset & \text{otherwise.} \end{cases} \tag{15}$$

If $\mathfrak{S}$ is a reduced proof scheme

$$\mathfrak{S} = \langle\langle s_1, \ldots, s_n\rangle, \langle C_1, \ldots, C_n\rangle, \langle(k_1, X_1 l_1, T_1), \ldots (k_t X_t l_t, T_t)\rangle\rangle$$

of P where

$$C_n = kXl \leftarrow q_1, \ldots, q_m, l'_1 Y_1 l''_1, \ldots, l'_r Y_r l''_r.$$

Then we construct clause

$$C_{\mathfrak{S}} = kXl \leftarrow s_1, \ldots, s_{n-1},\ 0R_{\mathfrak{S}}0$$

where $R_{\mathfrak{S}} = \bigcup_{i=1}^{t} Z_i$ and, for each $i = 1, \ldots, t$,

$$Z_i = \begin{cases} X_i - T_i & \text{if } |X_i| < l_i, \\ \emptyset & \text{otherwise.} \end{cases} \tag{16}$$

Let $BN_1(P)$ be the program whose clauses are precisely the set of $C_{\mathfrak{S}}$ such that $\mathfrak{S}$ is a reduced proof scheme of P.
$BN_1(P)$ is not quite the program that we want. In fact, we have to add some additional clauses to $BN_1(P)$ to get a CC-logic program $BN(P)$ and make one addition assumption about P before we can prove an analogue of Theorem 2.4 for CC-progams with P and $BN(P)$.
That is, first, CC-programs allow clauses of the form

$$C = 0R0 \leftarrow q_1, \ldots, q_m, k_1 X_1 l_1, \ldots, k_n X_n l_n. \tag{17}$$

We call clauses of the form of (17), *empty head clauses*. The problem with empty head clauses is that our definition of CC-proof scheme has no mechanism to reflect such clauses. That is, such clause cannot be used to put elements into a CC-stable model but they do restrict the set of models of programs that have such clauses. Hence our definition of $BN(P)$ is not sensitive to the existence of such clauses. However, we can easily construct a CC-logic program that is equivalent to P which does not have any empty head clauses. That is, we introduce an atom A which does not occur in P. Then for each clause C in P of the form of (17), we introduce a clause C_r for each $r \in R$,

$$C_r = A \leftarrow r, q_1, \ldots, q_m, k_1 X_1 l_1, \ldots, k_n X_n l_n, \neg A. \tag{18}$$

JEFF: We use here $\neg A$ i/s $0\{A\}0$. I Ihink this needs to be fixed
We call the resulting program $\overline{P}$. Note A cannot be in any CC-stable model of $\overline{P}$. That is, if $A \in M$, then M does not satisfy the body of any clause C_r. Hence there will be no clause D in $\overline{P}$ with A in the head such that $M \models body(D)$. It then follows that $A \notin T^{\omega}_{\overline{P},M}(\emptyset)$ and hence M is not a CC-stable model of $\overline{P}$. Now suppose that M is a CC-stable model of $\overline{P}$ such that $M \models body(C)$. Then $\{q_1, \ldots, q_m\} \subseteq M$ and $k_i \leq |M \cap X_i| \leq l_i$ for $i = 1, \ldots, n$. Then it is easy to see that it cannot be that $r \in M$ with $r \in R$. That is, if $r \in M \cap R$, then, since $M = T^{\omega}_{\overline{P},M}(\emptyset)$, there will be a k such that $r, q_1, \ldots, q_m \in T^{k}_{\overline{P},M}(\emptyset)$. But then C_r

would witness that $A \in T^{k+1}_{\overline{P},M}(\emptyset)$. Thus $M \cap R = \emptyset$ and hence $M \models C$. Thus every CC-stable model of $\overline{P}$ which satisfies $body(C)$ also satisfies C. It follows that M models P and that none of the clauses C_r that we introduced can be used to put elements into $T^{\omega}_{\overline{P},M}(\emptyset)$. Hence it is the case that

$$M = T^{\omega}_{\overline{P},M}(\emptyset) = T^{\omega}_{P,M}(\emptyset).$$

Thus M is a stable model of P.
One the other hand, if M is a CC-stable model of P, then $A \notin M$ since A does not occur in P. Moreover, if $M \models body(C)$, then $M \models head(C)$ and hence $M \cap R = \emptyset$. It then follows that $M \models C_r$ for all $r \in R$ since $M \not\models body(C_r)$. Thus M is a model of $\overline{P}$. Again, it will be the case that

$$M = T^{\omega}_{P,M}(\emptyset) = T^{\omega}_{\overline{P},M}(\emptyset)$$

so that M is a stable model of $\overline{P}$. Thus we have shown that P and $\overline{P}$ have the same set of CC-stable models.
Next we consider the clauses that we have to add to $BN_1(P)$ to obtain a CC-logic program $BN(P)$ which is equivalent to P. Suppose that $\mathfrak{S}$ is a reduced proof scheme

$$\mathfrak{S} = \langle\langle s_1, \ldots, s_n\rangle, \langle C_1, \ldots, C_n\rangle, \langle (k_1, X_1 l_1, T_1), \ldots (k_t X_t l_t, T_t)\rangle\rangle$$

of P, C is a clause of P of the form,

$$C = kXl \leftarrow q_1, \ldots, q_m, l'_1 A_1 l''_1, \ldots, l'_r A_r l''_r,$$

and $\vec{B} = (B_1, \ldots, B_r)$ is a sequence of sets such that

1. $\{q_1, \ldots, q_m\} \subseteq \{s_1, \ldots, s_n\}$ and
2. $|X \cap \{s_1, \ldots, s_n\}| > l$,
3. $|A_i \cap \{s_1, \ldots, s_n\}| \geq l'_i$ for $i = 1, \ldots, r$, and
4. for $i = 1, \ldots, r$, $B_i \subseteq A_i$ and $l'_i \leq |B_i| \leq l''_i$.

Then we construct clause

$$C_{\mathfrak{S},C,\vec{B}} \ = \ A \leftarrow s_1, \ldots, s_n, \ 0R_{\mathfrak{S},C}0, \neg A$$

where A is a new atom which does not occur in P and $R_{\mathfrak{S}} = (\bigcup_{i=1}^{t} Z_i) \cup (\bigcup_{j=1}^{r} D_i)$ where for each $i = 1, \ldots, t$,

$$Z_i = \begin{cases} X_i - T_i & \text{if } |X_i| < l_i, \\ \emptyset & \text{otherwise} \end{cases} \tag{19}$$

and

$$D_i = \begin{cases} A_i - B_i & \text{if } |X_i| < l_i, \\ \emptyset & \text{otherwise.} \end{cases} \tag{20}$$

We add a clause $C_{\mathfrak{S},C,\vec{B}}$ to $BN_1(P)$ for each such triple $\langle \mathfrak{S}, C, \vec{B}\rangle$ to get our desired program $BN(P)$. Clearly $BN(P)$ is a body-normal CC-logic program. Our next example explains why we need to add clauses of the form $C_{\mathfrak{S},C,\vec{B}}$ to $BN(P)$.

Example 2.5 Consider the CC-logic program

$C_1 : 1 \leftarrow$
$C_2 : 2 \leftarrow 1\{1,2\}2$
$C_3 : 0\{1,2\}1 \leftarrow$.

It is easy to see that P does not have any CC-stable models. Since clearly, clauses C_1 and C_2 will force 1 and 2 to be in $T_{P,M}(\emptyset)$ for any $M \subseteq \{1,2\}$. Thus the only possible CC-stable model is $M = \{1,2\}$. But then M satisfies $body(C_3)$ but does not satisfy the $head(C_3)$ so that M is not a model of P. Thus P has no CC-stable models.
There are 11 reduced CC-proof schemes of P. There are 3 CC-proof schemes of length 1.
$\mathfrak{S}_1 = \langle\langle 1\rangle, \langle C_1\rangle, \langle\rangle\rangle$,
$\mathfrak{S}_2 = \langle\langle 1\rangle, \langle C_3\rangle, \langle(0\{1,2\}1, \{1\})\rangle\rangle$,
$\mathfrak{S}_3 = \langle\langle 2\rangle, \langle C_3\rangle, \langle(0\{1,2\}1, \{2\})\rangle\rangle$.
There are reduced 6 reduced CC-proof schemes of length 2 with conclusion 2.
$\mathfrak{S}_4 = \langle\langle 1,2\rangle, \langle C_1, C_3\rangle, \langle(0\{1,2\}, \{2\})\rangle\rangle$,
$\mathfrak{S}_5 = \langle\langle 1,2\rangle, \langle C_3, C_3\rangle, \langle(0\{1,2\}, \{1\}), (0\{1,2\}, \{1\})\rangle\rangle$,
$\mathfrak{S}_6 = \langle\langle 1,2\rangle, \langle C_1, C_2\rangle, \langle(1\{1,2\}2, \{1\})\rangle\rangle$,
$\mathfrak{S}_7 = \langle\langle 1,2\rangle, \langle C_1, C_2\rangle, \langle(1\{1,2\}2, \{1,2\})\rangle\rangle$.
$\mathfrak{S}_8 = \langle\langle 1,2\rangle, \langle C_3, C_1\rangle, \langle(0\{1,2\}1, \{1\}), (1\{1,2\}2, \{1\})\rangle\rangle$,
$\mathfrak{S}_9 = \langle\langle 1,2\rangle, \langle C_3, C_1\rangle, \langle(0\{1,2\}1, \{1\}), (1\{1,2\}2, \{1,2\})\rangle\rangle$,
Finally there are 2 reduced proof schemes of length 2 with conclusion 1.
$\mathfrak{S}_{11} = \langle\langle 2,1\rangle, \langle C_3, C_1\rangle, \langle(0\{1,2\}1, \{2\})\rangle\rangle$, and
$\mathfrak{S}_{12} = \langle\langle 2,1\rangle, \langle C_3, C_3\rangle, \langle(0\{1,2\}1, \{1\}), (0\{1,2\}1, \{2\})\rangle\rangle$.

Thus $C_{\mathfrak{S}_1} = 1 \leftarrow$,
$C_{\mathfrak{S}_2} = 1 \leftarrow 0\{2\}0$,
$C_{\mathfrak{S}_3} = 2 \leftarrow 0\{1\}0)$,
$C_{\mathfrak{S}_4} = 2 \leftarrow 1, 0\{1\}0$,
$C_{\mathfrak{S}_5} = 2 \leftarrow 1, 0\{1,2\}0$,
$C_{\mathfrak{S}_6} = 2 \leftarrow 1$,
$C_{\mathfrak{S}_7} = 2 \leftarrow 1$,
$C_{\mathfrak{S}_8} = 2 \leftarrow 1, 0\{2\}0$,
$C_{\mathfrak{S}_9} = 1 \leftarrow 0\{2\}0$,
$C_{\mathfrak{S}_{11}} = 1 \leftarrow 0\{1\}0$, and
$C_{\mathfrak{S}_{12}} = 1 \leftarrow 0\{1,2\}0$.

It is the easy to see that $BN_1(P)$ which consists of $C_{\mathfrak{S}_1}, \ldots, C_{\mathfrak{S}_{11}}$ has one CC-

stable model, namely, $M = \{1,2\}$. Hence $BN_1(P)$ is not equivalent to P. Note it easy to see that all but clauses $C_{\mathfrak{S}_1}$, $C_{\mathfrak{S}_3}$ and $C_{\mathfrak{S}_6}$ are superfluous so that $BN_1(P)$ is equivalent to clauses:
$D_1 : 1 \leftarrow,$
$D_2 : 2 \leftarrow 0\{1\}0$ and
$D_3 : 2 \leftarrow 1.$
However, the clause C_3 and the empty sequence $\vec{B} = \langle\rangle$ together with any of the proof schemes $\mathfrak{S}_4, \ldots, C_{\mathfrak{S}_{11}}$ generate the following clauses in $BN(P)$.
$C_{\mathfrak{S}_4,C_3,\vec{B}} = A \leftarrow 1,2,0\{1\}0,\neg A,$
$C_{\mathfrak{S}_5,C_3,\vec{B}} = A \leftarrow 1,2,0\{1,2\}0,\neg A,$
$C_{\mathfrak{S}_6,C_3,\vec{B}} = A \leftarrow 1,2,\neg A,$
$C_{\mathfrak{S}_7,C_3,\vec{B}} = A \leftarrow 1,2,\neg A,$
$C_{\mathfrak{S}_8,C_3,\vec{B}} = A \leftarrow 1,2,0\{2\}0,\neg A,$
$C_{\mathfrak{S}_9,C_3,\vec{B}} = A \leftarrow 1,2,0\{2\}0,\neg A,$
$C_{\mathfrak{S}_{10},C_3,\vec{B}} = A \leftarrow 2,1,0\{1\})0,\neg A,$ and
$C_{\mathfrak{S}_{11},C_3,\vec{B}} = A \leftarrow 2,1,0\{1,2\})0,\neg A.$

It can not be that A is in any CC-stable model of $BN(P)$ because for any M which contains A, M does not satisfy any of the bodies of $C_{\mathfrak{S}_i,C_3,\vec{B}}$ for $i = 4,\ldots,11$. Hence A cannot be in $T^{\omega}_{BN(P),M}(\emptyset)$. Thus the only possible CC-models are subsets of $\{1,2\}$. But clauses $C_{\mathfrak{S}_1}$ and $C_{\mathfrak{S}_6}$ will force $\{1,2\} \subseteq T^{\omega}_{BN(P),M}(\emptyset)$ for any M so that the only possible CC-stable model of $BN(P)$ is $M = \{1,2\}$. Note that the clause $C_{\mathfrak{S}_6,C_3,\vec{B}} = A \leftarrow 1,2,\neg A$ prevents $\{1,2\}$ from being a CC-stable model of $BN(P)$ so that $BN(P)$ has no stable models and hence is equivalent to P. Moreover, it is easy to see that all the clauses with $i \neq 6$ are superfluous so that $BN(P)$ is equivalent to the following program:
$D_1 : 1 \leftarrow,$
$D_2 : 2 \leftarrow 0\{1\}0,$
$D_3 : 2 \leftarrow 1,$ and
$D_4 : A \leftarrow 1,2,\neg A.$

We then have the following analogue of Theorem 2.4. This the promised normal form theorem.

Theorem 2.6 *For any CC-logic program P which has no empty head clauses, P and $BN(P)$ have the same set of stable models.*

Proof. First we shall show that if $M \models P$, then $M \models BN(P)$. Assume that $M \models P$. Then we claim if $\mathfrak{S}$ is a reduced CC-proof scheme of P and $M \models body(C_{\mathfrak{S}})$, then $M \models C_{\mathfrak{S}}$ and hence $M \models BN(P)$. First consider the case where $\mathfrak{S}$ is of length 1. There are two cases.

Case 1. There is a clause $C = p \leftarrow l'_1 X_1 l''_1, \ldots, l'_n X_n l''_n$ is a clause in P such that

$$\mathfrak{S} = \langle\langle p\rangle, \langle C\rangle, \langle(l'_1 X_1 l''_1, Y_1), \ldots, (l'_n X_n l''_n, Y_n)\rangle\rangle$$

where for all $1 \leq i \leq n$, $l'_i = 0$ and Y_i is a subset of X_i such that $l'_i \leq |Y_i| \leq l''_i$
In this case,

$$C_{\mathfrak{S}} = p \leftarrow 0R_{\mathfrak{S}}0$$

where where $R_{\mathfrak{S}} = \bigcup_{i=1}^{t} Z_i$ and, for each $i = 1, \ldots, t$,

$$Z_i = \begin{cases} X_i - Y_i & \text{if } |X_i| < l_i, \\ \emptyset & \text{otherwise.} \end{cases} \tag{21}$$

Since $M \models \mathit{body}(C_{\mathfrak{S}})$, it must be the case that $M \cap R_{\mathfrak{S}} = \emptyset$. Thus if $l''_i < |X_i|$, $M \cap X_i \subseteq Y_i$ and hence $0 = l'_i \leq |M \cap X_i| \leq |Y_i| \leq l''_i$. Clearly, if $l''_i \geq |X_i|$, then $0 = l'_i \leq |M \cap X_i| \leq l''_i$. It then follows $M \models \mathit{body}(C)$. Since $M \models P$, it must be the case that $p \in M$ and hence $M \models C_{\mathfrak{S}}$.

Case 2. There is a clause $C = \$l'Xl'' \leftarrow l'_1X_1l''_1, \ldots, l'_nX_nl''_n$ in P

$$\mathfrak{S} = \langle\langle p\rangle, \langle C\rangle, \langle(l'Xl'', Y), (l'_1X_1l''_1, Y_1), \ldots, (l'_nX_nl''_n, Y_n)\rangle\rangle$$

where for all $1 \leq i \leq n$, $l'_i = 0$ and Y_i is a subset of X_i such that $l'_i \leq |Y_i| \leq l''_i$, Y is a subset of X such that $k \leq |Y| \leq l$ and $p \in Y$.
In this case,

$$C_{\mathfrak{S}} = kXl \leftarrow 0R_{\mathfrak{S}}0$$

where where $R_{\mathfrak{S}} = \bigcup_{i=1}^{t} Z_i$ and, for each $i = 1, \ldots, t$,

$$Z_i = \begin{cases} X_i - Y_i & \text{if } |X_i| < l_i, \\ \emptyset & \text{otherwise.} \end{cases} \tag{22}$$

Since $M \models \mathit{body}(C)$, it must be the case that $M \cap R_{\mathfrak{S}} = \emptyset$. Thus if $l''_i < |X_i|$, $M \cap X_i \subseteq Y_i$ and hence $0 = l'_i \leq |M \cap X_i| \leq |Y_i| \leq l''_i$. Clearly, if $l''_i \geq |X_i|$, then $0 = l'_i \leq |M \cap X_i| \leq l''_i$. It then follows $M \models \mathit{body}(C)$. Since $M \models P$, it must be the case that $M \models kXl$ and hence $M \models C_{\mathfrak{S}}$.

Next consider the case where $\mathfrak{S}$ has length $w + 1$, where $w \geq 1$. Again there are two cases.

Case 3. $\mathfrak{S}$ is of the form

$$\mathfrak{S} = \langle\langle s_1, \ldots, s_w, p\rangle, \langle C_1, \ldots, C_w, C\rangle, \\ \langle(k'_1X_1k''_1, Y_1), \ldots, (k'_rX_rk''_r, Y_r), (l'_1Z_1l''_1, T_1) \ldots, (l'_nZ_nl''_n, T_n)\rangle\rangle$$

where

$$\mathfrak{U} = \langle\langle s_1, \ldots, s_w\rangle, \langle C_1, \ldots, C_w\rangle, \langle(k'_1X_1k''_1, Y_1), \ldots, (k'_rX_rk''_r, Y_r)\rangle\rangle$$

is a CC-proof scheme in P and

$$C = p \leftarrow q_1, \ldots, q_m, l'_1Z_1l''_1, \ldots, l'_nZ_nl''_n$$

is a clause in P such that $\{q_1, \ldots, q_m\} \subseteq \{s_1, \ldots, s_w\}$ and for all $1 \leq i \leq n$, $|Z_i \cap \{s_1, \ldots, s_w\}| \geq k_i$ and T_i is a subset of Z_i such that $l'_i \leq |T_i| \leq l''_i$.
In this case,

$$C_{\mathfrak{S}} = p \leftarrow s_1, \ldots, s_w,\ 0R_{\mathfrak{S}}0$$

where where $R_{\mathfrak{S}} = \bigcup_{i=1}^{t} Z_i$ and, for each $i = 1, \ldots, t$,

$$Z_i = \begin{cases} X_i - Y_i & \text{if } |X_i| < l_i, \\ \emptyset & \text{otherwise.} \end{cases} \tag{23}$$

Case 4. $\mathfrak{S}$ is of the form

$$\mathfrak{S} = \langle\langle s_1, \ldots, s_w, p\rangle, \langle C_1, \ldots, C_w, C\rangle, \langle (l'Zl'', Y), (k'_1X_1k''_1, Y_1), \ldots, (k'_rX_rk''_r, Y_r), (l'_1Z_1l''_1, T_1) \ldots, (l'_nZ_nl''_n, T_n)\rangle\rangle$$

where

$$\mathfrak{U} = \langle\langle s_1, \ldots, s_w\rangle, \langle C_1, \ldots, C_w\rangle, \langle (k'_1X_1k''_1, Y_1), \ldots, (k'_rX_rk''_r, Y_r)\rangle\rangle$$

is a proof scheme in P and

$$C = l'Zl'' \leftarrow q_1, \ldots, q_m, l'_1Z_1l''_1, \ldots, l'_nZ_nl''_n$$

is a clause in P such that $\{q_1, \ldots, q_m\} \subseteq \{s_1, \ldots, s_w\}$, for all $1 \leq i \leq n$, $|Z_i \cap \{s_1, \ldots, s_w\}| \geq k_i$ and T_i is a subset of Z_i such that $l'_i \leq |T_i| \leq l''_i$ and Y is a subset of Z such that $l' \leq |Y| \leq l''$, and $p \in Y$.
In this case,

$$C_{\mathfrak{S}} = kXl \leftarrow s_1, \ldots, s_w,\ 0R_{\mathfrak{S}}0$$

where where $R_{\mathfrak{S}} = \bigcup_{i=1}^{t} Z_i$ and, for each $i = 1, \ldots, t$,

$$Z_i = \begin{cases} X_i - Y_i & \text{if } |X_i| < l_i, \\ \emptyset & \text{otherwise.} \end{cases} \tag{24}$$

Since $M \models \mathit{body}(C)$, it must be the case that $M \cap R_{\mathfrak{S}} = \emptyset$ and $\{s_1, \ldots, s_w\} \subseteq M$. Thus if $l''_i < |X_i|$, $M \cap \{s_1, \ldots, s_w\} \subseteq M \cap X_i \subseteq Y_i$ and hence $l'_i \leq |M \cap X_i| \leq |Y_i| \leq l''_i$. Clearly, if $l''_i \geq |X_i|$, then $l'_i \leq |M \cap \{s_1, \ldots, s_w\}| \leq |M \cap X_i| \leq l''_i$. It then follows $M \models \mathit{body}(C)$. Since $M \models P$, it must be the case that $m \models kXl$ and hence $M \models C_{\mathfrak{S}}$.
Next, consider clauses of the form $C_{\mathfrak{S},C,\vec{B}}$. That is, suppose $\mathfrak{S}$ is a reduced proof scheme

$$\mathfrak{S} = \langle\langle s_1, \ldots, s_n\rangle, \langle C_1, \ldots, C_n\rangle, \langle (k_1, X_1l_1, T_1), \ldots (k_tX_tl_t, T_t)\rangle\rangle$$

of P, C is a clause of P of the form,

$$C = kXl \leftarrow q_1, \ldots, q_m, l'_1A_1l''_1, \ldots, l'_rA_rl''_r,$$

and $\vec{B} = (B_1, \ldots, B_r)$ is a sequence of sets such that

1. $\{q_1, \ldots, q_m\} \subseteq \{s_1, \ldots, s_n\}$ and
2. $|X \cap \{s_1, \ldots, s_n\}| > l$,
3. $|A_i \cap \{s_1, \ldots, s_n\}| \geq l'_i$ for $i = 1, \ldots, r$, and
4. for $i = 1, \ldots, r$, $B_i \subseteq A_i$ and $l'_i \leq |B_i| \leq l''_i$.

In this case,

$$C_{\mathfrak{S},C,\vec{B}} \ := \ A \leftarrow s_1, \ldots, s_n, \ 0R_{\mathfrak{S},C,\vec{B}}0, \neg A$$

where A is a new atom which does not occur in P and $R_{\mathfrak{S}} = (\bigcup_{i=1}^{t} Z_i) \cup (\bigcup_{j=1}^{r} D_i)$ where for each $i = 1, \ldots, t$,

$$Z_i = \begin{cases} X_i - T_i & \text{if } |X_i| < l_i, \\ \emptyset & \text{otherwise} \end{cases} \tag{25}$$

and

$$D_i = \begin{cases} A_i - B_i & \text{if } |X_i| < l_i, \\ \emptyset & \text{otherwise.} \end{cases} \tag{26}$$

Since M is model of P, $A \notin M$. Now if $M \models body(C_{\mathfrak{S},C,\vec{B}})$, then we know that $\{s_1, \ldots, s_n\} \subseteq X$. Thus for all $i = 1, \ldots, r$,

$$l'_i \leq |A_i \cap \{s_1, \ldots, s_n\}| \leq |M \cap A_i|.$$

Moreover, it must be the case that $M \cap A_i \subseteq B_i$ since $X_i - B_i \subseteq R_{\mathfrak{S},C,\vec{B}}$ and $M \cap R_{\mathfrak{S},C,\vec{B}} = \emptyset$. Thus if $l''_i < |X_i|$ so that if $l''_i < |X_i|$, then $l'_i \leq |M \cap A_i| \leq l''_i$. Clearly if $l''_i \geq |X_i|$, then $l'_i \leq |M \cap A_i| \leq l''_i$. It follows that $M \models body(C)$. But this is impossible because, then the fact that $M \models P$ implies that $k \leq |M \cap X| \leq l$. However by assumption $|M \cap X| \geq |\{s_1, \ldots, s_n\} \cap X| > l$. Thus it must be the case that $M \not\models body((C_{\mathfrak{S},C,\vec{B}})$ for any such $\mathfrak{S}$, C and $\vec{B}$ and hence $M \models C_{\mathfrak{S},C,\vec{B}}$.
Next we show that for all models M of P,

$$T^{\omega}_{P,M}(\emptyset) = T^{\omega}_{BN(P),M}(\emptyset). \tag{27}$$

It will easily follow from (27) that if M is a CC-stable model of P, then M is a CC-stable model of $BN(P)$.
Assume that $M \models P$. By the arguement above, we know that $M \models BN(P)$. Let us note $T^{\omega}_{P,M}(\emptyset)$ equals the set of all $p \in At_P$ such that there there is a proof scheme

$$\mathfrak{S} = \langle\langle s_1, \ldots, s_n\rangle, \langle C_1, \ldots, C_n\rangle, \langle (k_1, X_1 l_1, T_1), \ldots (k_t X_t l_t, T_t)\rangle\rangle$$

of P with $s_n = p$ which is admitted by M. Now if $\mathfrak{S}$ is not reduced, it easy to see that we can trim $\mathfrak{S}$ to produce a reduced proof scheme with the same conclusion. Thus there is no loss in generality in assuming that $\mathfrak{S}$ is reduced. This given, we shall prove the following lemma.

Lemma 2.7 *The set of all p such that p is the conclusion of a reduced proof scheme admitted by M is contained in $T^{\omega}_{BN(P),M}(\emptyset)$*

Note that in our case, Lemma 2.7 implies

$$T^{\omega}_{P,M}(\emptyset) \subseteq T^{\omega}_{BN(P),M}(\emptyset).$$

Proof. Suppose that $p \in At_P$ is such that there there is a proof scheme

$$\mathfrak{S} = \langle\langle s_1, \ldots, s_n\rangle, \langle C_1, \ldots, C_n\rangle, \langle (k_1 X_1 l_1, T_1), \ldots (k_t X_t l_t, T_t)\rangle\rangle$$

of P with $s_n = p$ which is admitted by M. We shall prove by induction on the length n of $\mathfrak{S}$ that $p \in T^{\omega}_{BN(P),M}(\emptyset)$.
First consider the case where $\mathfrak{S}$ is of length 1. There are two subcases.

Case A. There is a clause $C = p \leftarrow l'_1 X_1 l''_1, \ldots, l'_n X_n l''_n$ is a clause in P such that

$$\mathfrak{S} = \langle\langle p\rangle, \langle C\rangle, \langle (l'_1 X_1 l''_1, Y_1), \ldots, (l'_n X_n l''_n, Y_n)\rangle\rangle$$

where for all $1 \leq i \leq n$, $l'_i = 0$ and Y_i is a subset of X_i such that $l'_i \leq |Y_i| \leq l''_i$
In this case,

$$C_{\mathfrak{S}} = p \leftarrow 0 R_{\mathfrak{S}} 0$$

where where $R_{\mathfrak{S}} = \bigcup_{i=1}^{t} Z_i$ and, for each $i = 1, \ldots, t$,

$$Z_i = \begin{cases} X_i - Y_i & \text{if } |X_i| < l_i, \\ \emptyset & \text{otherwise.} \end{cases} \tag{28}$$

Since M admits $\mathfrak{S}$, it must be the case that $M \cap R_{\mathfrak{S}} = \emptyset$. It then follows $M \models body(C_{\mathfrak{S}})$. Thus $C_{\mathfrak{S}}$ witnesses that $p \in T_{BN(P),M}(\emptyset)$.

Case B. There is a clause $C = l' X l'' \leftarrow l'_1 X_1 l''_1, \ldots, l'_n X_n l''_n$ in P

$$\mathfrak{S} = \langle\langle p\rangle, \langle C\rangle, \langle (l' X l'', Y), (l'_1 X_1 l''_1, Y_1), \ldots, (l'_n X_n l''_n, Y_n)\rangle\rangle$$

where for all $1 \leq i \leq n$, $l'_i = 0$ and Y_i is a subset of X_i such that $l'_i \leq |Y_i| \leq l''_i$, Y is a subset of X such that $k \leq |Y| \leq l$ and $p \in Y$.
In this case,

$$C_{\mathfrak{S}} = kXl \leftarrow 0 R_{\mathfrak{S}} 0$$

where where $R_{\mathfrak{S}} = \bigcup_{i=1}^{t} Z_i$ and, for each $i = 1, \ldots, t$,

$$Z_i = \begin{cases} X_i - Y_i & \text{if } |X_i| < l_i, \\ \emptyset & \text{otherwise.} \end{cases} \tag{29}$$

Since M admits $\mathfrak{S}$, $M \models body(C)$ and $p \in M$. Thus it must be the case that $M \cap R_{\mathfrak{S}} = \emptyset$ and hence $M \models body(C_{\mathfrak{S}})$. But then $C_{\mathfrak{S}}$ witnesses that $p \in T_{BN(P),M}(\emptyset)$.

Next consider the case where $\mathfrak{S}$ has length $w+1$, where $w \geq 1$. By induction, we can assume that the conclusion of any CC-proof scheme $\mathfrak{U}$ of P admitted by M where length of $\mathfrak{U}$ is less than or equal to w is in $T^{\omega}_{BN(P),M}(\emptyset)$. Again there are two cases.

Case C. $\mathfrak{S}$ is of the form

$$\begin{aligned}\mathfrak{S} = \langle\langle s_1,\dots,s_w,p\rangle,\langle C_1,\dots,C_w,C\rangle,&\\ \langle(k'_1X_1k''_1,Y_1),\dots,(k'_rX_rk''_r,Y_r),&\\ (l'_1Z_1l''_1,T_1)\dots,(l'_nZ_nl''_n,T_n)\rangle\rangle&\end{aligned}$$

where

$$\mathfrak{U} = \langle\langle s_1,\dots,s_w\rangle,\langle C_1,\dots,C_w\rangle,\langle(k'_1X_1k''_1,Y_1),\dots,(k'_rX_rk''_r,Y_r)\rangle\rangle$$

is a CC-proof scheme in P and

$$C = p \leftarrow q_1,\dots,q_m, l'_1Z_1l''_1,\dots,l'_nZ_nl''_n$$

is a clause in P such that $\{q_1,\dots,q_m\} \subseteq \{s_1,\dots,s_w\}$ and for all $1 \leq i \leq n$, $|Z_i \cap \{s_1,\dots,s_w\}| \geq k_i$ and T_i is a subset of Z_i such that $l'_i \leq |T_i| \leq l''_i$.
In this case,

$$C_{\mathfrak{S}} = p \leftarrow s_1,\dots,s_w,\ 0R_{\mathfrak{S}}0$$

where $R_{\mathfrak{S}} = \bigcup_{i=1}^t Z_i$ and, for each $i = 1,\dots,t$,

$$Z_i = \begin{cases} X_i - Y_i & \text{if } |X_i| < l_i,\\ \emptyset & \text{otherwise.}\end{cases} \tag{30}$$

Since M admits $\mathfrak{S}$, it must be the case that $M \cap R_{\mathfrak{S}} = \emptyset$. Since each of $s_1,\dots,s_w$ are the conclusions of self-consistent reduced proofs schemes of length $\leq w$ which are admitted by M, it follows from our induction hypothesis that $\{s_1,\dots,s_w\} \subseteq T^{\omega}_{BN(P),M}(\emptyset)$. Thus there must exist a k such that

$$\{s_1,\dots,s_w\} \subseteq T^{k}_{BN(P),M}(\emptyset).$$

Thus $C_{\mathfrak{S}}$ witness that

$$p \in T_{BN(P),M}(T^{k}_{BN(P),M}(\emptyset)) = T^{k+1}_{BN(P),M}(\emptyset).$$

Hence $p \in T^{\omega}_{BN(P),M}(\emptyset)$.

Case D. $\mathfrak{S}$ is of the form

$$\begin{aligned}\mathfrak{S} = \langle\langle s_1,\dots,s_w,p\rangle,\langle C_1,\dots,C_w,C\rangle,&\\ \langle(l'Zl'',Y),(k'_1X_1k''_1,Y_1),\dots,(k'_rX_rk''_r,Y_r),&\\ (l'_1Z_1l''_1,T_1)\dots,(l'_nZ_nl''_n,T_n)\rangle\rangle&\end{aligned}$$

where

$$\mathfrak{U} = \langle\langle s_1, \ldots, s_w\rangle, \langle C_1, \ldots, C_w\rangle, \langle (k'_1 X_1 k''_1, Y_1), \ldots, (k'_r X_r k''_r, Y_r)\rangle\rangle$$

is a proof scheme in P and

$$C = l' Z l'' \leftarrow q_1, \ldots, q_m, l'_1 Z_1 l''_1, \ldots, l'_n Z_n l''_n$$

is a clause in P such that $\{q_1, \ldots, q_m\} \subseteq \{s_1, \ldots, s_w\}$, for all $1 \leq i \leq n$, $|Z_i \cap \{s_1, \ldots, s_w\}| \geq k_i$ and T_i is a subset of Z_i such that $l'_i \leq |T_i| \leq l''_i$ and Y is a subset of Z such that $l' \leq |Y| \leq l''$, and $p \in Y$.
In this case,

$$C_{\mathfrak{S}} = kXl \leftarrow s_1, \ldots, s_w,\ 0R_{\mathfrak{S}}0$$

where where $R_{\mathfrak{S}} = \bigcup_{i=1}^{t} Z_i$ and, for each $i = 1, \ldots, t$,

$$Z_i = \begin{cases} X_i - Y_i & \text{if } |X_i| < l_i, \\ \emptyset & \text{otherwise.} \end{cases} \tag{31}$$

Since M admits $\mathfrak{S}$, it must be the case that $M \cap R_{\mathfrak{S}} = \emptyset$ and $p \in M$. As in Case C, we can argue that there must exist a k such that

$$\{s_1, \ldots, s_w\} \subseteq T^k_{BN(P),M}(\emptyset).$$

Thus $C_{\mathfrak{S}}$ witness that

$$p \in T_{BN(P),M}(T^k_{BN(P),M}(\emptyset)) = T^{k+1}_{BN(P),M}(\emptyset).$$

Hence $p \in T^{\omega}_{BN(P),M}(\emptyset)$. This completes the proof of the lemma. □
Next we have to show that if $M \models P$, then

$$T^{\omega}_{BN(P),M}(\emptyset) \subseteq T^{\omega}_{P,M}(\emptyset).$$

Since $M \models P$, we know that $A \notin M$ and $M \models BN(P)$. Now suppose that $p \in T^{\omega}_{BN(P),M}(\emptyset)$. Then again here is a CC-proof scheme **JEFF**

$$\mathfrak{U} = \langle\langle a_1, \ldots, a_r\rangle, \langle C_1, \ldots, C_r\rangle, \langle (k_1 X_1 l_1, Y_1), \ldots, (k_s X_s l_s, Y_s)\rangle\rangle$$

of $BN(P)$ with $a_r = p$ which is admitted by M. We shall prove by induction on the length of $\mathfrak{U}$, that $p \in TP, M^{\omega}(\emptyset)$. We have already shown that $M \not\models$ *body*$(C_{\mathfrak{S},C,\vec{B}})$ for any of the clauses $C_{\mathfrak{S},C,\vec{B}}$ that are in $BN(P)$. Thus there are no CC-proof schemes of $BN(P)$ admitted by M which contains any clause of the form $C_{\mathfrak{S},C,\vec{B}}$. Thus all clauses which occur a CC-proof scheme of $BN(P)$ admitted by M must be of the form $C_{\mathfrak{S}}$ for some CC-proof scheme of P.
First assume that $\mathfrak{U}$ is of length 1. Thus

$$\mathfrak{U} = \langle p, C_{\mathfrak{S}}, \langle (0R_{\mathfrak{S}}0, \emptyset)\rangle\rangle$$

where $\mathfrak{S}$ is a CC-proof scheme of P of length 1. There are two cases.

Case I. There is a clause $C = p \leftarrow l'_1 X_1 l''_1, \ldots, l'_n X_n l''_n$ is a clause in P such that

$$\mathfrak{S} = \langle\langle p\rangle, \langle C\rangle, \langle (l'_1 X_1 l''_1, Y_1), \ldots, (l'_n X_n l''_n, Y_n)\rangle\rangle$$

where for all $1 \leq i \leq n$, $l'_i = 0$ and Y_i is a subset of X_i such that $l'_i \leq |Y_i| \leq l''_i$. In this case,

$$C_{\mathfrak{S}} = p \leftarrow 0 R_{\mathfrak{S}} 0$$

where $R_{\mathfrak{S}} = \bigcup_{i=1}^{t} Z_i$ and, for each $i = 1, \ldots, t$,

$$Z_i = \begin{cases} X_i - Y_i & \text{if } |X_i| < l_i, \\ \emptyset & \text{otherwise.} \end{cases} \tag{32}$$

Since M admits $\mathfrak{U}$, it must be the case that $M \cap R_{\mathfrak{S}} = \emptyset$. Thus if $l''_i < |X_i|$, $M \cap X_i \subseteq Y_i$ and hence $0 = l'_i \leq |M \cap X_i| \leq |Y_i| \leq l''_i$. Clearly, if $l''_i \geq |X_i|$, then $0 = l'_i \leq |M \cap X_i| \leq l''_i$. It then follows $M \models \mathit{body}(C)$. Thus C witnesses that $p \in T_{P,M}(\emptyset)$.

Case II. There is a clause $C = l' X l'' \leftarrow l'_1 X_1 l''_1, \ldots, l'_n X_n l''_n$ in P

$$\mathfrak{S} = \langle\langle p\rangle, \langle C\rangle, \langle (l' X l'', Y), (l'_1 X_1 l''_1, Y_1), \ldots, (l'_n X_n l''_n, Y_n)\rangle\rangle$$

where for all $1 \leq i \leq n$, $l'_i = 0$ and Y_i is a subset of X_i such that $l'_i \leq |Y_i| \leq l''_i$, Y is a subset of X such that $k \leq |Y| \leq l$ and $p \in Y$.
In this case,

$$C_{\mathfrak{S}} = kXl \leftarrow 0 R_{\mathfrak{S}} 0$$

where where $R_{\mathfrak{S}} = \bigcup_{i=1}^{t} Z_i$ and, for each $i = 1, \ldots, t$,

$$Z_i = \begin{cases} X_i - Y_i & \text{if } |X_i| < l_i, \\ \emptyset & \text{otherwise.} \end{cases} \tag{33}$$

Since M admits $\mathfrak{U}$, it must be the case that $M \cap R_{\mathfrak{S}} = \emptyset$ and that $p \in M$. Thus if $l''_i < |X_i|$, $M \cap X_i \subseteq Y_i$ and hence $0 = l'_i \leq |M \cap X_i| \leq |Y_i| \leq l''_i$. Clearly, if $l''_i \geq |X_i|$, then $0 = l'_i \leq |M \cap X_i| \leq l''_i$. It then follows $M \models \mathit{body}(C)$. Thus C witnesses that $p \in T_{P,M}(\emptyset)$.

Next consider the case where $\mathfrak{U}$ has length $w + 1$, where $w \geq 1$. Thus

$$\mathfrak{U} = \langle a_1, \ldots, a_w, p\rangle, \langle C_{\mathfrak{S}_1}, \ldots, C_{\mathfrak{S}_{w+1}}, \langle (k_1 X_1, l_1, T_1), \ldots, (k_s X_s l_s, T_s)\rangle\rangle.$$

By induction, we can assume that the conclusion of any CC-proof scheme $\mathfrak{V}$ of $BN(P)$ admitted by M where the length of $\mathfrak{W}$ is less than or equal to w is in $T^{\omega}_{P,M}(\emptyset)$. Clearly each of $a_1, \ldots, a_w$ are the conclusions of CC-proof schemes of $BN(P)$ admitted by M and hence $\{a_1, \ldots, a_w\} \subseteq T^{\omega}_{P,M}(\emptyset)$ Thus there is a k such that

$$\{a_1, \ldots, a_w\} \subseteq T^{\omega}_{P,M}(\emptyset).$$

Again there are two cases.

Case III. $\mathfrak{S}_{w+1}$ is of the form

$$\mathfrak{S}_{w+1} = \langle\langle s_1, \ldots, s_m, p\rangle, \langle C_1, \ldots, C_m, C\rangle, \langle(k'_1 X_1 k''_1, Y_1), \ldots, (k'_r X_r k''_r, Y_r), (l'_1 Z_1 l''_1, T_1) \ldots, (l'_n Z_n l''_n, T_n)\rangle\rangle$$

where

$$\mathfrak{W} = \langle\langle s_1, \ldots, s_m\rangle, \langle C_1, \ldots, C_m\rangle, \langle(k'_1 X_1 k''_1, Y_1), \ldots, (k'_r X_r k''_r, Y_r)\rangle\rangle$$

is a CC-proof scheme in P and

$$C = p \leftarrow q_1, \ldots, q_s, l'_1 Z_1 l''_1, \ldots, l'_t Z_t l''_t$$

is a clause in P such that $\{q_1, \ldots, q_s\} \subseteq \{s_1, \ldots, s_m\}$ and for all $1 \leq i \leq t$, $|Z_i \cap \{s_1, \ldots, s_m\}| \geq k_i$ and T_i is a subset of Z_i such that $l'_i \leq |T_i| \leq l''_i$.
In this case,

$$C_{\mathfrak{S}_{w+1}} = p \leftarrow s_1, \ldots, s_m,\ 0R_{\mathfrak{S}_{w+1}}0$$

where $R_{\mathfrak{S}_{w+1}} = \bigcup_{i=1}^{t} Z_i$ and, for each $i = 1, \ldots, t$,

$$Z_i = \begin{cases} X_i - Y_i & \text{if } |X_i| < l_i, \\ \emptyset & \text{otherwise.} \end{cases} \tag{34}$$

It follows that $(0R_{\mathfrak{S}_{w+1}}0, \emptyset)$ is one of the constraints of $\mathfrak{U}$. Since M admits $\mathfrak{U}$, it must be the case that $M \cap R_{\mathfrak{S}} = \emptyset$. Moreover, $\{s_1, \ldots, s_m\}$ must be a subset of $\{a_1, \ldots, a_w\}$. Thus

$$\{q_1, \ldots, q_s\} \subseteq \{s_1, \ldots, s_m\} \subseteq \{a_1, \ldots, a_w\} \subseteq T^k_{P,M}(\emptyset).$$

Note that since for all $1 \leq i \leq t$, $|Z_i \cap \{s_1, \ldots, s_m\}| \geq k_i$, it must be the case that for all $1 \leq i \leq t$, $|Z_i \cap T^k_{P,M}(\emptyset)| \geq k_i$. But then C witnesses that

$$p \in T_{P,M}(T^k_{P,M}(\emptyset)) = T^{k+1}_{P,M}(\emptyset).$$

Hence $p \in T^{\omega}_{P,M}(\emptyset)$.

Case IV. $\mathfrak{S}_{w+1}$ is of the form

$$\mathfrak{S}_{w+1} = \langle\langle s_1, \ldots, s_m, p\rangle, \langle C_1, \ldots, C_m, C\rangle, \langle(l' Z l'', Y), (k'_1 X_1 k''_1, Y_1), \ldots, (k'_r X_r k''_r, Y_r), (l'_1 Z_1 l''_1, T_1) \ldots, (l'_n Z_n l''_n, T_n)\rangle\rangle$$

where

$$\mathfrak{W} = \langle\langle s_1, \ldots, s_w\rangle, \langle C_1, \ldots, C_w\rangle, \langle(k'_1 X_1 k''_1, Y_1), \ldots, (k'_r X_r k''_r, Y_r)\rangle\rangle$$

is a proof scheme in P and

$$C = l'Zl'' \leftarrow q_1, \ldots, q_m, l'_1Z_1l''_1, \ldots, l'_nZ_nl''_n$$

is a clause in P such that $\{q_1, \ldots, q_m\} \subseteq \{s_1, \ldots, s_w\}$, for all $1 \leq i \leq n$, $|Z_i \cap \{s_1, \ldots, s_w\}| \geq k_i$ and T_i is a subset of Z_i such that $l'_i \leq |T_i| \leq l''_i$ and Y is a subset of Z such that $l' \leq |Y| \leq l''$, and $p \in Y$.
In this case,

$$C_{\mathfrak{S}_{w+1}} = l'Zl'' \leftarrow s_1, \ldots, s_w,\ 0R_{\mathfrak{S}}0$$

where where $R_{\mathfrak{S}} = \bigcup_{i=1}^{t} Z_i$ and, for each $i = 1, \ldots, t$,

$$Z_i = \begin{cases} X_i - Y_i & \text{if } |X_i| < l_i, \\ \emptyset & \text{otherwise.} \end{cases} \tag{35}$$

It follows that $(0R_{\mathfrak{S}_{w+1}}0, \emptyset)$ and $(l'Zl'', M \cap Z)$ are among the constraints of $\mathfrak{U}$. Since M admits $\mathfrak{U}$, it must be the case that $p \in M$ and $M \cap R_{\mathfrak{S}} = \emptyset$. Moreover, $\{s_1, \ldots, s_m\}$ must be a subset of $\{a_1, \ldots, a_w\}$. Thus

$$\{q_1, \ldots, q_s\} \subseteq \{s_1, \ldots, s_m\} \subseteq \{a_1, \ldots, a_w\} \subseteq T^k_{P,M}(\emptyset).$$

Finally note that since for all $1 \leq i \leq t$, $|Z_i \cap \{s_1, \ldots, s_m\}| \geq k_i$, it must be the case that for all $1 \leq i \leq t$, $|Z_i \cap T^k_{P,M}(\emptyset)| \geq k_i$. But then C witnesses that

$$p \in T_{P,M}(T^k_{P,M}(\emptyset)) = T^{k+1}_{P,M}(\emptyset).$$

Hence $p \in T^{\omega}_{P,M}(\emptyset)$.

Thus we have proved that every CC-stable model of P is a CC-stable model of $BN(P)$. To complete our proof, we must show that every CC-stable model of $BN(P)$ is a CC-stable model of P.
So assume that M is a CC-stable model of $BN(P)$. In cannot be that $A \in M$. That is, if $A \in M$, then $M \not\models C_{\mathfrak{S},C,\vec{B}}$ for any clause $C_{\mathfrak{S},C,\vec{B}}$ in $BN(P)$. However, these are the only clauses in which A occurs in the head. Thus if $A \in M$, then $A \notin T^{\omega}_{BN(P),M}(\emptyset)$ and hence M is not a CC-stable model of $BN(P)$.
First we have to prove that M is a model of P. Since $M \models BN(P)$, we know that if $p \in M = T^{\omega}_{BN(P),M}(\emptyset)$, there is a CC-proof scheme **JEFF**

$$\mathfrak{U} = \langle\langle a_1, \ldots, a_r\rangle, \langle C_1, \ldots, C_r\rangle, \langle (k_1X_1l_1, Y_1), \ldots, (k_sX_sl_s, Y_s)\rangle\rangle$$

of $BN(P)$ with $a_r = p$ which is admitted by M. Note that since $A \notin M$, it cannot be the case that any of the rules $C_{\mathfrak{S},C,\vec{B}}$ can be used in a CC-proof scheme of $BN(P)$ admitted by M since all such rules have A in the head. Thus if a rule of the form $C_{\mathfrak{S},C,\vec{B}}$ was in a CC-proof scheme of $BN(P)$ admitted by M, it would follows that A is the conclusion of CC-proof scheme of $BN(P)$ admitted by M and hence A would be in M since M is a CC-stable model of $BN(P)$. We shall prove by induction on the length of $\mathfrak{U}$, that p is the conclusion of CC-proof scheme of P which is admitted by M.

First assume that $\mathfrak{U}$ is of length 1. Thus

$$\mathfrak{U} = \langle p, C_{\mathfrak{S}}, \langle (0R_{\mathfrak{S}}0, \emptyset)\rangle\rangle$$

where $\mathfrak{S}$ is a CC-proof scheme of P of length 1. There are two cases.

Case AI. There is a clause $C = p \leftarrow l'_1X_1l''_1, \ldots, l'_nX_nl''_n$ is a clause in P such that

$$\mathfrak{S} = \langle\langle p\rangle, \langle C\rangle, \langle (l'_1X_1l''_1, Y_1), \ldots, (l'_nX_nl''_n, Y_n)\rangle\rangle$$

where for all $1 \leq i \leq n$, $l'_i = 0$ and Y_i is a subset of X_i such that $l'_i \leq |Y_i| \leq l''_i$. In this case,

$$C_{\mathfrak{S}} = p \leftarrow 0R_{\mathfrak{S}}0$$

where $R_{\mathfrak{S}} = \bigcup_{i=1}^{t} Z_i$ and, for each $i = 1, \ldots, t$,

$$Z_i = \begin{cases} X_i - Y_i & \text{if } |X_i| < l_i, \\ \emptyset & \text{otherwise.} \end{cases} \tag{36}$$

Since M admits $\mathfrak{U}$, it must be the case that $M \cap R_{\mathfrak{S}} = \emptyset$. Thus if $l''_i < |X_i|$, $M \cap X_i \subseteq Y_i$ and hence $0 = l'_i \leq |M \cap X_i| \leq |Y_i| \leq l''_i$. Clearly, if $l''_i \geq |X_i|$, then $0 = l'_i \leq |M \cap X_i| \leq l''_i$. It then follows $M \models \mathit{body}(C)$. Thus

$$\langle\langle p\rangle, \langle C\rangle, \langle (l'_1X_1l''_1, M \cap X_1), \ldots, (l'_nX_nl''_n, M \cap X_n)\rangle\rangle$$

is CC-proof scheme of P with conclusion p admitted by M.

Case AII. There is a clause $C = l'Xl'' \leftarrow l'_1X_1l''_1, \ldots, l'_nX_nl''_n$ in P

$$\mathfrak{S} = \langle\langle p\rangle, \langle C\rangle, \langle (l'Xl'', Y), (l'_1X_1l''_1, Y_1), \ldots, (l'_nX_nl''_n, Y_n)\rangle\rangle$$

where for all $1 \leq i \leq n$, $l'_i = 0$ and Y_i is a subset of X_i such that $l'_i \leq |Y_i| \leq l''_i$, Y is a subset of X such that $k \leq |Y| \leq l$ and $p \in Y$.
In this case,

$$C_{\mathfrak{S}} = kXl \leftarrow 0R_{\mathfrak{S}}0$$

where where $R_{\mathfrak{S}} = \bigcup_{i=1}^{t} Z_i$ and, for each $i = 1, \ldots, t$,

$$Z_i = \begin{cases} X_i - Y_i & \text{if } |X_i| < l_i, \\ \emptyset & \text{otherwise.} \end{cases} \tag{37}$$

Since M admits $\mathfrak{U}$, it must be the case that $M \cap R_{\mathfrak{S}} = \emptyset$ and that $p \in M$. Thus if $l''_i < |X_i|$, $M \cap X_i \subseteq Y_i$ and hence $0 = l'_i \leq |M \cap X_i| \leq |Y_i| \leq l''_i$. Clearly, if $l''_i \geq |X_i|$, then $0 = l'_i \leq |M \cap X_i| \leq l''_i$. It then follows $M \models \mathit{body}(C)$. Moreover, since $M \models BN(P)$ and $M \models \mathit{body}(C_{\mathfrak{S}})$, it must be the case that $M \models kXL$. Thus $k \leq M \cap X \leq l$ and

$$\langle\langle p\rangle, \langle C\rangle, \langle (kXl, M \cap X), (l'_1X_1l''_1, M \cap X_1), \ldots, (l'_nX_nl''_n, M \cap X_n)\rangle\rangle$$

is CC-proof scheme of P with conclusion p admitted by M.

Next, consider the case where $\mathfrak{U}$ has length $w+1$, where $w \geq 1$. Thus

$$\mathfrak{U} = \langle\langle a_1, \ldots, a_w, p\rangle, \langle C_{\mathfrak{S}_1}, \ldots, C_{\mathfrak{S}_{w+1}}\rangle, \langle (k_1 X_1 l_1, T_1), \ldots, (k_s X_s l_s, T_s)\rangle\rangle.$$

By induction, we can assume that the conclusion c of any CC-proof scheme $\mathfrak{W}$ of $BN(P)$ admitted by M where the length of $\mathfrak{W}$ is less than or equal to w is also the conclusion of CC-proof scheme of P admitted by M. Clearly each of $a_1, \ldots, a_w$ are the conclusions of CC-proof schemes of $BN(P)$ admitted by M and hence, for each i, there is a CC-proof scheme, $\mathfrak{E}_i$, of P with conclusion a_i admitted by M where

$$\mathfrak{E}_i = \langle\langle b^i_1, \ldots, b^i_{m_i}, a_i\rangle, \langle D^i_1, \ldots, D^i_{m_i}, D^i\rangle, \\ \langle (k'_{1,i} X^i_1 k''_{1,i}, W^i_1), \ldots, (k'_{f_i,i} X^i_{f_i,i} k''_{f_i,i}, W_{f_i,i})\rangle\rangle.$$

Moreover since M is a CC-stable model of $BN(P)$, we have that $\{a_1, \ldots, a_w\} \subseteq T^{\omega}_{P,M}(\emptyset) = M$. Again there are two cases.

Case AIII. $\mathfrak{S}_{w+1}$ is of the form

$$\mathfrak{S}_{w+1} = \langle\langle s_1, \ldots, s_m, p\rangle, \langle C_1, \ldots, C_m, C\rangle, \\ \langle (k'_1 X_1 k''_1, Y_1), \ldots, (k'_r X_r k''_r, Y_r), \\ (l'_1 Z_1 l''_1, T_1) \ldots, (l'_n Z_n l''_n, T_n)\rangle\rangle$$

where

$$\mathfrak{V} = \langle\langle s_1, \ldots, s_m\rangle, \langle C_1, \ldots, C_m\rangle, \langle (k'_1 X_1 k''_1, Y_1), \ldots, (k'_r X_r k''_r, Y_r)\rangle\rangle$$

is a CC-proof scheme in P and

$$C = p \leftarrow q_1, \ldots, q_s, l'_1 Z_1 l''_1, \ldots, l'_t Z_t l''_t$$

is a clause in P such that $\{q_1, \ldots, q_s\} \subseteq \{s_1, \ldots, s_m\}$ and for all $1 \leq i \leq t$, $|Z_i \cap \{s_1, \ldots, s_m\}| \geq k_i$ and T_i is a subset of Z_i such that $l'_i \leq |T_i| \leq l''_i$.
In this case,

$$C_{\mathfrak{S}_{w+1}} = p \leftarrow s_1, \ldots, s_m,\ 0 R_{\mathfrak{S}_{w+1}} 0$$

where $R_{\mathfrak{S}_{w+1}} = \bigcup_{i=1}^{t} Z_i$ and, for each $i = 1, \ldots, t$,

$$Z_i = \begin{cases} X_i - Y_i & \text{if } |X_i| < l_i, \\ \emptyset & \text{otherwise.} \end{cases} \tag{38}$$

It follows that $(0 R_{\mathfrak{S}_{w+1}} 0, \emptyset)$ is one of the constraints of $\mathfrak{U}$. Since M admits $\mathfrak{U}$, it must be the case that $M \cap R_{\mathfrak{S}} = \emptyset$. Moreover, $\{s_1, \ldots, s_m\}$ must be a subset of $\{a_1, \ldots, a_w\}$. Thus $\{q_1, \ldots, q_s\} \subseteq \{s_1, \ldots, s_m\} \subseteq \{a_1, \ldots, a_w\}$. Note that since for all $1 \leq i \leq t$, $|Z_i \cap \{s_1, \ldots, s_m\}| \geq k_i$. Since M is a stable model $BN(P)$, it must be the case that $\{a_1, \ldots, a_w\} \subseteq M$ and hence

$$|Z_i \cap M| \geq |Z_i \cap \{a_1, \ldots, a_w\}| \geq |Z_i \cap \{s_1, \ldots, s_m\}| \geq k_i.$$

Moreover if $l''_i < |X_i|$, $M \cap X_i \subseteq Y_i$ and hence $0 = l'_i \leq |M \cap X_i| \leq |Y_i| \leq l''_i$. Clearly, if $l''_i \geq |X_i|$, then $0 = l'_i \leq |M \cap X_i| \leq l''_i$. It then follows $M \models body(C)$. But then we can concatonate the CC-proof scheme of P, $\mathfrak{E}_1, \ldots, \mathfrak{E}_w$ and add the clause C to get a CC-proof scheme of P with conclusion p as follows: **JEFF**

$$\begin{aligned}\langle\langle b^1_1, \ldots, b^1_{m_1}, a_1, \ldots, b^w_1, \ldots, b^w_{m_w}, a_w, p\rangle,\\ \langle D^1_1, \ldots, D^1_{m_1}, D^1, \ldots, D^w_1, \ldots, D^w_{m_w}, D^w, C, \rangle,\\ \langle(l'_1 Z_1 l''_1, M \cap Z_1), \ldots, (l'_t Z_t l''_t, M \cap Z_t),\\ (k'_{1,1} X^1_1 k''_{1,1}, W^1_1), \ldots, (k'_{f_1,1} X^1_{f_1,1} k''_{f_1,1}, W_{f_1,1}), \ldots\\ (k'_{1,w} X^w_1 k''_{1,w}, W^w_1), \ldots, (k'_{f_w,w} X^w_{f_w,w} k''_{f_w,w}, W_{f_w,w})\rangle\rangle.\end{aligned}$$

JEFFI believe it is Case V not IV, please check **Case V.** $\mathfrak{S}_{w+1}$ is of the form

$$\begin{aligned}\mathfrak{S}_{w+1} = \langle\langle s_1, \ldots, s_m, p\rangle, \langle C_1, \ldots, C_m, C\rangle,\\ \langle(l'Zl'', Y), (k'_1 X_1 k''_1, Y_1), \ldots, (k'_r X_r k''_r, Y_r),\\ (l'_1 Z_1 l''_1, T_1) \ldots, (l'_n Z_n l''_n, T_n)\rangle\rangle\end{aligned}$$

where

$$\mathfrak{W} = \langle\langle s_1, \ldots, s_w\rangle, \langle C_1, \ldots, C_w\rangle, \langle(k'_1 X_1 k''_1, Y_1), \ldots, (k'_r X_r k''_r, Y_r)\rangle\rangle$$

is a proof scheme in P and

$$C = l'Zl'' \leftarrow q_1, \ldots, q_m, l'_1 Z_1 l''_1, \ldots, l'_n Z_n l''_n$$

is a clause in P such that $\{q_1, \ldots, q_m\} \subseteq \{s_1, \ldots, s_w\}$, for all $1 \leq i \leq n$, $|Z_i \cap \{s_1, \ldots, s_w\}| \geq k_i$ and T_i is a subset of Z_i such that $l'_i \leq |T_i| \leq l''_i$ and Y is a subset of Z such that $l' \leq |Y| \leq l''$, and $p \in Y$.
In this case,

$$C_{\mathfrak{S}} = l'Zl'' \leftarrow s_1, \ldots, s_w,\ 0R_{\mathfrak{S}}0$$

where where $R_{\mathfrak{S}} = \bigcup_{i=1}^t Z_i$ and, for each $i = 1, \ldots, t$,

$$Z_i = \begin{cases} X_i - Y_i & \text{if } |X_i| < l_i, \\ \emptyset & \text{otherwise.} \end{cases} \tag{39}$$

It follows that $(0R_{\mathfrak{S}_{w+1}}0, \emptyset)$ and $(l'Zl'', M \cap Z)$ are among the constraints of $\mathfrak{U}$. Since M admits $\mathfrak{U}$, it must be the case that $p \in M$, $l' \leq |M \cap Z| \leq l''$ and

$M \cap R_{\mathfrak{S}} = \emptyset$. Moreover, $\{s_1, \ldots, s_m\}$ must be a subset of $\{a_1, \ldots, a_w\}$. Thus $\{q_1, \ldots, q_s\} \subseteq \{s_1, \ldots, s_m\} \subseteq \{a_1, \ldots, a_w\}$. Note that since for all $1 \leq i \leq t$, $|Z_i \cap \{s_1, \ldots, s_m\}| \geq k_i$. Since M is a stable model $BN(P)$, it must be the case that $\{a_1, \ldots, a_w\} \subseteq M$ and hence

$$|Z_i \cap M| \geq |Z_i \cap \{a_1, \ldots, a_w\}| \geq |Z_i \cap \{s_1, \ldots, s_m\}| \geq k_i.$$

Moreover if $l''_i < |X_i|$, $M \cap X_i \subseteq Y_i$ and hence $0 = l'_i \leq |M \cap X_i| \leq |Y_i| \leq l''_i$. Clearly, if $l''_i \geq |X_i|$, then $0 = l'_i \leq |M \cap X_i| \leq l''_i$. It then follows $M \models \mathit{body}(C)$. But then we can concatonate the CC-proof scheme of P, $\mathfrak{E}_1, \ldots, \mathfrak{E}_w$ and add the clause C to get a CC-proof scheme of P with conclusion p as follows: **JEFF**

$$\begin{aligned}
&\langle\langle b^1_1, \ldots, b^1_{m_1}, a_1, \ldots, b^w_1, \ldots, b^w_{m_w}, a_w, p\rangle, \\
&\qquad \langle D^1_1, \ldots, D^1_{m_1}, D^1, \ldots, D^w_1, \ldots, D^w_{m_w}, D^w, C, \rangle, \\
&\qquad \langle (l'Zl'', M \cap Z), (l'_1 Z_1 l''_1, M \cap Z_1), \ldots, (l'_t Z_t l''_t, M \cap Z_t), \\
&\qquad (k'_{1,1} X^1_1 k''_{1,1}, W^1_1), \ldots, (k'_{f_1,1} X^1_{f_1,1} k''_{f_1,1}, W_{f_1,1}), \ldots \\
&\qquad\qquad (k'_{1,w} X^w_1 k''_{1,w}, W^w_1), \ldots, (k'_{f_w,w} X^w_{f_w,w} k''_{f_w,w}, W_{f_w,w})\rangle\rangle.
\end{aligned}$$

We are now in a position to complete our proof that $M \models P$. That is, suppose that

$$C = p \leftarrow q_1, \ldots, q_w, k_1 X_1 l_1, \ldots, k_n X_n l_n$$

is a clause of C such that $M \models \mathit{body}(C)$. Then $q_1, \ldots, q_m$ are elements of M and hence there are CC-proof schemes, $\mathfrak{E}_i$, of P with conclusion a_i admitted by M where

$$\begin{aligned}
\mathfrak{E}_i = &\langle\langle b^i_1, \ldots, b^i_{m_i}, q_i\rangle, \langle D^i_1, \ldots, D^i_{m_i}, D^i\rangle, \\
&\qquad \langle (k'_{1,i} X^i_1 k''_{1,i}, W^i_1), \ldots, (k'_{f_i,i} X^i_{f_i,i} k''_{f_i,i}, W^i_{f_i,i})\rangle\rangle.
\end{aligned}$$

for $i = 1, \ldots, w$. Moreover, for all $1 \leq j \leq n$, $k_j \leq |M \cap X_j| \leq l_j$. It follows that **JEFF**

$$\begin{aligned}
\mathfrak{E} = &\langle\langle b^1_1, \ldots, b^1_{m_1}, q_1, \ldots, b^w_1, \ldots, b^w_{m_w}, q_w, p, \rangle, \\
&\qquad \langle D^1_1, \ldots, D^1_{m_1}, D^1, \ldots, D^w_1, \ldots, D^w_{m_w}, D^w, C, \rangle, \\
&\qquad \langle (k_1 X_1 l_1, M \cap X_1), \ldots, (k_n X_n l_n, M \cap X_t), \\
&\qquad (k'_{1,1} X^1_1 k''_{1,1}, W^1_1), \ldots, (k'_{f_1,1} X^1_{f_1,1} k''_{f_1,1}, W_{f_1,1}), \ldots \\
&\qquad\qquad (k'_{1,w} X^w_1 k''_{1,w}, W^w_1), \ldots, (k'_{f_w,w} X^w_{f_w,w} k''_{f_w,w}, W_{f_w,w})\rangle\rangle.
\end{aligned}$$

is a CC-proof scheme of P with conclusion p admitted by M. Now if $\mathfrak{E}$ is not reduced, we can trim it to produced a reduced CC-proof scheme $\mathfrak{F}$ with conclusion p admitted by M. We have already shown that the conclusion of any CC-proof scheme of P which is admitted by M is in $T^{\omega}_{BN(P),M}(\emptyset)$. Thus $p \in T^{\omega}_{BN(P),M}(\emptyset) = M$ and hence $M \models C$.
Next suppose that

$$C = kXl \leftarrow q_1, \ldots, q_w, k_1 X_1 l_1, \ldots, k_n X_n l_n$$

is a clause of C such that $M \models body(C)$. Then $q_1, \ldots, q_m$ are elements of M and hence there are CC-proof schemes, $\mathfrak{E}_i$, of P with conclusion a_i admitted by M where

$$\mathfrak{E}_i = \langle\langle b^i_1, \ldots, b^i_{m_i}, q_i\rangle, \langle D^i_1, \ldots, D^i_{m_i}, D^i\rangle, \\ \langle(k'_{1,i} X^i_1 k''_{1,i}, W^i_1), \ldots, (k'_{f_i,i} X^i_{f_i,i} k''_{f_i,i}, W^i_{f_i,i})\rangle\rangle.$$

for $i = 1, \ldots, w$. Moreover, for all $1 \leq j \leq n$, $k_j \leq |M \cap X_j| \leq l_j$.
There are now two cases.
Case I. $|M \cap X| \leq l$. Since P does not have any empty headed clauses, we know that $l > 0$ so let Y be any non-empty subset of X such that $k \leq |Y| \leq l$, $Y \subseteq (M \cap X)$ and let $p \in Y$. Then

$$\mathfrak{E} = \langle\langle b^1_1, \ldots, b^1_{m_1}, q_1, \ldots, b^w_1, \ldots, b^w_{m_w}, q_w, p\rangle, \\ \langle D^1_1, \ldots, D^1_{m_1}, D^1, \ldots, D^w_1, \ldots, D^w_{m_w}, D^w, C, \rangle, \\ \langle(kXl, Y), (k_1 X_1 l_1, M \cap X_1), \ldots, (k_n X_n l_n, M \cap X_t), \\ (k'_{1,1} X^1_1 k''_{1,1}, W^1_1), \ldots, (k'_{f_1,1} X^1_{f_1,1} k''_{f_1,1}, W_{f_1,1}), \ldots \\ (k'_{1,w} X^w_1 k''_{1,w}, W^w_1), \ldots, (k'_{f_w,w} X^w_{f_w,w} k''_{f_w,w}, W_{f_w,w})\rangle\rangle.$$

is a CC-proof scheme of P with conclusion p. It may not be the case that $\mathfrak{E}$ is admitted by M since it may not be the case that $Y = M \cap X$. Nevertheless, consider the clause,

$$C_{\mathfrak{E}} = b^1_1, \ldots, b^1_{m_1}, q_1, \ldots, b^w_1, \ldots, b^w_{m_w}, q_w, 0R_{\mathfrak{E}}0.$$

Since each of $b^1_1, \ldots, b^1_{m_1}, q_1, \ldots, b^w_1, \ldots, b^w_{m_w}, q_w$ are the conclusion of proof schemes of P admitted by M, it follows from Lemma 2.7 that **JEFF**

$$b^1_1, \ldots, b^1_{m_1}, q_1, \ldots, b^w_1, \ldots, b^w_{m_w}, q_w$$

are all elements of $T^{\omega}_{BN(P)}(\emptyset) = M$. It is also easy to check that since M admits $\mathfrak{E}_1, \ldots, \mathfrak{E}_w$, $M \models body(C)$ and the fact that $M \cap X \subseteq Y$ that it must be the case that $M \models body(C_{\mathfrak{E}})$. But then since $M \models BN(P)$, it must be the case that $M \models kXl$ and hence $M \models C$.

Case II. $|M \cap X| > l$.
We shall show that this case leads to a contradiction that $A \in M$. Hence we must be in Case I and $M \models C$.
Consider the proof scheme **JEFF**

$$\mathfrak{F} = \langle\langle b^1_1, \ldots, b^1_{m_1}, q_1, A \ldots, b^w_1, \ldots, b^w_{m_w}, q_w\rangle, \\ \langle D^1_1, \ldots, D^1_{m_1}, D^1, \ldots, D^w_1, \ldots, D^w_{m_w}, D^w, C\rangle, \\ \langle(k'_{1,1} X^1_1 k''_{1,1}, W^1_1), \ldots, (k'_{f_1,1} X^1_{f_1,1} k''_{f_1,1}, W_{f_1,1}), \ldots \\ (k'_{1,w} X^w_1 k''_{1,w}, W^w_1), \ldots, (k'_{f_w,w} X^w_{f_w,w} k''_{f_w,w}, W_{f_w,w})\rangle\rangle.$$

which is just the concatonation of $\mathfrak{C}_1, \ldots, \mathfrak{C}_w$, the clause C and sequence of set $\vec{B} = (M \cap X_1, \ldots, M \cap X_n)$. Now consider **JEFF**

$$C_{\mathfrak{F},C,\vec{B}} = A \leftarrow b_1^1, \ldots, b_{m_1}^1, q_1, \ldots, b_1^w, \ldots, b_{m_w}^w, q_w, 0R_{\mathfrak{F},C,\vec{B}}0, \neg A.$$

Again we can argue that $b_1^1, \ldots, b_{m_1}^1, q_1, \ldots, b_1^w, \ldots, b_{m_w}^w, q_w$ are all elements of $T^{\omega}_{BN(P)}(\emptyset) = M$. It is also easy to check that since M admits $\mathfrak{C}_1, \ldots, \mathfrak{C}_w$, $M \models body(C)$ that it must be the case that $M \models body(C_{\mathfrak{F},C,\vec{B}})$. But then since $M \models BN(P)$, it would be the case that $A \in M$.

Thus we have $M \models C$ for all $C \in P$. Hence $M \models P$. Now, we have already shown that if $M \models P$, then

$$T^{\omega}_{P,M}(\emptyset) = T^{\omega}_{BN(P),M}(\emptyset).$$

But since M is a CC-stable model of $NB(P)$, $T^{\omega}_{BN(P),M}(\emptyset) = M$ and hence M is a CC-stable model of P. □

Given our remarks preceeding Theorem 2.6, that for any CC-logic program P, we can construct a CC-logic program $\overline{P}$ which is equivalent to P, we then have the following corollary.

Corollary 2.8 *For any CC-logic program P and any atom $A \notin At_P$, there is body-normal CC-logic program $BN(\overline{P})$ with no empty headed clauses such that*

1. *the set of heads of clause of $BN(\overline{P})$ is contained in the set of heads of clauses of P together with $\{A\}$ and*
2. *P and $BN(\overline{P})$ have the same set of CC-stable models.*

3 Some complexity issues

We will now investigate some complexity issues related to CC-logic programs. In [NSS99], Niemelä, Simons and Soininen show that the stable model existence problem for CC-logic programs is NP-complete. In light of Theorem 2.6, one would expect that the existence problems for various restricted classes of CC-logic programs such a body normal CC-logic programs is already NP-complete. In fact, as we will see, a much smaller class of CC-logic programs has this property. In [FMT02], a class of *generator* CC-logic programs is introduced which consists of all CC-logic programs P such that each clause C of P is a single fact, i.e. C is of the form $p \leftarrow$, or of the form $kXl \leftarrow$. A *generator* for a set At is a generator CC-logic program P such that every atom in At occurs in some clause of P. The following fact has been proved by M. Truszczyński[FMT02].

Proposition 3.1 *Let P be a generator for the set of atoms At. Then every model of P is a CC-stable model of P.*

We observe that Proposition 3.1 follows from Proposition 2.5.
Next we observe the following

Proposition 3.2 *The existence problem for stable models of generator CC-logic programs is NP-complete.*

Proposition 3.2 follows from the existence of the reduction of the VERTEX COVER problem to the existence problem for CC-stable models of generator CC-logic programs. Indeed, let $G = \langle V, E \rangle$ be a graph and $k \in N$. Consider the following generator CC-logic program.

$1\{x, y\}2 \leftarrow$
$Vk \leftarrow$

Here the first clause is added for every edge $(x, y) \in E$. Moreover, V is identified with the set of atoms At. Call the resulting program $P_{G,k}$. It is then easy to see that models (and thus CC-stable models) of $P_{G,k}$ are vertex covers for G of size at most k.
Thus even the existence of models or CC-stable models for CC-logic programs and for generator CC-logic programs is NP-complete.
Finally, we will consider a slightly larger class of CC-logic programs P where the body of clause C of P contains no cardinality constraints. That is, C is of the form

$$p \leftarrow q_1, \ldots, q_m$$

or C is of the form

$$kXl \leftarrow q_1, \ldots, q_m.$$

We call such programs *semi-generator* CC-logic programs.
We will now show how to reduce the satisfiability problem for propositional logic to the existence problem for CC-stable models of semi-generator CC-logic programs. To this end, given a CNF formula $\Phi = C_1 \wedge \ldots \wedge C_m$, we will write a semi-generator CC-logic program P_Φ as follows. First let S denote the set of propositional letters that occur in Φ. For each $s \in S$, let $d(s) = \bar{s}$ and $d(\neg s) = s$. Next let p_0 be some fixed element in S and let $T = \{\bar{p} : p \in S\} \cup S$. The P_Φ consists of the following set of clauses.

(1) $1\{p, \bar{p}\}1 \leftarrow$

(2) $2\{p_0, \overline{p_0}\}2 \leftarrow d(\neg l_1^i), \ldots, d(\neg l_{n_i}^i)$

The clause (1) is added for every $p \in S$. The clause (2) is added for every clause $C_i = l_1^i \vee \cdots \vee l_{n_i}^i$ in Φ, $i = 1, \ldots, m$.
Note that the clauses of type (1) ensure that for any CC-stable model M of P_Φ, exactly one of p and $\bar{p}$ is in M for each $p \in S$. In particular, we can not have both p_0 and $\bar{p}_0$ in M. Thus if for all $s \in S$, if we interpret $s \in M$ as s being true and $\bar{s} \in M$ as s being false, then it is easy to see that clauses in (2) ensure that truth assignment determined by M must satisfy all the clauses C_i. Then it is clear that there is a one-to-one correspondence between stable models of P_Φ and valuations of S satisfying Φ. Thus we have proved the following result.

Proposition 3.3 *1. The problem consisting of pairs $\langle P, a\rangle$ where P is a semi-generator CC-logic program such that a belongs to some CC-stable model of P is NP-complete.*

2. The problem consisting of pairs $\langle P, a\rangle$ where P is a semi-generator CC-logic program such that a belongs to all CC-stable models of P is co-NP-complete.

4 Conclusions and further research

HAVE TO BE WRITTEN

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