Scenario Optimization for Multi-Stage Stochastic Programming Problems

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Abstract. The field of multi-stage stochastic programming provides a rich modelling framework to tackle a broad range of real-world decision problems. In order to numerically solve such programs - once they get reasonably large - the infinite-dimensional optimization problem has to be discretized. The stochastic optimization program generally consists of an optimization model and a stochastic model. In the multi-stage case the stochastic model is most commonly represented as a multi-variate stochastic process. The most common technique to calculate an useable discretization is to generate a scenario tree from the underlying stochastic process. Scenario tree generation is examplified by reviewing one specific algorithm based on multi-dimensional facility location applying backward stagewise clustering.

1 Introduction

A large class of decision problems involve decision stages and uncertainty. Examples are multi-stage portfolio optimization or asset liability management problems, energy production models as well as models in telecommunication, transportation, supply chain management (for a recent overview see [1]). A common feature of these models is the fact that a stochastic process describing the uncertain environment (asset prices, insurance claims, energy demand, communication load and so on) is the most important part of the input data. Typically these stochastic processes are estimated from historical data and calibrated using some prior information. For the subsequent decision model however, one needs a numerically tractable approximation, which is small enough to allow reasonable calculation times and large enough to capture the important features of the problem.

The main goal in modelling relevant stochastic process by scenario trees is the following: assume that a discrete-time continuous space stochastic process $(\xi_t)_{t=0,1,2,\ldots,T}$ is given, where $\xi_0 = x_0$ represents the today's value and is constant. The distribution of this process may be the result of a parametric or non-parametric estimation based on historical data. The state space may be univariate (the $\mathbb{R}^1$) or multivariate (the $\mathbb{R}^k$). We look for an approximate simple stochastic process $\tilde{\xi}_t$, which takes only finitely many values and which is as close as possible to the original process (ξ_t) and at the same time has a predetermined structure as a tree. Denote the finite state space of $\tilde{\xi}_t$ by S_t, i.e.

$$\mathbb{P}\{\tilde{\xi}_t \in S_t\} = 1.$$

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Let $c(t) = \#(S_t)$ be the cardinality of S_t. We have that $c(0) = 1$. If $x \in S_t$, we call the branching factor of x the quantity

$$b(x,t) = \#\{y : \mathbb{P}\{\tilde{\xi}_{t+1} = y | \tilde{\xi}_t = x\} > 0\}.$$

In an obvious way, the process $(\tilde{\xi}_t)_{t=0,\ldots,T}$ may be represented as a tree, where the root is $(x_0, 0)$ and the node (x, t) and $(y, t+1)$ is connected by an arc, if $\mathbb{P}\{\tilde{\xi}_t = x, \tilde{\xi}_{t+1} = y\} > 0$. The collection of all branching factors $b(x,t)$ determines the size of the tree. Typically, we choose the branching factors beforehand and independent of x. In this case, the structure of the tree is determined by the vector $[b(1), b(2), b(3), \ldots, b(T)]$. For example, a [5,3,3,2] tree has height 4 and $1 + 5 + 5 \cdot 3 + 5 \cdot 3 \cdot 3 + 5 \cdot 3 \cdot 3 \cdot 2 = 156$ nodes. The number of arcs is always equal the number of nodes minus 1.

The main approximation problem is an optimization problem of one of the following types and is most often determined by the chosen scenario generation method:

The given-structure problem. Which discrete process $(\tilde{\xi}_t), t = 0, \ldots, T$ with given branching structure $[b(1), b(2), b(3), \ldots, b(T)]$ is closest to a given process $(\xi_t), t = 0, \ldots, T$? The notion of closeness has to be defined in an appropriate manner.

The free-structure problem. Here again the process $(\xi_t), t = 0, \ldots, T$ has to be approximated by $(\tilde{\xi}_t), t = 0, \ldots, T$, but its branching structure is free except for the fact that the total number of nodes is predetermined. This hybrid combinatorial optimization problem is more complex than the given-structure problem.

A summary of this methods developed until 2000 can be found in [2]. Methods published since include [3][4] for moment matching strategies, [5][6][7] for probability metric minimization and [8][9] for an integration quadratures approach.

A methodology to compute valuable discretizations based on probability metric minimization is the so called *stagewise backward (tree) clustering*, which is based on a set of simulated underlying paths and generates the necessary tree for optimization purposes. Table 1 summarizes this algorithm for the univariate case with T stages (root $t = 0$) and $n_1, n_2, \ldots, n_T$ nodes per stage.

Future research includes setting up an integrated framework to compare and test different scenario generation methodologies.

References

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Cluster n_T centers in $\mathbb{R}^T$ with distance $d(\cdot)$
Pop Tth component of clusters, i.e. $\mathbb{R}^T \rightarrow \mathbb{R}^{T-1}$
▷ Backward distance minimization
for $i \leftarrow T-1$ **downto** 1 **do**
    Cluster n_i centers in $\mathbb{R}^i$ with distance $d(\cdot)$
    Pop ith component of cluster $\forall$ clusters
end for
▷ Forward tree buildup
for $i \leftarrow 1$ **to** T **do**
    Push ith component to cluster $\forall$ clusters
end for

Table 1. Algorithm

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