Unifying Functional Interpretations

Paulo Oliva

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Department of Computer Science
Queen Mary, University of London
pbo@dcs.qmul.ac.uk

Abstract. The purpose of this article is to present a parametrised functional interpretation. Depending on the choice of the parameter relations one obtains well-known functional interpretations, such as Gödel's Dialectica interpretation, Diller-Nahm's variant of the Dialectica interpretation, Kohlenbach's monotone interpretations, Kreisel's modified realizability and Stein's family of functional interpretations. A functional interpretation consists of a formula translation and a proof translation. We show that all these interpretation only differ on two choices: firstly, on "how much" of the counter-examples for A became witnesses for $\neg A$ when defining the formula translation, and, secondly, "how much" of the witnesses of A one is interested in when defining the proof translation.

Keywords. Functional interpretations, Dialectica interpretation, modified realizability, monotone functional interpretations, majorizability, proof mining.

1 Introduction

In [1] Gödel developed his *Dialectica interpretation* (also known as *functional interpretation*) with the goal of proving relative consistency of first-order arithmetic. The consistency of arithmetic was reduced to that of a quantifier-free calculus based on the language of finite types. He successfully showed that quantifier dependencies can be totally captured by functional dependency, so that logic is eliminated in favour of objects of higher types. Around the same time, Kreisel observed that such proof techniques give in fact much more than just relative conservation results. In the process of replacing logic by functionals, the interpretation automatically makes explicit information "hidden" in the logical structure of the proof. In [2] Kreisel then gives a clear presentation of Gödel's Dialectica interpretation and uses it to define the *constructive truth* of mathematical theorems.

In the same paper Kreisel also sketches an "alternative interpretation", which was further developed in [3] and came to be called *modified realizability*. Kreisel observes that those proof interpretations can be used for proving independence results, and the different versions of the interpretation might provide independence proofs for different principles.

It is normally held that one of the weakenings of Gödel's Dialectica interpretation is that it assumes decidability of prime formulas, known as the *contraction problem*. This happens because when interpreting e.g. $A \to A \wedge A$ the Dialectica interpretation must, in loose terms, pick one counter-example out of two, which can be done by checking which one is indeed a counter-example. It might as well be that both are indeed counter-examples, which implies that the choice at this point is not unique, making the Dialectica interpretation non-canonical (cf. [4], section 2.3.1). A variant of the Dialectica interpretation in which this problem is circumvented was then suggested in [5], and is known as the *Diller-Nahm variant of Dialectica interpretation*. The trick was simply to collect all such counter-examples, postponing the actual decision. In [6] Stein showed that this idea could be generalised, and he defines a family of interpretation, parametrised by the type level from which the counter-examples are collected.

In [7], Kohlenbach observed that Howard's majorizability relation [8] can be used to define a "monotone" version of the Dialectica interpretation, where majorants, rather than precise witnesses, are obtained from proof. This allows for new (even ineffective) principles to be interpreted, and for a new solution to the contraction problem.

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The goal of this article is to show that all these functional interpretation can be viewed as special cases of a single parametrised interpretation via a careful instantiation of two parameter relations. We are able to prove a single soundness theorem, stating sufficient conditions on the parameter relation for such soundness theorem to hold. We also characterise the interpretation via parametrised logical principle.

1.1 Heyting arithmetic in all finite types

The set of *finite types* $\mathcal{T}$ is inductively defined as follows:

$$\rho, \sigma \quad :\equiv \quad o \quad | \quad \rho \to \sigma \quad | \quad \rho \times \sigma \quad | \quad \rho^*.$$

The language of IL^ω contains variables for all finite types, and constants serving as constructors and destructors on the non-basic types, e.g. two families, for each type σ, $\mathsf{len}(\cdot) : \sigma^* \to o$ and $(\cdot)_{(\cdot)} : \sigma^* \times o \to \sigma$ for the length and lookup of finite sequences of type σ^* (respectively). The atomic formulas are simply equalities of basic type $s = t$ and $\perp$. Formulas are built out of atomic formulas via the logical constructions $\wedge, \vee, \to$ and quantifiers $\forall x^\sigma$ and $\exists x^\sigma$. Negation is defined as $\neg A \equiv A \to \perp$. As for the logical rules, we have chosen to work with a natural deduction system (as shown in Appendix A) for two reasons. Firstly, this will give a better understanding of how each logical construct influences on the interpretation. Secondly, natural deduction systems have not been widely used in the context of proof interpretations, and we try to contra-balance this here.

Heyting arithmetic in all finite types HA^ω is an extension of IL^ω with constants for zero, successor, higher-order recursion, together with the appropriate quantifier-free axioms, and the induction rule:

$$\frac{A(0) \qquad A(n) \to A(n+1)}{A(n)}\ \mathsf{IND}$$

Without loss of generality we can assume that the subproofs of $A(0)$ and $A(n) \to A(n+1)$ do not contain undischarged assumptions.

We will assume that higher type equality in HA^ω is treated neutrally, as defined in [9]. This means that only equality between objects of basic type is part of the language, and only universal axioms are used. In the soundness proofs, however, we will make use of the rule of extensionality, as used by Spector [10], which should therefore be added to the verifying system.

2 Functional Interpretations

Let T^ω be an extension of the quantifier-free fragment of HA^ω. A *functional interpretation* of HA^ω into T^ω consists of two mappings. One *formula translation*

$$A \quad \mapsto \quad |A|^x_y,$$

where x and y are two (possibly empty) disjoint sequences of free-variables of the resulting formula; and a *proof translation* (also called *soundness*)

$$\mathsf{HA}^\omega \vdash A \quad \mapsto \quad \mathsf{T}^\omega \vdash B,$$

for some formula B such that $\mathsf{HA}^\omega \vdash B \to \exists x \forall y |A|^x_y$. We will call *standard soundness* the proof translation in which B is the formula $\forall y |A|^t_y$, for some term t.

The sequence of variables x marks the computational information required by the formula, or the constructive content of A. The substitution of a term t for such variable, i.e. $|A|^x_y\,[t/x]$ will be denoted by $|A|^t_x$. We will call any term t for which $\forall y |A|^t_y$ holds a *witness* for A. Moreover, the type of x, which we shall denote by τ_A, we call the *type of the formula A*. The sequence of variables y marks the position of the possible counter-examples for concrete potential witnesses t. Hence, in order to show that t is not a witness for A we just need to produce a sequence of terms s such that $\neg |A|^t_s$.

Therefore, the proof translation component of the interpretation gives a way of translating a proof of A into a proof of some formula B which implies the existence of witnesses for A. We want proof interpretations which are also *complete* in the sense that the existence of witnesses for A also implies back the truth of A, i.e. $\exists x \forall y |A|^x_y \to A$ in some reasonable model.

The goal of this article is to show that *both* formula and proof translations can be parametrised, so that instantiations of those parameters will give rise to most of the known functional interpretations. We will start with a parametrisation of the formula translation, and we show that a standard soundness, i.e. proof mapping, can be given by assuming some simple properties of the parameter. We then introduce a second parameter in the proof translation, and show that further restrictions on the second parameter allow us to prove the parametrised proof translation for the parametrised formula translation.

3 Parametrised Formula Translation

We will give an interpretation of HA^ω into a theory T^ω, which contains an uninterpreted bounded universal quantifier $\forall x \sqsubset t A(x)$. This should be viewed as an *abbreviation* rather than as new formula construct, and the symbol $\sqsubset$ is merely part of the abbreviation. As we will see, the bounded quantification $\forall x \sqsubset t A(x)$ will allow us to parametrise the amount of counter-example that becomes witness when interpreting negation (or more generally, when interpreting the premise of an implication). More clearly, we should answer the following question: given the interpretation $|A|^x_y$ of a formula A, what should the interpretation of $\neg A$ be? The Dialectica interpretation's approach is to say that the witnesses of $\neg A$ are functionals which take potential witnesses for A and turn them into counter-example of those potential witnesses, i.e. an f satisfying $\neg |A|^x_{fx}$ is a witness for $\neg A$. The counter-examples of $\neg A$ are just the real witnesses of A. So,

$$|\neg A|^f_x :\equiv \neg |A|^x_{fx}.$$

We can take a less radical position and say that f does not produce counter-examples for A, but just gives a "bound" on those, i.e.

$$|\neg A|^f_x :\equiv \neg (\forall y \sqsubset fx |A|^x_y).$$

By choosing $\forall x \sqsubset t A(x)$ to be an abbreviation for $A(t)$ we get back Gödel's interpretation of negation $|\neg A|^f_x :\equiv \neg |A|^x_{fx}$. On the other hand, if we take $\forall x \sqsubset t A(x)$ to be an abbreviation for $\forall x A(x)$ instead we have that $|\neg A|^f_x :\equiv \neg \forall y |A|^x_y$, meaning that any term is a witness for a negated statement, provided that $\neg \forall y |A|^x_y$ holds, for all x. This corresponds to the approach taken by modified realizability, which says negated formulas do not ask for any specific witnesses. We will show that with some basic assumptions on the choice of the abbreviation $\forall x \sqsubset t A(x)$ we can define a parametrised interpretation and prove a general soundness theorem. This implies that instantiations of the abbreviation which satisfy the required assumptions will give rise to different functional interpretations.

Before listing our assumptions on what $\forall x \sqsubset t A(x)$ can be an abbreviation for, let us define the class of formulas which will be the image of the parametrised formula translation, i.e. for all formulas A, the formula $|A|$ will be in this class.

Definition 1. *Let the abbreviation $\forall x \sqsubset t A(x)$ be fixed. The class of* $\sqsubset$-bounded formulas *(we denote arbitrary formulas in this class by $A_{\rm b}$ and $B_{\rm b}$) are those built out of prime formulas via conjunction ($A_{\rm b} \wedge B_{\rm b}$), implication ($A_{\rm b} \to B_{\rm b}$) and premise-bounded implication ($\forall x \sqsubset t A_{\rm b}(x) \to B_{\rm b}$).*

Definition 2 (Parametrised Formula Translation). *To each formula A we associate a $\sqsubset$-bounded formula $|A|^x_y$ as follows.*

$$|A_{\mathsf{at}}| :\equiv A_{\mathsf{at}}, \text{ when } A_{\mathsf{at}} \text{ is an atomic formula.}$$

That is to say, for atomic formulas the tuples of witnesses and counter-examples are both empty. Assume we have already defined $|A|^x_y$ and $|B|^v_w$, we define

$$
\begin{array}{lcl}
|A \wedge B|^{x,v}_{y,w} & :\equiv & |A|^x_y \wedge |B|^v_w, \\
|A \vee B|^{x,v,n}_{y,w} & :\equiv & (n = 0 \rightarrow |A|^x_y) \wedge (n \neq 0 \rightarrow |B|^v_w), \\
|A \rightarrow B|^{g,f}_{x,w} & :\equiv & \forall y \sqsubset gxw \, |A|^x_y \rightarrow |B|^{fx}_w, \\
|\forall z^\rho A(z)|^f_{y,z} & :\equiv & |A(z)|^{fz}_y, \\
|\exists z^\rho A(z)|^{x,z}_y & :\equiv & |A(z)|^x_y.
\end{array}
$$

Recall that x, y, v, w, f and g are sequences of variables. Therefore, if y is the empty sequence we let $|A \rightarrow B|^f_{x,w} :\equiv |A|^x \rightarrow |B|^{fx}_w$.

We will make use of the following three condition on the choice of what $\forall x \sqsubset t A(x)$ can be an abbreviation for. Let T^ω be an extension[1] of HA^ω. For all formulas Γ, A and B (with free variables a) and closed terms p, q, r_0, r_1, s, t there must exist closed terms $\mathsf{a}_1, \mathsf{a}_2$ and a_3 such that

(A1) $\mathsf{T}^\omega \vdash |\Gamma \rightarrow \Gamma|^{\mathsf{a}_1 a, \lambda l.l}_{l,u}$,
(A2) if $\mathsf{T}^\omega \vdash |\Gamma \wedge \Gamma \rightarrow A|^{r_0 a, r_1 a, ta}_{l_0, l_1, y}$ then $\mathsf{T}^\omega \vdash |\Gamma \rightarrow A|^{\mathsf{a}_2 a, \lambda l.tall}_{l,y}$.
(A3) if $\mathsf{T}^\omega \vdash |\Gamma \rightarrow (A \wedge (A \rightarrow B))|^{pa,sa,qa,ta}_{l,y,x,w}$ then $\mathsf{T}^\omega \vdash |\Gamma \rightarrow B|^{\mathsf{a}_3 a, (ta)\circ(sa)}_{l,w}$.

Intuitively, $\mathsf{a}_1 alu$ produces a bound for u; $\mathsf{a}_2 aly$ provides a common bound for $r_0 aly$ and $r_1 aly$; and $\mathsf{a}_3 alw$ gives a bound on the potentially infinite set

$$\{w : w \sqsubset paly(sal)w \wedge y \sqsubset qal(sal)w\}.$$

It is clear that condition (A3) supersides (A2) when $\sqsubset$-bounded quantifiers (i.e. $\forall x \sqsubset t A(x)$) can be used to represent standard numeric bounded quantifier (i.e. $\forall n \leq t A(n)$). As we will see, however, this is not the case e.g. with the Dialectica interpretation.

We present now a standard proof translation for the parametrised formula translation. This should be viewed as a preparation for the next step: a parametrisation of the proof translation. The following theorem will be shown to be a special case of the parametrised proof translation.

Theorem 1 (Standard Soundness). *Assume T^ω is an extension of HA^ω for which the* (A) *conditions hold. If*

$$\mathsf{HA}^\omega + \{\Gamma\} \vdash A,$$

then there are sequences of closed terms t and r of appropriate types such that

$$\mathsf{T}^\omega \vdash \forall a, v, y \, |\Gamma \rightarrow A|^{ra,ta}_{v,y},$$

where a is the tuple of free-variables of Γ and A.

Proof. See Appendix B.
□

The notation $\{\Gamma\}$ is used in order to distinguish axioms of the theory HA^ω from implicative assumptions Γ, which are considered undischarged assumptions. The functional interpretation will provide witnesses for the conclusion and "potential" counter-examples for these assumptions. Of course, one could also view the axioms of HA^ω as undischarged assumption, in which case potential counter-example for the axioms will also be provided.

3.1 Characterisation

We show now that the following principles are sufficient for proving the equivalence between the truth of A and the existence of witnesses for A. Markov principle for $\sqsubset$-bounded formulas

$$\mathsf{MP}_\sqsubset \quad : \quad (\forall x A_\mathrm{b}(x) \rightarrow B_\mathrm{b}) \rightarrow \exists b (\forall x \sqsubset b A_\mathrm{b}(x) \rightarrow B_\mathrm{b}),$$

the schema of choice

[1] For simplicity we will not go into the details of how much arithmetic the verifying system T^ω should contain. It will be clear, however, that for the results presented here T^ω could also be a subsystem of HA^ω.

$$\mathsf{AC} \quad : \quad \forall x \exists y A(x,y) \to \exists f \forall x A(x, fx)$$

and independence of premise for $\forall \sqsubset$-bounded formulas

$$\mathsf{IP}_{\sqsubset} \quad : \quad (\forall x A_{\mathrm{b}}(x) \to \exists y B(y)) \to \exists y(\forall x A_{\mathrm{b}}(x) \to B(y)).$$

Theorem 2. *For any formula A in the language of HA^ω*

$$\mathsf{HA}^\omega + \mathsf{MP}_{\sqsubset} + \mathsf{AC} + \mathsf{IP}_{\sqsubset} \vdash A \leftrightarrow \exists x \forall y |A|^x_y.$$

Proof. By induction on the logical structure of A. The only non-trivial case is when A has the form $A \to B$, in which case we have:

$$\begin{array}{rcl} A \to B & \overset{\text{IH}}{\Longleftrightarrow} & \exists x \forall y |A|^x_y \to \exists v \forall w |B|^v_w \\ & \overset{\text{IL}}{\Longleftrightarrow} & \forall x(\forall y |A|^x_y \to \exists v \forall w |B|^v_w) \\ & \overset{\mathsf{IP}_{\sqsubset}}{\Longleftrightarrow} & \forall x \exists v(\forall y |A|^x_y \to \forall w |B|^v_w) \\ & \overset{\text{IL}}{\Longleftrightarrow} & \forall x \exists v \forall w(\forall y |A|^x_y \to |B|^v_w) \\ & \overset{\mathsf{MP}_{\sqsubset}}{\Longleftrightarrow} & \forall x \exists v \forall w \exists b(\forall y \sqsubset b |A|^x_y \to |B|^v_w) \\ & \overset{\mathsf{AC}}{\Longleftrightarrow} & \exists f, g \forall x, w(\forall y \sqsubset gxw |A|^x_y \to |B|^{fx}_w) \\ & \overset{\text{Def}}{\Longleftrightarrow} & \exists f, g \forall x, w |A \to B|^{f,g}_{x,w}. \quad \Box \end{array}$$

4 Instantiations of $\forall x \sqsubset aA(x)$

We show next that by simply instantiating the parameter relation $\sqsubset$ in the unifying functional interpretation we obtain well-known functional interpretations, both the formula translation and the corresponding standard soundness theorem. In each case we explicitly give the families of terms $\mathsf{a}_1, \mathsf{a}_2$ and a_3 and show that conditions (A1), (A2) and (A3) hold for such choices.

4.1 Kreisel's modified realizability

Modified realizability is defined as follows:

Definition 3 (Modified realizability [2, 3]). *For each formula A of HA^ω we associate a new formula x mr A (x is a sequence of fresh variable) inductively as follows:*

$$x \ \mathsf{mr} \ A_{\mathsf{at}} :\equiv A_{\mathsf{at}}, \text{ when } A_{\mathsf{at}} \text{ is an atomic formula.}$$

Assume we have already defined x mr A and v mr B, we define

$$\begin{array}{rll} x, y & \mathsf{mr} \ A \wedge B & :\equiv x \ \mathsf{mr} \ A \wedge v \ \mathsf{mr} \ B, \\ x, v, n & \mathsf{mr} \ A \vee B & :\equiv (n = 0 \to x \ \mathsf{mr} \ A) \wedge (n \neq 0 \to v \ \mathsf{mr} \ B), \\ f & \mathsf{mr} \ A \to B & :\equiv \forall x(x \ \mathsf{mr} \ A \to fx \ \mathsf{mr} \ B), \\ f & \mathsf{mr} \ \forall z A(z) & :\equiv \forall z(fz \ \mathsf{mr} \ A(z)) \\ x, z & \mathsf{mr} \ \exists z A(z) & :\equiv x \ \mathsf{mr} \ A(z). \end{array}$$

In order to obtain modified realizability from the unifying functional interpretation we take $\forall x \sqsubset^{\mathsf{r}} aA(x)$ to mean $\forall x A(x)$. In such case, the definition of implication

$$|A \to B|^{f,g}_{x,w} :\equiv \forall y \sqsubset^{\mathsf{r}} gxw \, |A|^x_y \to |B|^{fx}_w,$$

given in Definition 2 when instantiated becomes

$$|A \to B|^{f}_{x,w} :\equiv \forall y \, |A|^x_y \to |B|^{fx}_w.$$

Notice that conditions (A1), (A2) and (A3) clearly hold for such choice, no matter what terms $\mathsf{a}_1, \mathsf{a}_2$ and a_3 one chooses.

Theorem 3. *In the Definition 2, let $\forall x \sqsubset^{\mathsf{r}} a A(x)$ be an abbreviation for $\forall x A(x)$. Then for all formulas A in the language of HA^ω*

$$\mathsf{HA}^\omega \vdash x \,\mathsf{mr}\, A \leftrightarrow \forall y |A|^x_y.$$

Proof. By induction on the logical structure of A. The case in which A is atomic is trivial. For the composite cases, assume $x \,\mathsf{mr}\, A \leftrightarrow \forall y |A|^x_y$ and $v \,\mathsf{mr}\, B \leftrightarrow \forall w |A|^v_w$.

$$\begin{aligned} x, v \,\mathsf{mr}\, A \wedge B &\equiv x \,\mathsf{mr}\, A \wedge v \,\mathsf{mr}\, B \\ &\leftrightarrow \forall y |A|^x_y \wedge \forall w |B|^v_w \\ &\leftrightarrow \forall y, w(|A|^x_y \wedge |B|^v_w) \\ &\equiv \forall y, w |A \wedge B|^{x,v}_{y,w} \end{aligned}$$

$$\begin{aligned} x, v, n \,\mathsf{mr}\, A \vee B &\equiv (n = 0 \to x \,\mathsf{mr}\, A) \wedge (n \neq 0 \to v \,\mathsf{mr}\, B) \\ &\leftrightarrow (n = 0 \to \forall y |A|^x_y) \wedge (n \neq 0 \to \forall w |B|^v_w) \\ &\leftrightarrow \forall y, w((n = 0 \to |A|^x_y) \wedge (n \neq 0 \to |B|^v_w)) \\ &\equiv \forall y, w |A \vee B|^{x,v,n}_{y,w} \end{aligned}$$

$$\begin{aligned} f \,\mathsf{mr}\, A \to B &\equiv \forall x(x \,\mathsf{mr}\, A \to fx \,\mathsf{mr}\, B) \\ &\leftrightarrow \forall x(\forall y |A|^x_y \to \forall w |B|^{fx}_w) \\ &\leftrightarrow \forall x, w(\forall y\, |A|^x_y \to |B|^{fx}_w) \\ &\equiv \forall x, w |A \to B|^f_{x,w} \end{aligned}$$

$$\begin{aligned} f \,\mathsf{mr}\, \forall z A(z) &\equiv \forall z(fz \,\mathsf{mr}\, A(z)) \\ &\leftrightarrow \forall z(\forall y |A(z)|^{fz}_y) \\ &\equiv \forall z, y |\forall z A(z)|^f_{z,y} \end{aligned}$$

$$\begin{aligned} x, z \,\mathsf{mr}\, \exists z A(z) &\equiv x \,\mathsf{mr}\, A(z) \\ &\leftrightarrow \forall y |A(z)|^x_y \\ &\equiv \forall y |\exists z A(z)|^{x,z}_y \end{aligned}$$

4.2 Gödel's functional interpretation

Gödel's functional interpretation (also known as the Dialectica interpretation) is normally defined as follows:

Definition 4 (Dialectica interpretation [1, 11]). *For each formula A of HA^ω we associate a new formula A^D of the form $\forall x \exists y A_D$, where A_D is quantifier-free, inductively as follows:*

$$(A_{\mathsf{at}})^D :\equiv A_{\mathsf{at}}, \text{ when } A_{\mathsf{at}} \text{ is an atomic formula.}$$

Assume we have $A^D \equiv \exists x \forall y A_D(x, y)$ and $B^D \equiv \exists v \forall w B_D(v, w)$, we define

$$\begin{aligned} (A \wedge B)^D &:\equiv \exists x, v \forall y, w(A_D(x, y) \wedge B_D(v, w)), \\ (A \vee B)^D &:\equiv \exists n, x, v \forall y, w(n = 0 \to A_D(x, y) \wedge n \neq 0 \to B_D(v, w)), \\ (A \to B)^D &:\equiv \exists f, g \forall x, w(A_D(x, gxw) \to B_D(fx, w)), \\ (\forall z A(z))^D &:\equiv \exists f \forall z, w A_D(fz, x, z) \\ (\exists z A(z))^D &:\equiv \exists z, x \forall y A_D(x, y, z). \end{aligned}$$

In each case it is hopefully clear what the quantifier-free formula $(\cdot)_D$ is. E.g. In the case of disjunction we have $(A \vee B)_D \equiv (n = 0 \to A_D(x, y)) \wedge (n \neq 0 \to B_D(v, w))$.

In order to obtain Gödel's original functional interpretation from the unifying functional interpretation we take $\forall x \sqsubset^{\mathsf{g}} t A(x)$ to mean $A(t)$. In such case, the definition of implication

$$|A \to B|^{f,g}_{x,w} :\equiv \forall y \sqsubset gxw\, |A|^x_y \to |B|^{fx}_w,$$

can again on the meta-level be simplified to

$$|A \to B|^f_{x,w} :\equiv |A|^x_{gxw} \to |B|^{fx}_w,$$

and that is what we use in the following.

Condition (A1) holds by taking $\mathsf{a}_1 alu := u$. The fact that condition (A2) holds now is not as trivial as in the case of modified realizability. Notice that it is enough to produce a term a_2 satisfying

$$|\Gamma|^l_{\mathsf{a}_2 aly} \to |\Gamma|^l_{r_0 aly} \wedge |\Gamma|^l_{r_1 aly}.$$

This can be achieved e.g. if for each formula Γ one can produce a term t_Γ satisfying

$$\mathsf{T}^\omega \vdash |\Gamma(a)|^l_u \leftrightarrow t_\Gamma alu = 0.$$

If this is the case the we can define a_2 as

$$\mathsf{a}_2 aly := \begin{cases} r_0 aly & \text{if } t_\Gamma al(r_0 aly) \neq 0 \\ r_1 aly & \text{otherwise.} \end{cases}$$

As for condition (A3), we simply take $\mathsf{a}_3 alw := pal(qal(sal)w)(sal)w$. It is easy to see that if

$$|\Gamma|^l_{palyxw} \to (|A|^{sal}_y \wedge (|A|^x_{qalxw} \to |B|^{talx}_w))$$

holds then, by taking $x := sal$ and $y := qal(sal)w$, we get

$$|\Gamma|^l_{pal(qal(sal)w)(sal)w} \to |B|^{tal(sal)}_w$$

i.e. $|\Gamma \to B|^{\mathsf{a}_3 a,(ta)\circ(sa)}_{l,w}$.

Theorem 4. *In the Definition 2, let $\forall x \sqsubset^{\mathsf{g}} aA(x)$ be an abbreviation for $A(a)$. Then for all formulas A in the language of HA^ω*

$$\mathsf{HA}^\omega \vdash A_D(x,y) \leftrightarrow |A|^x_y.$$

Proof. With the simplification outlined above for the case of implication, one can immediately see that the definition of $A_D(x,y)$ coincedes precisely with the definition of $|A|^x_y$. □

4.3 Diller-Nahm functional interpretation

The Diller-Nahm interpretation is normally presented as a variant of Gödel's Dialectica interpretation where decidability of prime formulas is no longer necessary. The drawback is that instead of producing witnessing terms given a proof of A the Diller-Nahm interpretation only produces a finite collection of candidate witnesses, with the assurance that one of those is indeed a witness.

Definition 5 (Diller-Nahm interpretation [5]). *For each formula A of HA^ω we associate a new formula $A^\wedge$ of the form $\forall x \exists y A_\wedge$, where $A_\wedge$ is 0-bounded – i.e. contains only bounded numerical quantifiers, inductively as follows:*

$$(A_{\mathsf{at}})^\wedge :\equiv A_{\mathsf{at}}, \text{ when } A_{\mathsf{at}} \text{ is atomic formula.}$$

Assume we have $A^\wedge \equiv \exists x \forall y A_\wedge(x,y)$ and $B^\wedge \equiv \exists v \forall w B_\wedge(v,w)$, we define

$$\begin{aligned}
(A \wedge B)^\wedge &:\equiv \exists x, v \forall y, w (A_\wedge(x,y) \wedge B_\wedge(v,w)),\\
(A \vee B)^\wedge &:\equiv \exists n, x, v \forall y, w ((n = 0 \to A_\wedge(x,y)) \wedge (n \neq 0 \to B_\wedge(v,w))),\\
(A \to B)^\wedge &:\equiv \exists f, b, g_{(\cdot)} \forall x, w (\forall n \leq bxw A_\wedge(x, g_n xw) \to B_\wedge(fx, w)),\\
(\forall z A(z))^\wedge &:\equiv \exists f \forall z, w A_\wedge(fz, x, z),\\
(\exists z A(z))^\wedge &:\equiv \exists z, x \forall y A_\wedge(x, y, z).
\end{aligned}$$

The only difference to Gödel's original functional interpretation is in the treatment of implication. The functional b and the sequence of functionals $g_{(\cdot)}$ are a convenient way to quantify over finite multi-sets. For simplicity we abbreviate e.g. $\exists b^o \exists g_{(\cdot)}^{o \to \tau} \forall n \leq b A(g_n)$ by the use of finite sequence as $\exists g^{\tau^*} \forall y \in g A(y)$. Using this shorthand the treatment of implication can be rewritten as

$$(A \to B)^\wedge :\equiv \exists f, g \forall x, w (\forall y \in gxw A_\wedge(x, y) \to B_\wedge(fx, w)).$$

Therefore, the Diller-Nahm interpretation can be viewed as an instantiation of the unifying functional interpretation where $\forall x^\tau \sqsubset^\wedge t^{\tau^*} A(x)$ is an abbreviation for $\forall x \in t A(x)$.

The term a_1 in condition (A1) can be taken to be a functional which produces a singleton tuple out of a given parameter, i.e. $\mathsf{a}_1 alu := \langle u \rangle$. In order to satisfy condition (A2) we need terms a_2 computing the union of two finite multi-sets, which is simply the concatenation of two finite sequences. As for condition (A3), the term $\mathsf{a}_3 alw$ can be taken to be the union of the finite multi-sets $paly(sal)w$, for each $y \in qal(sal)w$, where $qal(sal)w$ are also finite multi-sets.

4.4 Stein's family of interpretations

In [6], a family of interpretation between Diller-Nahm and modified realizability was defined, parametrised by a number $n > 0$. The parameter n basically dictates the types of the universal quantifiers which are left "untouched" by the interpretations, as done in the definition of modified realizability. Universal quantifiers of type level bigger than n will get pulled out from premises of implications, following the idea of Diller and Nahm interpretation.

The interpretation we define below is basically Stein's definition, in the sense that for each formula A, our definition of A^n is intuitionistically (but not syntactically) equivalent to his definition.

For the rest of this section we use the following notations: given a tuple of variable x, we will denote by $\underline{x}$ the sub-tuple containing the variables in x which have type level $\geq n$, whereas $\overline{x}$ denotes the sub-tuple of the variables in x which have type level $< n$. The actual value of n will be clear from the context.

Definition 6 (Stein's family of f.i.'s [6]). *For each positive natural number n, the interpretation of a formula A of HA^ω is a new formula A^n of the form $\exists x \forall \underline{y} \forall \overline{y} A_n$, where A_n contains only universal quantifiers of type level $< n$, and no existential quantifier. The assignment is done inductively as follows:*

$$(A_{\mathsf{at}})^n :\equiv A_{\mathsf{at}}, \text{ when } A_{\mathsf{at}} \text{ is an atomic formula.}$$

Assume we have $A^n \equiv \exists x \forall \underline{y} \forall \overline{y} A_n(x, y)$ and $B^n \equiv \exists v \forall \underline{w} \forall \overline{w} B_n(v, w)$, we define

$$\begin{array}{ll}
(A \wedge B)^n & :\equiv \exists x, v \forall \underline{y}, \underline{w} \forall \overline{y}, \overline{w} (A_n(x, y) \wedge B_n(v, w)), \\
(A \vee B)^n & :\equiv \exists m, x, v \forall \underline{y}, \underline{w} \forall \overline{y}, \overline{w} ((m = 0 \to A_n(x, y)) \wedge (m \neq 0 \to B_n(v, w))), \\
(A \to B)^n & :\equiv \exists f, g \forall \underline{x}, \underline{w} \forall \overline{x}, \overline{w} (\forall i^{n-1} \forall \overline{y} A_n(x, gxwi, \overline{y}) \to B_n(fx, w)), \\
(\forall z A(z))^n & :\equiv \exists f \forall \underline{z}, \underline{y} \forall \overline{z}, \overline{y} A_n(fz, y, z), \\
(\exists z A(z))^n & :\equiv \exists z, x \forall \underline{y} \forall \overline{y} A_n(x, y, z).
\end{array}$$

This interpretation can be viewed as a generalization of Diller and Nahm's idea of collecting witnesses into finite sequences. In the case of Stein's family of interpretations, one actually collects families of witnesses, where the indexing i of the family ranges over the pure type $(n-1)$. Therefore, in the treatment of implication, the resulting value of gxw is actually a function of type $(n-1) \to \tau$. For the sake of simplicity and intuition we write quantifications of the form $\forall i^{n-1} A_n(x, gxwi)$ as $\forall y \in \mathsf{img}(gxw) A_n(x, y)$. We can then more clearly write the treatment of implication as

$$(A \to B)^n :\equiv \exists f, g \forall \underline{x}, \underline{w} \forall \overline{x}, \overline{w} (\forall \underline{y} \in \mathsf{img}(gxw) \forall \overline{y} A_n(x, y) \to B_n(fx, w)),$$

We show now that also Stein's family of interpretations can be obtained via the unifying functional interpretation by defining the following abbreviation

$$\forall x^\tau \sqsubset_\tau^n g^{(n-1) \to \tau} A(x) :\equiv \begin{cases} \forall x A(x) & \text{if } \mathsf{level}(\tau) < n \\ \forall x \in \mathsf{img}(g) A(x) & \text{otherwise.} \end{cases}$$

It is again easy to see that this choice complies with conditions (A1) and (A2), all we need is a surjective functional $i \mapsto i_0 \times i_1$ of type $(n-1) \to (n-1) \times (n-1)$. In the case of condition (A3) it is sufficient to produce a term a_3 satisfying

$$\forall y^\sigma \sqsubset \mathsf{a}_3 hzA(y) \to \forall x^\tau \sqsubset z \forall y^\sigma \sqsubset hxA(y).$$

The only non-trivial situation is when $\sigma \geq n$ and $\tau < n$, i.e.

$$\forall y^\sigma \in \mathsf{a}_3 hzA(y) \to \forall x \forall y^\sigma \in hxA(y),$$

which stands for (z is not used)

$$\forall i^{n-1} A(\mathsf{a}_3 hi) \to \forall x^{<n} \forall j^{n-1} A(hxj).$$

We can then take $\mathsf{a}_3 hi := hi_0 i_1$ using again any surjective functional $i \mapsto i_0 \times i_1$.

Theorem 5. *In the Definition 2, let* $\forall x \sqsubset^n_\tau aA(x)$ *be an abbreviation as described above. Then for all formulas* A *(let* $A^n \equiv \exists x \forall \underline{y} \forall \overline{y} A_n(x,y)$*) in the language of* HA^ω

$$\mathsf{HA}^\omega \vdash A_n(x,y) \leftrightarrow |A|^x_y.$$

Proof. We only present here the case of implication. Assume $A^n \equiv \exists x \forall \underline{y} \forall \overline{y} A_n(x,y)$ and $B^n \equiv \exists v \forall \underline{w} \forall \overline{w} B_n(v,w)$. Then

$$\begin{aligned}
(A \to B)_n(f,g,x,w) &\equiv \forall i^{n-1} \forall \overline{y} A_n(x, gxwi, \overline{y}) \to B_n(fx,w) \\
&\equiv \forall \underline{y} \in \mathsf{img}(gxw) \forall \overline{y} A_n(x,y) \to B_n(fx,w) \\
&\equiv \forall y \sqsubset^n gxwA_n(x,y) \to B_n(fx,w) \\
&\overset{\text{IH}}{\leftrightarrow} \forall y \sqsubset^n gxw|A|^x_y \to |B|^{fx}_w \\
&\equiv |A \to B|^{f,g}_{x,w}.
\end{aligned}$$

□

5 Parametrised Proof Translation

We describe now how the Soundness Theorem (Theorem 1) can also be generalised, via a second family of parameter relations $\prec$. This gives rise to a family of soundness theorems for a family of functional interpretations. We will again only use the relation in the context $\exists x \prec aA(x)$, so that $\prec$ does not need to be a relational symbol in the language. We will assume, however, that the abbreviation $\exists x \prec tA(x)$ must "behave" as an existential quantifier, in the following sense

(E1) For all formulas $A(x)$, $B(y)$ and $C(x,y)$, and terms t, s, if
- $\mathsf{T}^\omega \vdash \exists x \prec tA(x)$,
- $\mathsf{T}^\omega \vdash \exists y \prec sB(y)$, and
- $\mathsf{T}^\omega + A(x) + B(y) \vdash C(x,y)$

then $\mathsf{T}^\omega \vdash \exists x \prec t, y \prec sC(x,y)$.

For instance, we can choose $\exists x \prec aA(x)$ to mean $A(a)$. In fact, this instantiation will give us back Theorem 1 as a special case of our parametrised soundness theorem. We will show, however, that $\exists x \prec tA(x)$ can also be taken to be $\exists x(x \leq^* t \wedge A(x))$, where $\leq^*$ is Howard's majorizability relation (see [8]). This was first observed in [7], and gives rise to so-called *monotone* versions of the Dialectica interpretation and modified realizability. As we will see, according to the framework set up in Section 2, these monotone variants are a combination of the standard formula translations with a *monotone proof translation.*

Besides condition (E1), we will assume also the following on the choice of the abbreviation $\exists x \prec tA(x)$:

(E2) For each formula A, closed term s and term $t[f]$, if

$\mathsf{T}^\omega \vdash \exists f \prec s \forall a, y |A|^{t[f]a}_y$

then there exists a closed term t^* such that

$\mathsf{T}^\omega \vdash \exists F \prec t^* \forall a, y |A|^{Fa}_y$.

We call t^* a $\prec$-*majorizing term* for t. In particular, when the tuple f is empty we have

$$\mathsf{T}^\omega \vdash \forall a, y|A|^{ta}_y \quad \Rightarrow \quad \mathsf{T}^\omega \vdash \exists f \prec t^* \forall a, y|A|^{fa}_y$$

We will also consider bounded versions of the conditions (A1), (A2) and (A3). For all formulas Γ, A, B and C, and closed terms r_0, r_1, s, t, p, q there are closed terms $\mathsf{a}_1^*, \mathsf{a}_2^*, \mathsf{a}_3^*$ such that

(A1)* $\mathsf{T}^\omega \vdash \exists \nu \prec \mathsf{a}_1^* \forall a, l, u|\Gamma \to \Gamma|^{\nu a, \lambda l.l}_{l,u}$,

(A2)* if

$$\mathsf{T}^\omega \vdash \exists g_0 \prec r_0, g_1 \prec r_1, f \prec t\, \forall a, l_0, l_1, y|\Gamma \wedge \Gamma \to A|^{g_0 a, g_1 a, fa}_{l_0, l_1, y}$$

then

$$\mathsf{T}^\omega \vdash \exists \chi \prec \mathsf{a}_2^* \exists f \prec t^* \forall a, l, y|\Gamma \to A|^{\chi a, fa}_{l,y},$$

where t^* is the majorizing term for $\lambda a, l.tall$.

(A3)* if

$$\mathsf{T}^\omega \vdash \exists g \prec p, f \prec s, h \prec q, j \prec t \forall l, y, x, w|\Gamma \to (A \wedge (A \to B))|^{ga, fa, ha, ja}_{l,y,x,w}$$

then

$$\mathsf{T}^\omega \vdash \exists \xi \prec \mathsf{a}_3, f \prec s, j \prec t \forall l, w|\Gamma \to B|^{\xi a, (ja)\circ(fa)}_{l,w}.$$

We show that such weakenings together with the two (E) conditions are sufficient for proving the following parametrised version of the Soundness Theorem.

Theorem 6 (Parametrised Soundness). *Assume T^ω is an extension of HA^ω for which the* (A)* *and* (E) *conditions hold. If*

$$\mathsf{HA}^\omega + \{\Gamma\} \vdash A,$$

then there are sequences of closed terms t^* and r^* of appropriate types such that

$$\mathsf{T}^\omega \vdash \exists g \prec r^* \exists f \prec t^* \forall a, v, y\, |\Gamma \to A|^{ga, fa}_{v,y},$$

where a is a tuple of free-variables of Γ and A. Recall that we use $\{\Gamma\}$ to indicate that the assumptions Γ are being viewed as undischarged assumptions, rather than axioms of the theory, which (could be, but) are not (in general) witnessed.

Proof. See Appendix C.

Notice that the parametrised soundness gives a family of soundness theorems (depending on the choice of $\exists x \prec aA(x)$) for the parametrised formula translation (which is parametrised by $\forall x \sqsubset aA(x)$). Moreover, observe that in the monotone soundness we do not require terms $\mathsf{a}_1, \mathsf{a}_2$ and a_3 to be part of the language, but only majorizing terms $\mathsf{a}_1^*, \mathsf{a}_2^*$ and a_3^* for those, according to the choice for $\exists x \prec aA(x)$.

5.1 Instantiating the parametrised soundness

Table 1 summarises how different instantiations of $\forall x \sqsubset tA(x)$ and $\exists x \prec tA(x)$ give rise to different functional interpretations.

$\forall x^\rho \sqsubset aA(x)$	$\exists x \prec aA(x)$	**Formulas + proof translation**
$A(a)$	$A(a)$	Dialectica interpretation
$\forall i \leq \lvert a\rvert\, A(a_i)$	$A(a)$	Diller-Nahm variant of Dialectica
$\forall i^{n-1}\, A(ai)$ or$_{\rho \geq n}$ $\forall x A(x)$	$A(a)$	Stein's family of interpretations
$\forall x A(x)$	$A(a)$	Modified realizability
$A(a)$	$\exists x \leq^* aA(x)$	Monotone Dialectica interpretation
$\forall i \leq \lvert a\rvert\, A(a_i)$	$\exists x \leq^* aA(x)$	*no given name*
$\forall i^{n-1}\, A(ai)$ or$_{\rho \geq n}$ $\forall x A(x)$	$\exists x \leq^* aA(x)$	*no given name*
$\forall x A(x)$	$\exists x \leq^* aA(x)$	Monotone realizability

Table 1. Instantiations of the parametrised functional interpretation

For instance, instantiating $\sqsubset$ with $\sqsubset^{\mathsf{g}}$ and $\prec$ with $\leq^*$ gives the so-called *monotone Dialectica interpretation*, whereas taking $\sqsubset$ to be $\sqsubset^{\mathsf{r}}$ gives the *monotone realizability interpretation* (cf. [7]). In the first case, we no longer need characteristic terms a_2 for deciding prime formulas, but only a majorant a_2^* for those, which can be taken to be the constant 1 functional. Moreover, as shown in [7], ineffective principles such as weak König's lemma, which are not interpretable with the standard soundness theorem, become interpretable under the monotone soundness.

6 Programs from Proofs

One should notice that the parametrised functional interpretation can be applied directly to analyse proofs, leaving the instantiation to a later stage, once the (parametrised) witnessing term and (parametrised) verifying proof have been obtained. This can be achieved by adding to the language new formulas constructs $\forall x \sqsubset tA(x)$ and $\exists x \sqsubset tA(x)$, families of constants $\mathsf{a}_1^*, \mathsf{a}_2^*, \mathsf{a}_3^*$ and axiom schemes corresponding to the conditions (A) and (E).

Starting with a proof of A what one obtains is then a proof of $\exists f \prec t \forall a, y |A|_y^{fa}$, for some closed term t. The extracted term t will potentially contain the new constants added, and the proof of $|A|_y^t$ will potentially make use of the new formula constructs. Extracting the "abstract" witnessing term t allows for a comparison between the terms extracted via different functional interpretation, namely, we know that terms extracted via different interpretations will have the same structure, and will only differ on the choices of $\mathsf{a}_1^*, \mathsf{a}_2^*$ and a_3^*. This will be clear cut when analysing proofs of theorem whose interpretation do not contain $\sqsubset$, e.g. implication-free theorems and theorems in prenex form. For such formulas A the interpretation $|A|_y^x$ will be syntactically the same, no matter what the choice of the abbreviation $\forall x \sqsubset tA(x)$, although the extracted term t and the proof of $\exists f \prec t \forall a, y |A|_y^x$ will possibly change.

6.1 Dealing with classical proofs

The difference between the interpretation becomes evident when e.g. dealing with classical logic. After applying the negative translation (for elimination of classical logic) to proofs of theorems $\forall x \exists y A_{\mathsf{qf}}(x, y)$ one gets a proof of $\forall x \neg\neg \exists y A_{\mathsf{qf}}(x, y)$, which is a shorthand for

$$\forall x((\exists y A_{\mathsf{qf}}(x, y) \to \perp) \to \perp)).$$

Via the parametrised f.i. applied to a proof of such theorem one obtains a term t and a proof of

$$\exists f \prec t \forall x \neg \forall y \sqsubset fx \neg A_{\mathsf{qf}}(x, y).$$

If we instantiate the parametrised interpretation with the Dialectica abbreviation $\forall x \sqsubset^{\mathsf{g}} tA(x)$ we get that t is actually a bound (in the sense of $\prec$) on the witnessing function for the original theorem. Moreover, if $\exists x \prec sA(x)$ is chosen to mean $A(s)$, then t is the actual witnessing function for the theorem.

Similarly, instantiating with the Diller-Nahm abbreviation we would get a function which for any x produces a finite tuple of possible witnesses for y. In the case of Stein's abbreviation, e.g. with n bigger than the type level of y, the term t would produce a family (indexed by the pure type $n-1$) of possible witnesses for y.

Applying modified realizability directly would simply produce a new proof of $\forall x \neg\neg \exists y A_{\mathsf{qf}}(x, y)$, without any extra information on the witnessing function. In such cases we can make use of the A-translation, which transforms proofs of $\forall x \neg\neg \exists y A_{\mathsf{qf}}(x, y)$ into proofs of $\forall x \exists y A_{\mathsf{qf}}(x, y)$ before modified realizability is used.

References

1. Gödel, K.: Über eine bisher noch nicht benützte Erweiterung des finiten Standpunktes. Dialectica **12** (1958) 280–287
2. Kreisel, G.: Interpretation of analysis by means of constructive functionals of finite types. In Heyting, A., ed.: Constructivity in Mathematics. North Holland, Amsterdam (1959) 101–128
3. Kreisel, G.: On weak completeness of intuitionistic predicate logic. The Journal of Symbolic Logic **27** (1962) 139–158
4. Hyland, J.M.E.: Proof theory in the abstract. Annals of Pure and Applied Logic **114** (2002) 43–78
5. Diller, J., Nahm, W.: Eine Variant zur Dialectica-interpretation der Heyting Arithmetik endlicher Typen. Arch. Math. Logik Grundlagenforsch **16** (1974) 49–66
6. Stein, M.: Interpretationen der Heyting-Arithmetik endlicher Typen. Arch. Math. Logik Grundlag **19** (1979) 175–189
7. Kohlenbach, U.: Analysing proofs in Analysis. In Hodges, W., Hyland, M., Steinhorn, C., Truss, J., eds.: Logic: from Foundations to Applications, European Logic Colloquium (Keele, 1993), Oxford University Press (1996) 225–260
8. Howard, W.A.: Hereditarily majorizable functionals of finite type. In Troelstra, A.S., ed.: Metamathematical investigation of intuitionistic Arithmetic and Analysis. Volume 344 of Lecture Notes in Mathematics. Springer, Berlin (1973) 454–461
9. Troelstra (ed.), A.S.: Metamathematical Investigation of Intuitionistic Arithmetic and Analysis. Volume 344 of Lecture Notes in Mathematics. Springer, Berlin (1973)
10. Spector, C.: Provably recursive functionals of analysis: a consistency proof of analysis by an extension of principles in current intuitionistic mathematics. In Dekker, F.D.E., ed.: Recursive Function Theory: Proc. Symposia in Pure Mathematics. Volume 5., American Mathematical Society, Providence, Rhode Island (1962) 1–27
11. Avigad, J., Feferman, S.: Gödel's functional ("Dialectica") interpretation. In Buss, S.R., ed.: Handbook of proof theory. Volume 137 of Studies in Logic and the Foundations of Mathematics. North Holland, Amsterdam (1998) 337–405

Appendix A: Logical rules

The logical framework we use is intuitionistic predicate logic in the language of finite types, on top of which we build HA^ω. We have opted for the following natural deduction formalism:

$$\{A\}_\alpha \qquad\qquad \frac{\perp}{A}\,\perp\mathrm{E}$$

$$\frac{A_0 \qquad A_1}{A_0 \wedge A_1}\,\wedge\mathrm{I} \qquad\qquad \frac{A_0 \wedge A_1}{A_i}\,\wedge\mathrm{E}$$

$$\frac{A_i}{A_0 \vee A_1}\,\vee\mathrm{I} \qquad\qquad \frac{A_0 \vee A_1 \qquad \begin{matrix}[A_0]_{\alpha_0}\\ \vdots \\ B\end{matrix} \qquad \begin{matrix}[A_1]_{\alpha_1}\\ \vdots \\ B\end{matrix}}{B}\,\vee\mathrm{E}_{\alpha_0,\alpha_1}$$

$$\frac{\begin{matrix}[A]_\alpha\\ \vdots \\ B\end{matrix}}{A \to B}\,\to\mathrm{I}_\alpha \qquad\qquad \frac{A \qquad A \to B}{B}\,\to\mathrm{E}$$

$$\frac{A(x^\sigma)}{\forall x^\sigma A(x)}\,\forall\mathrm{I} \qquad\qquad \frac{\forall x^\sigma A(x)}{A(s^\sigma)}\,\forall\mathrm{E}$$

$$\frac{A(s^\sigma)}{\exists x^\sigma A(x)}\,\exists\mathrm{I} \qquad\qquad \frac{\exists x^\sigma A(x) \qquad \begin{matrix}[A(x)]_\alpha\\ \vdots \\ B\end{matrix}}{B}\,\exists\mathrm{E}_\alpha$$

In the rule $\forall$I the variable x must not occur free in (non-discharged) assumptions of the derivation, whereas in the rule $\exists$E, x must not occur free in assumptions on which B depends, except for $A(x)$. Furthermore $\exists$E has the restriction that x is not free in B.

We are using the convention that undischarged assumptions are marked by $\{\cdot\}$. This is changed into $[\cdot]$ when the assumptions are discharged. Except in the proofs of Theorems 1 and 6 we do not explicitly show undischarged assumptions.

Appendix B: Proof of Theorem 1

As stated in the theorem, let us assume for simplicity that a is a tuple of variables containing all the free-variables of Γ, A and B. Also for simplicity we assume that Γ is a single formula, which might be used multiple times in the derivation of A. In the following we indicate how a proof $\pi : (\Gamma \vdash A)$ can be transformed into a proof $\tilde{\pi}$ of $|\Gamma \to A|^{ra,ta}_{v,y}$, for closed terms r and t, by induction on the structure of π. During the translation, rules of π might get translated into *derivable rules* in the derivation $\tilde{\pi}$.

Assume Γ, A and B have interpretations $|\Gamma|^l_u, |A|^x_y$ and $|B|^v_w$, respectively.

Axioms. All the axioms of HA^ω are universal, i.e. of the form $\forall n A_{\mathsf{qf}}(n)$, so that $|\forall n A_{\mathsf{qf}}(n)|_n \equiv A_{\mathsf{qf}}(n)$. To each of those we associate the proof

$$\frac{\forall n A_{\mathsf{qf}}(n)}{|\forall n A_{\mathsf{qf}}(n)|_n}\ \forall\mathrm{E}$$

i.e. the axioms of HA^ω imply their interpretation.

Assumption. To each undischarged assumption $\{\Gamma\}$ we associate a derivation of $|\Gamma \to \Gamma|^{\mathsf{a}_1 a, \lambda l.l}_{l,u}$, which is assumed to exist by condition (A1).

Falsity elimination. Consider an arbitrary instance

$$\frac{\begin{array}{c}\{\Gamma\}_\alpha \\ \vdots\ \pi \\ \bot\end{array}}{A}\ \bot\,\mathrm{E}$$

of this rule. By induction hypothesis we have a derivation $\tilde{\pi}$ of $|\Gamma \to \bot|^{ra}_l$, for some closed term r. We can then easily derive $|\Gamma \to A|^{r'a,0}_{l,y}$, where 0 is the zero functional of appropriate type and $r'aly := ral$. In the following we will omit the definition of the primed terms when those are simple projections, as it is the case in r'.

Conjunction introduction. Assume for simplicity that both branches use the same undischarged assumption Γ.

$$\frac{\begin{array}{c}\{\Gamma\}_\alpha \\ \vdots\ \pi_0 \\ A_0\end{array} \qquad \begin{array}{c}\{\Gamma\}_\alpha \\ \vdots\ \pi_1 \\ A_1\end{array}}{A_0 \wedge A_1}\ \wedge\mathrm{I}$$

By induction hypothesis we have derivations of

(i) $|\Gamma \to A_0|^{q_0 a, t_0 a}_{l_0, y_0}$
(ii) $|\Gamma \to A_1|^{q_1 a, t_1 a}_{l_1, y_1}$

for some closed terms q_0, q_1, t_0, t_1. From (i) and (ii) we can build a derivation of

$$|\Gamma \wedge \Gamma \to A_0 \wedge A_1|^{q_0' a, q_1' a, t_0' a, t_1' a}_{l_0', l_1', y_0, y_1},$$

where $q_0', q_1', t_0', t_1', l_0'$ and l_1' are appropriate projections. By condition (A2) we have the existence of a closed term a_2 and a derivation of

$$|\Gamma \to A_0 \wedge A_1|^{\mathsf{a}_2 a, \lambda l. t_0' all, \lambda l. t_1' all}_{l, y_0, y_1}.$$

Conjunction elimination. By induction hypothesis we have closed terms q, t_0 and t_1 and a proof of

$$|\Gamma \to A_1 \wedge A_2|^{qa, t_0 a, t_1 a}_{v, y_0, y_1}.$$

This easily gives a derivation of $|\Gamma \to A_0|^{qa,t_0a}_{v,y_0}$, where $q'avy_0 := qavy_00$ (y_1 is substituted by the constant 0 of appropriate type).

Implication introduction. This is immediately treated by the induction hypothesis.

Implication elimination. Let us look at the following instance of this rule

$$\dfrac{\begin{array}{c}\{\Gamma\}_\alpha\\ \vdots\,\pi_1\\ A\end{array}\qquad\begin{array}{c}\{\Gamma\}_\alpha\\ \vdots\,\pi_2\\ A\to B\end{array}}{B}\to\mathrm{E}$$

By induction hypothesis we have closed terms p_1, p_2, q, s and t and derivations for

(i) $|\Gamma \to A|^{p_1a,sa}_{l_1,y}$
(ii) $|\Gamma \to (A \to B)|^{p_2a,qa,ta}_{l_2,x,w}$.

From those we can derive

$$|(\Gamma \wedge \Gamma) \to (A \wedge (A \to B))|^{p_1a,p_2a,sa,qa,ta}_{l_1,l_2,y,x,w}$$

By condition (A2) we get

$$|\Gamma \to (A \wedge (A \to B))|^{\mathsf{a}_2a,\lambda l.sall,\lambda l.qall,\lambda l.tall}_{l,y,x,w}$$

Finally, by condition (A3) we have a closed term a_3 and a proof of

$$|\Gamma \to B|^{\mathsf{a}_3a,(\lambda l.tall)\circ(\lambda l.sall)}_{l,w}.$$

Universal introduction and elimination. Let us look at the following instance of the rule

$$\dfrac{\begin{array}{c}\{\Gamma\}_\alpha\\ \vdots\,\pi\\ A(z)\end{array}}{\forall z A(z)}\,\forall\mathrm{I}$$

By induction hypothesis we have closed terms r and t and a derivation of

$$|\Gamma \to A(z)|^{raz,taz}_{l,y}.$$

This can be simply viewed as a derivation of $|\Gamma \to \forall z A(z)|^{ra,ta}_{l,y,z}$. Similarly with the universal elimination.

Disjunction introduction. Assume we have a derivation of $|\Gamma \to A_0|^{qa,t_0a}_{l,y_0}$. We can then obtain a derivation of $|\Gamma \to A_0 \vee A_1|^{qa,t_0a,0,0}_{l,y_0,y_1}$, where 0 is the zero functional of appropriate type, and y_1 is a fresh variable. Similarly with the case $\Gamma \vdash A_1$.

Disjunction elimination. Let us consider now the rule

$$\dfrac{\begin{array}{c}\{\Gamma\}_\beta\\ \vdots\,\pi_2\\ A_0\vee A_1\end{array}\quad\begin{array}{c}[A_0]_{\alpha_0}\\ \vdots\,\pi_0\\ B\end{array}\quad\begin{array}{c}[A_1]_{\alpha_1}\\ \vdots\,\pi_1\\ B\end{array}}{B}\vee\mathrm{E}_{\alpha_0,\alpha_1}$$

Instead of applying the induction hypothesis to the two derivations $\pi_0 : (A_0 \vdash B)$ and $\pi_1 : (A_1 \vdash B)$ we will consider a slight extensions of those, namely

$$\begin{array}{c}\dfrac{\{n=0\}_{\alpha_0}\quad\{n=0\to A_0\}_{\beta_0}}{A_0}\to\mathrm{E}\\ \vdots\,\pi_{0,e}\\ B\end{array}\qquad\qquad\begin{array}{c}\dfrac{\{n\neq 0\}_{\alpha_1}\quad\{n\neq 0\to A_1\}_{\beta_1}}{A_1}\to\mathrm{E}\\ \vdots\,\pi_{1,e}\\ B\end{array}$$

Applying the IH to these extended derivations will give us tuples of closed terms q_0, q_1, s_0 and s_1 and derivations of

(i) $(n = 0) \to |(n = 0 \to A_0) \to B|^{q_0 an, s_0 an}_{x_0, w}$
(ii) $(n \neq 0) \to |(n \neq 0 \to A_1) \to B|^{q_1 an, s_1 an}_{x_1, w}$

By the decidability of equality of basic type we get

$$|((n = 0 \to A_0) \wedge (n \neq 0 \to A_1)) \to B|^{q_0 an, q_1 an, \mathsf{cond}(n, s_0 an, s_1 an)}_{x_0, x_1, w},$$

i.e.

$$|A_0 \vee A_1 \to B|^{q_0 a, q_1 a, \lambda n.\mathsf{cond}(n, s_0 a, s_1 a)}_{x_0, x_1, n, w}.$$

More generally, if Γ had been used in the sub-proofs π_0 and π_1 then we would actually have

$$|\Gamma \to (A_0 \vee A_1 \to B)|^{p_0 a, q_0 a, q_1 a, \lambda n.\mathsf{cond}(n, s_0 a, s_1 a)}_{x_0, x_1, n, w}.$$

Again by IH, the derivation π_2 is transformed into a proof of

$$|\Gamma \to A_0 \vee A_1|^{p_1 a, t_0 a, t_1 a, ta}_{l, y_0, y_1},$$

for some closed terms p_1, t_0, t_1 and t. By conditions (A2) and (A3) (similarly to the treatment of $\to$E) we get

$$|\Gamma \to B|^{\mathsf{a}_3 a, \mathsf{cond}(\lambda l.tall, (\lambda l.s_0 all) \circ (\lambda l.t_0 all), (\lambda l.s_1 all) \circ (\lambda l.t_1 all))}_{l, w}$$

Existential introduction. Let us consider the following instance of the rule

$$\dfrac{\begin{matrix} \{\Gamma\}_\alpha \\ \vdots\ \pi \\ A(s) \end{matrix}}{\exists z A(z)}\ \exists\mathrm{I}$$

By induction hypothesis we get closed terms q and t and a proof of

$$|\Gamma \to A(s)|^{qa, ta}_{l, y},$$

which can be viewed as a proof of $|\Gamma \to \exists z A(z)|^{qa, ta, s}_{l, y}$.

Existential elimination. Consider the rule

$$\dfrac{\begin{matrix} \{\Gamma\}_\alpha \\ \vdots\ \pi_0 \\ \exists z A(z) \end{matrix} \qquad \begin{matrix} \{\Gamma\}_\alpha, [A(z)]_\beta \\ \vdots\ \pi_1 \\ B \end{matrix}}{B}\ \exists\mathrm{E}_\beta$$

By induction hypothesis we get derivations of

(i) $|\Gamma \to A(ral)|^{q_0 a, sa}_{l_0, y}$
(ii) $|\Gamma \to (A(z) \to B)|^{q_1, paz, taz}_{l_1, x, w}$

Let $z := ral$ in (ii). By conditions (A2) and (A3) we get $|\Gamma \to B|^{\mathsf{a}_3 a, (ta(ral)) \circ (sa)}_{l, w}$.

Induction rule. Recall that we might assume that the subproofs in the induction rule do not contain undischarged assumptions. Assume we have already analysed the two subproofs obtaining

$$\dfrac{\dfrac{\dfrac{\begin{matrix} \emptyset \\ \vdots\ \tilde{\pi}_0 \\ |A(0)|^s_y \end{matrix}}{\forall y |A(0)|^{\mathsf{R}(s,t,0)}_y}\ \forall\mathrm{I} \qquad \dfrac{\dfrac{\dfrac{\begin{matrix} \emptyset \\ \vdots\ \tilde{\pi}_1 \\ \forall y' \sqsubset q[x,y] |A(n)|^x_{y'} \to |A(n+1)|^{tx}_y \end{matrix}}{\forall y (\forall y' \sqsubset q[x,y] |A(n)|^x_{y'} \to |A(n+1)|^{tx}_y)}\ \forall\mathrm{I}}{\forall x (\forall y |A(n)|^x_y \to \forall y |A(n+1)|^{tx}_y)}}{\forall y |A(n)|^{\mathsf{R}(s,t,n)}_y \to \forall y |A(n+1)|^{\mathsf{R}(s,t,n+1)}_y}}{\forall y |A(n)|^{\mathsf{R}(s,t,n)}_y}\ \mathrm{IND}}{|A(n)|^{\mathsf{R}(s,t,n)}_y}\ \forall\mathrm{E}$$

having that $\mathsf{R}(s, t, 0) = s$ and $\mathsf{R}(s, t, n+1) = t\, \mathsf{R}(s, t, n)$. Here we have used on more condition on the abbreviation $\forall y \sqsubset q\, A(y)$, namely that $\forall y A(y) \to \forall y \sqsubset q\, A(y)$. $\Box$

Appendix C: Proof of Theorem 6

The proof proceeds by induction on the structure of the proof $\mathsf{HA}^\omega + \{\Gamma\} \vdash A$. We will use most of the notation introduced in Appendix B.

Axioms. All the axioms of HA^ω are universal, i.e. of the form $\forall n A_{\mathsf{qf}}(n)$, so that nothing needs to be done. i.e. the axioms of HA^ω imply their interpretation.

Assumption. Let us consider first the case of an undischarged assumption Γ. By assumption (A1)* we have

$$\exists \nu \prec \mathsf{a}_1^* \forall a, l, u | \Gamma \to \Gamma|_{l,u}^{\nu a, \lambda l.l}.$$

Let $t := \lambda a, l.l$. By (E2) we then get

$$\exists \nu \prec \mathsf{a}_1^*, f \prec t^* \forall a, l, u | \Gamma \to \Gamma|_{l,u}^{\nu a, fa},$$

which is the derivation in the new proof associated with the undischarged assumption Γ in the original proof.

Falsity elimination. Consider an arbitrary instance

$$\dfrac{\begin{array}{c}\{\Gamma\}_\alpha \\ \vdots\, \pi \\ \bot\end{array}}{A}\ \bot\,\mathrm{E}$$

of this rule. By induction hypothesis we have a derivation $\tilde{\pi}$ of

$$\exists g \prec r \forall a, l | \Gamma \to \bot|_l^{ga},$$

for some closed term r. By the assumption (E1) we can derive $\exists g \prec r \forall a, l | \Gamma \to A|_{l,y}^{t[g]a, sa}$, where $s := \lambda a.0$ (the zero functional of appropriate type) and $t[f] := \lambda a, l, y.fal$. By (E2) we get

$$\exists g \prec t^*, f \prec s^* \forall a, l, y | \Gamma \to A|_{l,y}^{ha, fa}.$$

Conjunction introduction. In this case

$$\dfrac{\begin{array}{c}\{\Gamma\}_\alpha \\ \vdots\, \pi_0 \\ A_0\end{array} \qquad \begin{array}{c}\{\Gamma\}_\alpha \\ \vdots\, \pi_1 \\ A_1\end{array}}{A_0 \wedge A_1}\ \wedge\mathrm{I}$$

by induction hypothesis we have closed terms t_0, t_1, q_0 and q_1 such that

$$\begin{array}{l}\exists f_0 \prec t_0, g_0 \prec q_0 \forall a, l_0, y_0 | \Gamma \to A_0|_{l_0, y_0}^{g_0 a, f_0 a} \\ \exists f_1 \prec t_1, g_1 \prec q_1 \forall a, l_1, y_1 | \Gamma \to A_1|_{l_1, y_1}^{g_1 a, f_1 a}.\end{array}$$

By (E1), from these we can derive

$$\exists f_0 \prec t_0, f_1 \prec t_1, g_0 \prec q_0, g_1 \prec q_1 \forall a, l_0, l_1, y_0, y_1 | \Gamma \wedge \Gamma \to A_0 \wedge A_1|_{l_0, l_1, y_0, y_1}^{g_0 a, g_1 a, f_0 a, f_1 a}.$$

By (A2)* we have

$$\exists \chi \prec \mathsf{a}_2, f_0 \prec t_0^*, f_1 \prec t_1^* \forall a, l, y_0, y_1 | \Gamma \to A_0 \wedge A_1|_{l, y_0, y_1}^{\chi a, f_0 a, f_1 a},$$

where t_i^* majorizes $\lambda l.tll$.

Conjunction elimination. By induction hypothesis we have closed terms q, t_0 and t_1 and a proof of

$$\exists g \prec q, f_0 \prec t_0, f_1 \prec t_1 \forall a, v, y_0, y_1 | \Gamma \to A_1 \wedge A_2|_{v, y_0, y_1}^{ga, f_0 a, f_1 a}.$$

By (E1), this gives a derivation of $\exists g \prec q', f_i \prec t_i \forall a, v, y_i | \Gamma \to A_i|^{ga, f_i a}_{v, y_i}$, for $i \in \{0, 1\}$, where q' is an appropriate projection of q.

Implication introduction. This is immediately treated by the induction hypothesis.

Implication elimination. Let us consider the implication elimination

$$\dfrac{\begin{array}{c}\{\Gamma\}_\alpha \\ \vdots\, \pi \\ A\end{array} \qquad A \to B}{B} \to \mathrm{E}$$

assuming for simplicity that only the derivation of A has assumptions (otherwise (E2)* must also be used). By induction hypothesis we have

$$\begin{array}{l} \exists g \prec p, f \prec s \forall a, l, y | \Gamma \to A|^{ga, fa}_{l, y} \\ \exists h \prec q, j \prec t \forall a, x, u | A \to B|^{ha, ja}_{x, u} \end{array}$$

which by (E1) gives

$$\exists g \prec p, f \prec s, h \prec q, j \prec t \forall a, l, y, x, u | \Gamma \to (A \wedge (A \to B))|^{ga, fa, ha, ja}_{l, y, x, u}.$$

Applying condition (A3)* we get

$$\exists \xi \prec \mathsf{a}_3, f \prec s, j \prec t \forall a, l, u | \Gamma \to B|^{\mathsf{a}_3 a, (ja) \circ (fa)}_{l, u}.$$

Finally, by (E2) we obtain

$$\exists \xi \prec \mathsf{a}_3, f \prec r^* \forall a, l, u | \Gamma \to B|^{\mathsf{a}_3 a, fa}_{l, u}.$$

where $r[j, f] := \lambda a.(ja) \circ (fa)$.

Universal introduction and elimination. Let us look at the following instance of the rule

$$\dfrac{\begin{array}{c}\{\Gamma\}_\alpha \\ \vdots\, \pi \\ A(z)\end{array}}{\forall z A(z)} \forall \mathrm{I}$$

By induction hypothesis we have closed terms r and t and a derivation of

$$\exists g \prec r, f \prec t \forall a, z, l, y | \Gamma \to A(z)|^{gaz, faz}_{l, y}.$$

This can be simply viewed as a derivation of

$$\exists g \prec r, f \prec t \forall a, z, l, y | \Gamma \to \forall z A(z)|^{ga, fa}_{l, y, z}.$$

Similarly with the universal elimination.

Disjunction introduction. Assume we have a derivation of

$$\exists g \prec q, f_0 \prec t_0 \forall a, l, y_0 | \Gamma \to A_0|^{ga, f_0 a}_{l, y_0}.$$

By (E1) we can then obtain a derivation of

$$\exists g \prec q, f_0 \prec t_0 \forall a, l, y_0, y_1 | \Gamma \to A_0 \vee A_1|^{ga, f_0 a, 0, 0}_{l, y_0, y_1},$$

where 0 is the zero functional of appropriate type, and y_1 is a fresh variable. By (E2) this gives

$$\exists g \prec q, f_0 \prec t_0, f_1 \prec t_1, k \prec s \forall a, l, y_0, y_1 | \Gamma \to A_0 \vee A_1|^{ga, f_0 a, f_1 a, ka}_{l, y_0, y_1},$$

Similarly with the case $\Gamma \vdash A_1$.

<u>*Disjunction elimination*</u>. Let us consider now the rule

$$\dfrac{\begin{array}{c}\{\Gamma\}_\beta \\ \vdots\,\pi_2 \\ A_0 \vee A_1\end{array} \quad \begin{array}{c}[A_0]_{\alpha_0} \\ \vdots\,\pi_0 \\ B\end{array} \quad \begin{array}{c}[A_1]_{\alpha_1} \\ \vdots\,\pi_1 \\ B\end{array}}{B}\ \vee\mathrm{E}_{\alpha_0,\alpha_1}$$

Applying the IH we get tuples of closed terms r_0, r_1, p_0, p_1, terms s_0, s_1 and the derivations of

(i) $\exists g_0 \prec q_0, f_0 \prec s_0 \forall a, x_0, w | A_0 \to B|^{g_0 a, f_0 a}_{x_0, w}$
(ii) $\exists g_1 \prec q_1, f_1 \prec s_1 \forall a, x_1, w | A_1 \to B|^{g_1 a, f_1 a}_{x_1, w}$

By the decidability of equality of basic type and assumption (E1) we get

$$\begin{cases} \exists g_0 \prec q_0, g_1 \prec q_1, f_0 \prec s_0, f_1 \prec s_1 \forall a, x_0, x_1, n, w \\ \qquad |((n = 0 \to A_0) \wedge (n \neq 0 \to A_1)) \to B|^{g_0 a, g_1 a, \lambda n.\mathsf{cond}(n, f_0 a, f_1 a)}_{x_0, x_1, n, w}, \end{cases}$$

i.e.

$$\exists g_0 \prec q_0, g_1 \prec q_1, f_0 \prec s_0, f_1 \prec s_1 \forall a, x_0, x_1, n, w | A_0 \vee A_1 \to B|^{g_0 a, g_1 a, \lambda n.\mathsf{cond}(n, f_0 a, f_1 a)}_{x_0, x_1, n, w}.$$

Again by IH, the derivation π_2 is transformed into a proof of

$$\exists h \prec p, j_0 \prec t_0, j_1 \prec t_1, j \prec t \forall a, l, y_0, y_1 | \Gamma \to A_0 \vee A_1|^{ha, j_0 a, j_1 a, ja}_{l, y_0, y_1},$$

for some closed terms p, t_0, t_1 and t. Finally, by conditions (E1), (E2) and (A3)* we get

$$\exists \xi \prec \mathsf{a}_3^*, f \prec r^* \forall a, l, w | \Gamma \to B|^{\xi a, fa}_{l, w},$$

where $r[f_0, f_1, j_0, j_1, j] := \lambda a.\mathsf{cond}(ja, (f_0 a) \circ (j_0 a), (f_1 a) \circ (j_1 a))$.

<u>*Existential introduction*</u>. Let us consider the following instance of the rule

$$\dfrac{\begin{array}{c}\{\Gamma\}_\alpha \\ \vdots\,\pi \\ A(s[a])\end{array}}{\exists z A(z)}\ \exists\mathrm{I}$$

By induction hypothesis we get closed terms q and t and a proof of

$$\exists g \prec q, f \prec t \forall a, l, y | \Gamma \to A(s[a])|^{ga, fa}_{l, y},$$

which can be viewed as a proof of $\exists g \prec q, f \prec t \forall a, l, y | \Gamma \to \exists z A(z)|^{qa, ta, (\lambda a. s[a])a}_{l, y}$. By (E2) we have a closed term s^* and a proof of

$$\exists g \prec q, f \prec t, h \prec s^* \forall a, l, y | \Gamma \to \exists z A(z)|^{qa, ta, ha}_{l, y}.$$

<u>*Existential elimination*</u>. Consider the rule

$$\dfrac{\begin{array}{c}\{\Gamma\}_\alpha \\ \vdots\,\pi_0 \\ \exists z A(z)\end{array} \quad \begin{array}{c}[A(z)]_\beta \\ \vdots\,\pi_1 \\ B\end{array}}{B}\ \exists\mathrm{E}_\beta$$

By induction hypothesis we get derivations of

(i) $\exists g \prec q, f \prec s \forall a, l, y | \Gamma \to A(ral)|^{ga, fa}_{l, y}$
(ii) $\exists h \prec p, j \prec t \forall a, x, w | A(z) \to B|^{haz, jaz}_{x, w}$

Let $z := ral$ in (ii). By conditions (E1), (E2) and (A3)* we get

$$\exists g \prec \mathsf{a}_3^*, f \prec t^* \forall a, l, w | \Gamma \to B|^{\mathsf{a}_3^* a, fa}_{l, w}.$$

Induction rule. We assume to have already analysed the two subproofs of the induction rule, obtaining

(i) $\exists f \prec s \forall a, y |A(0)|_y^{fa}$
(ii) $\exists g \prec q, h \prec t \forall a, n, x, y |A(n) \to A(n+1)|_{x,y}^{ga,ta}$.

By condition (E1), from (ii) we get

$$\exists h \prec t \forall a, n, x (\forall y |A(n)|_y^x \to \forall y |A(n+1)|_y^{hx}),$$

which gives

$$\exists f \prec s, h \prec t \forall a, n (\forall y |A(0)|_y^{\mathsf{R}(f,h,0)a} \wedge (\forall y |A(n)|_y^{\mathsf{R}(f,h,n)a} \to \forall y |A(n+1)|_y^{\mathsf{R}(f,h,n+1)a})).$$

By induction (again using condition (E1)) we get

$$\exists f \prec s, h \prec t \forall a, n, y |A(n)|_y^{\mathsf{R}(f,h,n)a},$$

which by condition (E2) implies

$$\exists f \prec r^* \forall a, n, y |A(n)|_y^{fa},$$

where $r[f,h] := \lambda n. \mathsf{R}(f,h,n)$. Here we have again used one more condition on the abbreviation $\forall y \sqsubset q\, A(y)$, namely that $\forall y A(y) \to \forall y \sqsubset q\, A(y)$. $\square$