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Non-Transitive Consumer Behavior

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Abstract. Rational choice when preferences are not required to be transitive and complete has been discussed in the literature for years. In this article transitivity and completeness of the preference relation is also not assumed. It will be shown that nevertheless the existence of a competitive equilibrium can be proven when those properties are replaced by a domination property which allows that there could be cicles among those alternatives which are of less importance for the individual and which he or she would never choose if better ones are available.
Moreover, one can show that the compensated demand function is continuous under very weak conditions, and because of this, Shephard's lemma follows without assuming transitivity and completeness of the underlying preferences.

Extended Abstract

1. Introduction

Preferences of the individuals in an economy are usually assumed to be transitive and complete. In this article models of consumer behavior will be considered when preferences do not possess these properties. This conception is realistic since we often are not aware which of two alternatives we prefer, especially when they are of less importance for us or when it is not possible to compare them directly with each other. An appealing example when indifference is not transitive is the following one:
An individual may be indifferent between 1000 Euro and 1001 Euro, and he may be also indifferent between 1001 Euro and 1002 Euro, and so on, but usually he will not be indifferent between 1000 Euro and 10000 Euro.
In the following analysis two models of consumer behavior will be considered when preferences are not assumed to be transitive and complete. We will see that nevertheless appealing results can be deduced, as for instance continuity of the demand functions or

the existence of a competitive equilibrium.

2. Existence of a Competitive Equilibrium

The existence of a competitive equilibrium without assuming transitivity and completeness of the underlying preferences has been firstly investigated by Gale and Mas-Colell [1], Mas-Colell [7] and Shafer and Sonnenschein [8]. These authors develop their analysis for abstract economies. More recently, an article of Kim and Richter [6] is also concerned with this problem. In this article instead of transitivity and completeness of the preferences another property, defined in Definition 1, will be applied.
Therefore, let us consider a nonempty set of alternatives X, and a relation R on X, representing the preferences of the individual. xRy means, in the opinion of the individual, x is at least as good as y.

Definition 1: (Domination of alternatives): For any relation R on X, a finite set $\{x^1, ..., x^m\} \subseteq B$ is said to be "dominated in B" if there exists an $y \in B$ such that yRy and yRx^i for $j = 1, ..., m$.
The intuitive interpretation of Definition 1 is the following one: The individual is aware that he appreciates the alternative y at least as much as the other alternative x^1,...,x^k, but he may be undecided between some of the latter.

In order to recall what is meant by a competitive equilibrium in an economy we have preliminarily to recall further definitions and results. Therefore, let us consider an economic agent who in every price-income situation $(p,\ M)$, $(p,\ M)\in\ \mathbb{R}^n_{++} \times \mathbb{R}_+$[1]), chooses a commodity bundle $h(p, M)$ out of a budget set $B(p, M)= \{x \in \mathbb{R}^n_+ \mid px \leq M\}$. This function $h : \mathbb{R}^n_{++} \times \mathbb{R}_+ \longrightarrow \mathbb{R}^n_+$, $x = h(p, M)$, describing consumer behavior, will be called "demand function". More generally one can introduce a "demand correspondence"

[1]) $x \in \mathbb{R}^n_+ \Longleftrightarrow x \geq 0, x = (x_1, ..., x_n.)$, $x \in \mathbb{R}^n_{++} \Longleftrightarrow x > 0$.

$h : \mathbb{R}^n_{++} \times \mathbb{R}_+ \longrightarrow 2^{\mathbb{R}^n_+}$, where $2^{\mathbb{R}^n_+}$ denotes the power set of $\mathbb{R}^n_+$. In our interpretation this means, that the individual can choose more than one commodity bundle out of $B(p, M)$ in the price-income situation (p, M). Throughout this paper we assume $h(p, M) \neq \emptyset$, for all $(p, M) \in \mathbb{R}^n_{++} \times \mathbb{R}_+$. We will call a consumer "rational" when in every price-income situation he or she chooses the best alternatives out of $B(p, M)$. This also presumes, that the consumer reveals his or her preferences by the actions. Formally it means

Definition 2: Given a demand correspondence $h : \mathbb{R}^n_{++} \times \mathbb{R}_+ \longrightarrow 2^{\mathbb{R}^n_+}$, and a relation R on $X \subseteq \mathbb{R}^n_+$. Then h is called "rational" with respect to R iff for all $(p, M) \in \mathbb{R}^n_{++} \times \mathbb{R}_+$, $h(p, M) = \{x \in X \mid x \in B(p, M) \wedge \forall y \in X \cap B(p, M) : xRy\}$.

We will now consider a finite, non-empty set I of individuals and an exchange economy $\mathfrak{E}$ where every individual i has an initial equipment $e^i \in \mathbb{R}^n_+$ which he can sell. Given the price system $p \in \mathbb{R}^n_{++}$ the individual can earn $pe^i = \sum_{j=1}^n p_j e^i_j$ amounts of money.
In order to make clear that the demand function of every individual depends on his or her preference relation $\succeq_i$ on $\mathbb{R}^n_+$ and the initial equipment e^i we follow Hildenbrand and Kirman [5] and write $\varphi(\succeq_i, e^i, p)$ instead of $h^i(p, M)$.
Every individual $i \in I$ acts rationally with respect to $\succeq_i$, and his or her behavior is described by a demand correspondence $\varphi : p \longrightarrow \varphi(\succeq_i, e^i, p)$, where $p \in \mathbb{R}^n_{++}$ is a price vector and $\varphi(\succeq_i, e^i, p) = \{x \in \mathbb{R}^n_+ \mid px \leq pe^i \wedge \forall y \in \mathbb{R}^n_+ : py \leq pe^i \Rightarrow x \succeq_i y\}$.
A Walras-equilibrium $((x^{*i})_{i \in I}, p^*)$ of an economy $\mathfrak{E} = ((\succeq_i)_{i \in I}, (e^i)_{i \in I}, \mathbb{R}^n_+)$ is defined by

i) $x^{*_i} \in \varphi(\succeq_i, e^i, p^*)$ and
ii) $\sum_{i \in I} x^{*_i} \leq \sum_{i \in I} e^i$.

We will present conditions which imply the existence of a competitive equilibrium. The proof bases heavily on the upper hemicontinuity of the demand correspondence which has been proved in [2]. Upper hemicontinuity of a demand function can be defined in the following way [5]:

Let $F : S \longrightarrow 2^T$, where $S, T \subseteq \mathbb{R}^n$, then F is called "upper hemicontinuous" at $x^0 \in S$, if for every $< x^k >, x^k \in S$, with $\lim_{k\to\infty} x^k = x^0$, and for every sequence $< y^k >$ with $y^k \in F(x^k)$, there exists a convergent subsequence $< y^{k_j} >$ of $< y^k >$, such that $\lim_{j\to\infty} y^{k_j} = y \in F(x^0)$. F is called upper hemicontinuous if it is upper hemicontinuous at every $x \in S$.

Now I will recall Theorem 3 in [2], where upper hemicontinuity of a demand correspondence is shown, when the domain of h is $\mathfrak{B}^*$, i.e. the family of all competitive budget sets $B(p, M) = \{x \mid x \in \mathbb{R}^n_+ \wedge px \leq M\}$ for all $p \in \mathbb{R}^n_{++}$ and $M \in \mathbb{R}_+$. Instead of $h(B(p, M))$ one can write $h(p, M)$.

Lemma 1 *(see Theorem 3 in [2], p.75[2)]) Consider R on $\mathbb{R}^n_+$ such that R is continuous[3)] on $\mathbb{R}^n_+$ and let $R(x)$ be convex for all $x \in \mathbb{R}^n_+$. Furthermore, let every finite subset $A \subseteq B$ with cardinality $n + 1$ be dominated in B for every $B \in \mathfrak{B}^*$. Then there exists an upper hemicontinuous choice correspondence $h : \mathbb{R}^n_{++} \times \mathbb{R}_+ \longrightarrow 2^{\mathbb{R}^n_+}$ rational with respect to R.*

We will now demonstrate the existence of a competitive equilibrium when we assume domination of alternatives instead of completeness and transitivity of the preferences. The proof bases heavily on Hildenbrand's and Kirman's Proposition 3.1 ([5], p.93) where these authors show certain properties of the demand functions which imply the existence of a competitive equilibrium. It will be shown that these properties also hold under the conditions of this article.

Theorem 2 *Consider an economy $\mathfrak{E}$ with every agent $i \in I$ having a continuous and*

[2)] In the proof of Theorem 3 in [2], p.75, line 16 from above write "there exists $z^0 \in B(p^0, M^0)$" instead of "there exists $z^0 \in h(p^0, M^0)$".

[3)] R is continuous on X, if for all $x \in X$, $R(x) = \{z \mid z \in X \wedge zRx\}$ and $R^{-1}(x) = \{z \mid z \in X \wedge xRz\}$ is closed in X.

monotonic relation[4] $\succeq_i$ *on* $I\!R^n_+$, *and let* $R^i(x) = \{y \mid y \in I\!R^n_+ \wedge y \succeq_i x\}$ *be convex for all* $x \in I\!R^n_+$. *Moreover, let for every* $B \in \mathfrak{B}^*$ *every set* $A \subseteq B$ *with* $\mid A \mid = n+1$ *be dominated in* B, *and additionally assume* $\sum_{i \in I} e^i > 0$. *Then there exists a Walras-equilibrium* $\Big((x^{*_i})_{i\in I}, (p^*)\Big)$ *in* $\mathfrak{E}$. *If additionally,* $\succeq_i$ *satisfies the following strict convexity condition,* $x \sim_i y \wedge x \neq y \Rightarrow \lambda x + (1-\lambda)y \succ_i y, \forall \lambda \in]0,1[$[5], *for all* $x, y \in I\!R^n_+$, *then* $x^i = \varphi(\succeq_i, e^i, p^*)$ *is fulfilled.*

Proof. Application of Lemma 1 together with the definition of φ yields that $\varphi(\succeq_i, e^i, \cdot)$ is well-defined on $\mathbb{R}^n_{++}$ and rational with respect to $\succeq_i$. Moreover, $\varphi(\succeq_i, e^i, p)$ is compact, since $R^i(x,p) = \{y \in \mathbb{R}^n_+ \mid py \leq pe^i \wedge y \succeq_i x\}$ is compact, and therefore $\varphi(\succeq_i, e^i, p) = \bigcap_{x \in \mathbb{R}^n_+} R^i(x,p)$ is compact. From the definition of φ we can also immediately conclude that $\varphi(\succeq_i, e^i, \cdot)$, is homogeneous of degree zero in prices. Upper hemicontinuity of $\varphi(\succeq_i, e^i, \cdot)$, immediately follows from Lemma 1. Next, one has to show the following property:
For every sequence $< p^k >, p^k \in \mathbb{R}^n_{++}$:
$\Big[\lim_{k\to\infty} p^k = \bar{p} \geq 0 \wedge \bar{p} \neq 0 \wedge \bar{p} \not> 0\Big] \Rightarrow \inf\{\| x \| \mid x \in \varphi(\succeq_i, e^i, p^k)\} \to \infty$.
However, if we examine the proof of Hildenbrand and Kirman in order to show this property we can see that transitivity or completeness of $\succeq_i$ is nowhere used, and thus it also holds under the present conditions.
Since $R^i(x)$ is convex, we also have that $R^i(x,p)$ is convex, and therefore
$\varphi(\succeq_i, e^i, p) = \bigcap_{x \in \mathbb{R}^n_+} R^i(x,p)$ is convex. Finally, if $\succeq_i$ satisfies the above strict convexity condition, then φ is a function.
Based on the above properties which are only concerned with demand correspondences or - if $\succeq_i$ is strictly convex - with demand functions, the existence of a Walras-equilibrium follows (see [5], Theorem 3.1 and 3.2). ■

We thus have seen that in the existence theorem of a Walras-equilibrium we can replace transitivity and connectedness of the preference relation by the domination of alternatives.

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4) A relation $\succeq$ on X is monotonic if $x \geq y \wedge x \neq y \Rightarrow \lambda x + (1-\lambda)y \succ y, \forall \lambda \in]0,1[$ where $\succ$ is the asymmetric part of $\succeq$.

5) $x \sim y$ means, $x \succeq y \wedge y \succeq x; x \succ y$ means, $x \succeq y \wedge \neg(y \succeq x)$.

3. Compensated Demand

Closely related to (direct) demand functions are compensated demand functions, defined by $g(p,x) = \arg\min \{pz \mid z \in X \wedge z \succeq x\}$, for $(p,x) \in \mathbb{R}^n_{++} \times X$, where $X \subseteq \mathbb{R}^n_+$. Hence, $g(p,x)$ consists of those alternatives which according to the opinion of the individual are at least as good as x and which are the cheapest ones in the price situation p. One can show that compensated demand functions, or more generally, compensated demand correspondences are defined under quite weak assumptions [3]. Therefore, let us recall the following basic model of consumer behavior:

(A1) $X \subseteq \mathbb{R}^n_+, X \neq \emptyset$, is supposed to be a closed set alternatives.
(A2) $\succeq$ is a reflexive relation on X.
(A3) $\succeq$ is upper semicontinuous on X, i.e. the set $R(x) = \{y \in X \mid y \succeq x\}$ is closed in X for every $x \in X$.

Under these conditions the compensated demand correspondence $g : \mathbb{R}^n_{++} \times X \longrightarrow 2^X, g(p,x) \subseteq X$ is well defined. One can even show that under these weak assumptions (A1) to (A3), $g(\cdot, x^0)$ is upper hemicontinuous with respect to p. For the proof recall that $m(p,x) = min\{pz \mid z \in X \wedge z \succeq x\}$ is known as income compensation function.

Theorem 3 *Under (A1) to (A3), $g : \mathbb{R}^n_{++} \times X \longrightarrow 2^X, g(p,x^0) \subseteq X$, is upper hemicontinuous with respect to p for every $x^0 \in X$.*

Proof. Let $< p^k >, p^k \in \mathbb{R}^n_{++}$ be a sequence, such that $\lim_{k\to\infty} p^k = \tilde{p} \in \mathbb{R}^n_{++}$ and let $x^k \in g(p^k, x^0)$. Since $\succeq$ is reflexive the definition of $g(p^k, x^0)$ yields $p^k x^k \leq p^k x^0$. Since $< p^k >$ is convergent it is also bounded, and hence, in view of the previous inequality, $< x^k >$ is also bounded. Thus there exists a subsequence $< x^{k_j} >$ of $< x^k >$, and $\tilde{x} \in \mathbb{R}^n_+$ such that $\lim_{j\to\infty} x^{k_j} = \tilde{x}$. Since $p^{k_j} x^{k_j} \leq p^{k_j} x^0$, we obtain $\lim_{j\to\infty} p^{k_j} x^{k_j} \leq \tilde{p} x^0$.

By definition of $g(p^{k_j}, x^0), x^{k_j} \succeq x^0$, and hence in view of upper semicontinuity of $\succeq$, $\tilde{x} \succeq x^0$. By definition of $g(p^k, x^0)$ we have $p^k x^k = m(p^k, x^0)$. By a former result (see [3], Lemma 1) $m(\cdot, x^0)$ is continous with respect to p, and thus $\tilde{p}\tilde{x} = \lim_{j\to\infty} p^{k_j} \lim_{j\to\infty} x^{k_j} = \lim_{j\to\infty} p^{k_j} x^{k_j} = \lim_{j\to\infty} m(p^{k_j}, x^0) = m(\tilde{p}, x^0)$.
Since we also have $\tilde{x} \succeq x^0$, we thus obtain $\tilde{x} \in g(\tilde{p}, x^0)$. This concludes our proof. ∎

The above result is rather important, because based on it one can show that Shephard's Lemma holds, when $g(p, x)$ is single-valued [4]. Single-valuedness of $g(p, x)$ for instance follows, when $\succeq$ is strictly convex.

4. Summary

In this article conditions were presented which imply the existence of a competitive equilibrium without assuming transitivity and completeness of the underlying individual preferences. These attributes are replaced by a domination property which is concerned with the mostly preferred alternatives in the budget sets.
Moreover, one can show that under conditions still weaker than the above ones the compensated demand correspondence also possesses important properties, which imply the validity of Shephard's Lemma.

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