Generalized metatheorems on the extractability of uniform bounds in functional analysis (extended abstract)

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1 Introduction

In [6], the second author proved general metatheorems for the extraction of effective uniform bounds from ineffective existence proofs in functional analysis, more precisely from proofs in classical analysis $\mathcal{A}^\omega$(:= weakly extensional[1] Peano arithmetic WE-PA in all finite types + quantitifer-free choice + the axiom schema of dependent choice DC) extended with (variants of) abstract bounded metric spaces (X, d), and bounded hyperbolic spaces (X, d, W) ([6,9,4]) as well as abstract normed linear spaces $(X, \|\cdot\|)$ with a bounded convex subset $C \subseteq X$.

The theories $\mathcal{A}^\omega[X, d]$, $\mathcal{A}^\omega[X, d, W]$, $\mathcal{A}^\omega[X, \|\cdot\|, C]$ and further variants – based on CAT(0)-spaces, uniformly convex spaces and inner product spaces – result from extending $\mathcal{A}^\omega$ to the set $\mathbf{T}^X$ of all finite types over the two ground types 0 and X and adding the necessary constants such as d_X and $\|\cdot\|_X$, and (purely universal) axioms for metric, resp. normed linear spaces. In particular, the theories contain an axiom expressing the boundedness of (X, d), resp. the boundedness of the convex subset C.

Extending Kohlenbach's monotone variant of Gödel's[3] and Spector's[8] functional ('Dialectica') interpretation for $\mathcal{A}^\omega$ to these theories one can extract effective bounds from given ineffective existence proofs, where, using a majorization argument, the extracted bounds are shown to be independent of parameters ranging over the bounded metric space, resp. over the bounded convex subset of the normed linear space. The significance of this rests on the fact that this yields

[1] The restriction in the availability of extensionality for types other than 0 by including only Spector's [8] quantifier-free rule of extensionality is of crucial importance for the results outlined below to hold. For the applicability of these results in e.g. metric fixed point theory this does not cause as serious limitation as here usually all the functions involved are provably extensional. See [6] for a thorough discussion of this issue.

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independence from parameters without imposing any compactness conditions on (X, d) or C. For details, see [6].

Two different approaches to extending the Howard-Bezem[5,1] strong majorization relation to the new type X were employed for metric spaces and normed linear spaces. For metric spaces, whose metric is bounded by some $b \in \mathbb{N}$, the majorization relation was defined to be always true for the type X and the metric constant d_X will be majorized by the constant-b function. The extraction of uniform bounds then consist of two main steps: (1) extraction of effective bounds using functional interpretation and (2) majorization in the types $\mathbf{T}^X$ to eliminate the dependency from the new constants of $\mathcal{A}^\omega[X, d]$ and achieve the independence from parameters in X (among other things).

This approach does not work for (non-trivial) normed linear spaces as normed linear spaces always are unbounded. Therefore in [6] for normed linear spaces the majorization relation for the type X was defined via the norm, i.e. x^* s-maj$_X$ $x :\equiv \|x^*\| \geq \|x\|$. The independence of extracted bounds from a (norm-)bounded convex subset C can then be achieved similarly to the bounded metric case. For normed linear spaces, one constructs uniform bounds from the terms extracted by functional interpretation in three steps: (1) majorization of the extracted terms in the types $\mathbf{T}^X$, (2) elimination of the dependency from the new constants of $\mathcal{A}^\omega[X, \|\cdot\|, C]$ using an ineffective operator $()_\circ$ and a relation $\sim_\rho$, and finally (3) majorization in the types $\mathbf{T}$ to eliminate ineffective instances of the $()_\circ$-operator.

Of particular importance is the relation $\sim_\rho$. The relation $\sim_\rho$ relates functionals of type $\rho \in \mathbf{T}^X$ to functionals of type $\widehat{\rho}$, where the mapping $\widehat{\cdot}$ is defined as:

$$(*)\ \widehat{0} := 0,\ \widehat{X} := 0,\ \widehat{\rho \to \tau} := \widehat{\rho} \to \widehat{\tau},$$

for metric spaces and with $\widehat{X} := 1$ for normed linear spaces. Defining the relation $\sim_\rho$ for the constants of $\mathcal{A}^\omega[X, d]$, resp. $\mathcal{A}^\omega[X, \|\cdot\|, C]$ one may inductively translate extracted terms in $\mathcal{A}^\omega[X, d]$, resp. $\mathcal{A}^\omega[X, \|\cdot\|, C]$ into terms of independent of the new constants such as $d_X, \|\cdot\|$ etc. For details on the operator $()_\circ$ and its role in this translation, see [6].

2 Main results

In order to treat *unbounded* metric spaces and *unbounded* convex subsets of metric spaces, we adapt the approach to majorization for normed linear spaces and combine the three steps into one. The main idea is to define a new so-called a-majorization relation $\gtrsim^a$, which is parametrized by an element $a \in X$ and combines Howard-Bezem's strong majorization relation with the relation $\sim_\rho$ as it is defined for metric spaces:

Definition 1. *We define a ternary relation* $\gtrsim^a_\rho$ *between objects* x, y *and* a *of type* $\widehat{\rho}, \rho$ ($\widehat{\rho}$ *defined as in* $(*)$ *above) and* X *respectively as follows:*[2]

- $x^0 \gtrsim^a_0 y^0 :\equiv x \geq_0 y$,
- $x^0 \gtrsim^a_X y^X :\equiv (x)_{\mathbb{R}} \geq_{\mathbb{R}} d_X(y, a)$,
- $x \gtrsim^a_{\rho\to\tau} y :\equiv \forall z', z(z' \gtrsim^a_\rho z \to xz' \gtrsim^a_\tau yz) \wedge \forall z', z(z' \gtrsim^a_{\widehat{\rho}} z \to xz' \gtrsim^a_{\widehat{\tau}} xz)$.

Restricted to the types $\mathbf{T}$, the relation $\gtrsim^a$ coincides with Bezem's strong majorization relation. Using our new majorization relation we can extend Bezem's type structure $\mathcal{M}^\omega$ of strongly majorizable functionals of types in $\mathbf{T}$ to all types is $\mathbf{T}^X$ where for the new ground type X we take our metric (hyperbolic, resp. normed) space as the domain. Whereas for the new type the relation $\gtrsim^a$ depends on a, the type structure of all $\gtrsim^a$**-majorizable** functionals is in fact independent of the choice of $a \in X$. For metric spaces, the chosen element $a \in X$ serves as a reference point, similar to the element 0_X for normed linear spaces, but with the crucial difference that the element $a \in X$ is a variable and not a fixed element of the space like the constant 0_X. For normed linear spaces we consider the metric $d(x, y) \equiv \|x - y\|$ and always choose $a = 0_X$ so that $\mathbb{N} \ni n \gtrsim^a_X x$ iff $(n)_{\mathbb{R}} \geq \|x\|$.

Defining majorants for the constants of $\mathcal{A}^\omega[X, d]_{-b}$, $\mathcal{A}^\omega[X, \|\cdot\|, C]_{-b}$ (where the $-b$ signifies that the axiom expressing the boundedness of (X, d), resp. the convex subset C, is dropped) may again involve the generally ineffective $()_\circ$-operator. However, by letting majorants for elements of type X be natural numbers instead of elements of type X, we avoid ineffective instances of the $()_\circ$-operator. Here, the $()_\circ$-operator is only applied to natural numbers, where it is effectively computable.

The a-majorization relation $\gtrsim^a$ provides a natural extension of Bezem's strong majorization relation to metric and normed linear spaces. To a-majorize an element $x \in X$ means to provide a bound on its distance to the reference point $a \in X$, which in the case of normed linear spaces means a bound on the norm of x. To a-majorize a sequence $z^{0\to X}$ of elements means to provide for each n an upper bound of the largest distance to a among the first n elements of the sequence. To a-majorize a function $f : X \to X$ means to provide a bound on the "displacement" of f, i.e. that is given a bound on $n \geq d(x, a)$ a bound (in n) on $d(f(x), a)$.

From generalized metatheorems for $\mathcal{A}^\omega[X, d]_{-b}$, $\mathcal{A}^\omega[X, \|\cdot\|, C]_{-b}$ and further variants of these theories one may e.g. derive the following theorem for hyperbolic spaces (X, d, W):

Definition 2. *A function* $f : X \to X$ *on a metric space* (X, d) *is called*

- *nonexpansive ('f n.e.') if* $d(f(x), f(y)) \leq d(x, y)$ *for all* $x, y \in X$,
- *Lipschitz continuous if* $d(f(x), f(y)) \leq K \cdot d(x, y)$ *for some* $K > 0$ *and for all* $x, y \in X$,

[2] Here we refer to the representation of real numbers by number theoretic functions from [6], i.e. objects of type 1. $(x^0)_{\mathbb{R}}$ represents the canonical embedding of $\mathbb{N}$ into $\mathbb{R}$ under this representation.

– *Hölder-Lipschitz continuous if* $d(f(x), f(y)) \le K \cdot d(x,y)^\alpha$ *for some* $K > 0$, $0 < \alpha \le 1$ *and for all* $x, y \in X$.

Theorem 1 ([2]).

1. *Let* P *(resp.* K*) be a* $\mathcal{A}^\omega$*-definable Polish space (resp. compact Polish space)*[3]. *Assume we prove, in* $\mathcal{A}^\omega[X,d,W]_{-b}$*:*

$$\forall x \in P \forall y \in K \forall z^X \forall f^{X \to X}(f \text{ n.e.} \wedge \forall u^0 B_\forall(x,y,z,f,u) \to \exists v^0 C_\exists(x,y,z,f,u))$$

where 0_X *does not occur in* $B_\forall$ *and* $C_\exists$[4].
Then there is a computable functional $\Phi : \mathbb{N}^\mathbb{N} \times \mathbb{N} \to \mathbb{N}$ *s.t. for all* $g_x \in \mathbb{N}^\mathbb{N}$ *representing an element* $x \in P$ *and all* $y \in K$

$$\forall z^X \forall f^{X \to X}(f \text{ n.e.} \wedge d(z, f(z)) \le b \wedge \forall u^0 \le \Phi(g_x, b)\ B_\forall \to \exists v^0 \le \Phi(g_x, b)\ C_\exists)$$

holds in all (non-empty) hyperbolic spaces (X, d, W).
2. *The Corollary also holds for an additional parameter* $\forall {z'}^X$ *and the additional premise* $d(z, z') \le b$.
3. *Furthermore, the Corollary holds for an additional parameter* $\forall a^{0 \to X}$ *and the additional premise* $\forall n(d(z, a(n)) \le b)$ *or even* $\forall n(d(z, a(n)) \le g(n))$*, where the extracted bound then additionally depends on* g.
4. *1., 2. and 3. also hold if we replace 'f n.e.' with f satisfying*

$$d(x,y) \le n \to d(f(x), f(y)) \le \Omega(n)\ (**)$$

for all $x, y \in X$*, where* Ω *is a function* $\Omega : \mathbb{N} \to \mathbb{N}$*. The extracted bound will then additionally depend on* Ω*. This condition covers Lipschitz continuous and Hölder-Lipschitz continuous functions (with constants* K*, resp.* $K, \alpha : \mathbb{Q}^*_+$*, where* $\alpha \le 1$*), as well as uniformly continuous functions (with a modulus of uniform continuity* ω*). Thus, for Lipschitz and Lipschitz-Hölder continuous functions the bound depends on the given constants, for uniformly continuous functions the bound depends on the given modulus of uniform continuity.*[5]

In contrast to the results from [6], the theorem above does not require the whole space X to be bounded but only a bound b on $d(z, f(z))$. Nevertheless the bound Φ is independent not only from $y \in K$ but even from $z \in X$ and $f : X \to X$. This fact allows one to extend the applications to metric fixed point

[3] We assume the Polish, resp. compact Polish space are given in so-called standard representation, see [6].

[4] A ∀-formula, resp. an ∃-formula is a purely universal, resp. existential formula, where moreover the types of the quantified variables are of suitable restricted type. For details see [6].

[5] In contrast to the assumptions on f such as being nonexpansive, Lipschitzean or uniformly continuous (which imply the extensionality of f) the condition $(**)$ does not imply extensionality. So in the context of $(**)$ our restriction to weak extensionality is significant.

theory given in [6] considerably and explains a number of results obtained in case studies where bounds Φ of the kind predicted by the theorem had been explicitly extracted (e.g. [7]).

Similar results can be obtained for the case of normed spaces $(X, \|\cdot\|)$ and convex subsets $C \subseteq X$ (then, in addition to $b \geq \|z - f(z)\|$, one needs that $b \geq \|z\|$), uniformly convex spaces as well as inner product spaces.

Finally, we note that instead of a single space X we can also add finitely many abstract (metric, hyperbolic or normed) spaces $X_1, \ldots, X_n$ simultaneously to $\mathcal{A}^\omega$. All this will be carried out in detail in [2].

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