**Computing earliest arrival flows with multiple sources**

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(joint work with Nadine Baumann)

Earliest arrival flows are motivated by applications related to evacuation. In typical evacuation situations, the most important task is to get people out of an endangered building or area as fast as possible. Since it is usually not known how long a building can withstand a fire before it collapses or how long a dam can resist a flood before it breaks, it is advisable to organize an evacuation such that as much as possible is saved no matter when the inferno will actually happen. In the more abstract setting of network flows over time, the latter requirement is captured by so-called earliest arrival flows. Before we discuss this in more detail, we first give a short and descriptive introduction into flows over time.

**Flows over time.** We consider a network $N = (V, A)$ with capacities $u_e \geq 0$ and transit times $\tau_e \geq 0$ on the arcs $e \in A$. The capacity of an arc bounds the flow rate (i.e., flow per time) at which flow can enter the arc. The transit time of an arc specifies the amount of time it takes for flow to travel from the tail to the head of the arc. Moreover, there is a set of source nodes $S^+ \subseteq V$ and a set of sink nodes $S^- \subseteq V \backslash S^+$. Each source $s \in S^+$ has a supply $v(s) > 0$ and each sink $t \in S^-$ a demand $-v(t) > 0$ such that $\sum_{w \in S^+ \cup S^-} v(w) = 0$. A *flow over time* specifies for each arc $e$ and each point in time the flow rate at which flow enters the arc (and leaves the arc again $\tau_e$ time units later). Flow conservation constraints require that at every point in time and for every intermediate node $w \in V \setminus (S^+ \cup S^-)$ the flow entering and leaving node $w$ must cancel out each other.

Flows over time have been introduced by Ford and Fulkerson [6]. Given a network with a single source node $s$, a single sink node $t$, and a time horizon $\theta \geq 0$, they consider the problem of sending as much flow as possible from $s$ to $t$ within time $\theta$. It turns out that a maximal $s$-$t$-flow over time can be determined by a static min-cost flow computation where transit times of arcs are interpreted as cost coefficients.

Ford and Fulkerson [6] also introduce the concept of *time-expanded networks* that consist of one copy of the node set of the given network for each time unit (we call such a copy a *time layer*). For each arc $e$ of the original network with transit time $\tau_e$ the time-expanded network contains copies connecting any two time layers at distance $\tau_e$. On the positive side, most flow over time problems can be solved by static flow computations in time-expanded networks. On the negative side, time-expanded networks are huge in theory and in practice. In particular, the size of a time expanded network is linear in the given time horizon $\theta$ and therefore exponential (but still pseudopolynomial) in the input size.

Hoppe and Tardos [11] consider the *quickest transshipment problem* which is defined as follows. Given a network with several source and sink nodes with given supplies and demands, find a flow over time with minimal time horizon $\theta$ that



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satisfies all supplies and demands. Hoppe and Tardos give a strongly polynomial algorithm for this problem which, however, relies on submodular function minimization and is highly nontrivial.

**Earliest arrival flows.** Shortly after Ford and Fulkerson introduced flows over time, the more elaborate *$s$-$t$-earliest arrival flow problem* was studied by Gale [7]. Here the goal is to find a single $s$-$t$-flow over time that simultaneously maximizes the amount of flow reaching the sink $t$ up to any time $\theta \geq 0$. A flow over time fulfilling this requirement is said to have the *earliest arrival property* and is called *earliest arrival flow*. Gale [7] showed that $s$-$t$-earliest arrival flows always exist. Minieka [14] and Wilkinson [17] both gave pseudopolynomial-time algorithms for computing earliest arrival flows based on the Successive Shortest Path Algorithm. Hoppe and Tardos [10] present a fully polynomial time approximation scheme for the earliest arrival flow problem that is based on a clever scaling trick.

In a network with several sources and sinks with given supplies and demands, flows over time having the earliest arrival property do not necessarily exist [3]. We give a simple counterexample with one source and two sinks. For the case of several sources with given supplies and a single sink, however, earliest arrival flows do always exist [15]. This follows, for example, from the existence of lexicographically maximal flows in time-expanded networks; see, e.g., [14]. We refer to this problem as the *earliest arrival transshipment problem*. Hajek and Ogier [8] give the first polynomial time algorithm for the earliest arrival transshipment problem with zero transit times. Fleischer [3] gives an algorithm with improved running time. Fleischer and Skutella [5] use condensed time-expanded networks to approximate the earliest arrival transshipment problem for the case of arbitrary transit times. They give an FPTAS that approximates the time delay as follows: For every time $\theta \geq 0$ the amount of flow that should have reached the sink in an earliest arrival transshipment by time $\theta$, reaches the sink at latest at time $(1+\varepsilon)\theta$. Tjandra [16] shows how to compute earliest arrival transshipments in networks with time dependent supplies and capacities in time polynomial in the time horizon and the total supply at sources. The resulting running time is thus only pseudopolynomial in the input size.

Earliest arrival flows are motivated by applications related to evacuation. In the context of emergency evacuation from buildings, Berlin [1] and Chalmet et al. [2] study the quickest transshipment problem in networks with multiple sources and a single sink. Jarvis and Ratliff [12] show that three different objectives of this optimization problem can be achieved simultaneously: (1) Minimizing the total time needed to send the supplies of all sources to the sink, (2) fulfilling the earliest arrival property, and (3) minimizing the average time for all flow needed to reach the sink. Hamacher and Tufecki [9] study an evacuation problem and propose solutions which further prevents unnecessary movement within a building.

**Our contribution.** While it has previously been observed that earliest arrival transshipments exist in the general multiple-source single-sink setting, the problem of computing one efficiently has been open. All previous algorithms rely on time expansion of the network into exponentially many time layers. We solve this open problem and present an efficient algorithm which, in particular, does not rely on time expansion.

Using a necessary and sufficient criterion for the feasibility of transshipment over time problems given by Klinz [13], we first recursively construct the earliest arrival pattern, that is, the piece-wise linear function that describes the time-dependent maximum flow value. As a by-product, we present a new proof for the existence of earliest arrival flows that does not rely on time expansion. We finally show how to turn the earliest arrival pattern into an earliest arrival flow by slightly extending the network and applying the quickest transshipment algorithm of Hoppe and Tardos [11].

The running time of our algorithm is polynomial in the input size plus the number of breakpoints of the earliest arrival pattern. Since the earliest arrival pattern is more or less explicitly part of the output of the earliest arrival transshipment problem, we can say that the running time of our algorithm is polynomially bounded in the input plus output size.

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