A Game-Theoretic Approach to Line Planning*

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Abstract. We present a game-theoretic model for the line planning problem in public transportation, in which each line acts as player and aims to minimize a cost function which is related to the traffic along its edges. We analyze the model and in particular show that a potential function exists. Based on this result, we present a method for calculating equilibria and present first numerical results using the railway network of *Deutsche Bahn*.

Keywords. Line Planning, Network Game, Equilibrium

1 Introduction

In line planning, a *public transportation network (PTN)* is modeled by vertices for each stop (or train station) and edges for each direct connection between stops (or tracks between stations). A *line* is given as a path in the PTN and the *frequency* indicates, how often the bus or train goes within a certain time interval. The goal is to choose lines from a given *line pool* that satisfy certain criteria and minimize an objective function. The usual restrictions consider that the demand of the passengers is satisfied, i.e. that enough resources are provided to transport the customers that want to travel in the PTN. Furthermore, the amount of traffic may be limited e.g. by safety regulations. Problems of this kind have been treated with different objective functions: In [1] the lines are chosen with respect to the cost of operating the lines, but also customer-oriented objectives have been considered (see [2] for maximizing the number of direct travelers and [3,4,5,6] for recent approaches minimizing traveling times).

In our approach, we present a new model for line planning, namely from a game theoretic point of view. The lines act as players, the strategies of the players correspond to the frequencies of the lines. The payoff of the game represents the objective of the players which is to minimize the expected delay. This delay is dependent on the overall traffic and hence on the frequencies of all lines in the

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network. The remainder of the paper is structured as follows. In order to keep the notation clear, we first present the model for lines between only a single origin and a single destination (see Section 2) and show in Section 3 that this model is a special case of generalized Nash equilibrium games with a polyhedron as feasible set. In particular this allows us to prove the existence of an exact potential function. In Section 4 we extend our results to multiple origin-destination pairs. First numerical results within a real-world application are presented in Section 5. The paper is concluded by suggestions for further research.

2 The Line Planning Game Model

We consider a network $G = (V, E)$ with vertices $v \in V$ and edges $e \in E$, where V and E are nonempty and finite. A *line* P in G is given by a finite path of edges $e \in E$: $P = (e_1, ..., e_k)$. We denote the *line pool* $\mathcal{P}$ as a set of lines P in G from a single *origin* s to a single *destination* t. Multiple origin-destination pairs will be considered in Section 4.

The *frequency* of a line P is denoted by f_P. The frequencies in the complete network are represented by the *frequency vector*, given by $f \in \mathbb{R}_+^{|\mathcal{P}|}$. Furthermore, the frequency (or load) on an edge $e \in E$ is given by the sum of the frequencies on lines that are containing e,

$$f_e = \sum_{P:e\in P} f_P \ . \tag{1}$$

As common in the literature about line planning, we consider the following two restrictions. First, a *minimal frequency* $f^{min} \geq 0$ from s to t has to be covered to meet the demand of the customers, i.e. we require

$$\sum_{P\in\mathcal{P}} f_P \geq f^{min} \ . \tag{2}$$

If this condition is not satisfied all lines receive a payoff M, with M being a large number working as a penalty. The second bound is the real-valued *maximal frequency* $0 \leq f_e^{max} < \infty$ that is assigned to each edge $e \in E$, i.e. it has to hold

$$f_e \leq f_e^{max} \ \forall \ e \in E \ . \tag{3}$$

The maximal frequency establishes a capacity constraint usually given by security issues. If $f_e > f_e^{max}$ for an edge e, all lines that contain e receive a payoff of $N < M$. We allow N to be any real value smaller than M, nevertheless in the line planning problem it makes sense to choose N being a large number to punish if constraints (3 are exceeded.
Further, we will call a frequency vector f *feasible* if the both the constraints (2) and (3) are satisfied, i.e. if both bounds f^{min} and $f_e^{max}, e \in E$, are respected. The *set of feasible frequency vectors* is given by

$$\mathbb{F}^{LPG} = \left\{ f \in \mathbb{R}_+^{|\mathcal{P}|} : \sum_{P\in\mathcal{P}} f_P \geq f^{min} \ \wedge \sum_{P:e\in P} f_P \leq f_e^{max} \ \forall \ e \in E \right\} \ .$$

Finally, we have to specify the payoff function of the game. To this end, we first define the cost of a line P as the sum of costs on the edges belonging to that line,

$$c_P(f) = \sum_{e \in P} c_e(f_e) \ ,$$

where the cost functions $c_e(\cdot)$ describe the expected average delay on edge e, which depends on the frequency or load on e. We assume the cost functions c_e to be continuous and nonnegative for nonnegative loads, i.e. $c_e(x) \geq 0$ for $x \geq 0$. We need no further assumption on the cost functions, although in line planning the costs are usually nondecreasing.
The *payoff function* (or benefit) of a line P is for nonnegative frequency vectors f given by

$$b_P(f) = \begin{cases} c_P(f) & if \sum_{P_k \in \mathcal{P}} f_{P_k} \geq f^{min} \wedge \forall\ e \in P : f_e \leq f_e^{max} \\ N & if \sum_{P_k \in \mathcal{P}} f_{P_k} \geq f^{min} \wedge \exists\ e \in P : f_e > f_e^{max} \\ M & if \sum_{P_k \in \mathcal{P}} f_{P_k} < f^{min} \end{cases} \ .$$

Summarizing, the line planning game Γ is given by the tuple

$$\Gamma = (G, \mathcal{P}, f^{min}, f^{max}, c, N, M) \ .$$

To illustrate the payoff function of one single player (or line) P, we fix the frequencies f_{P_k} of all other players $P_k \neq P$. We obtain the frequency vector f_{-P}, by deleting the P^{th} component in the frequency vector f. The payoff function depending just on f_P is illustrated in Figure 1. The payoff consists of three

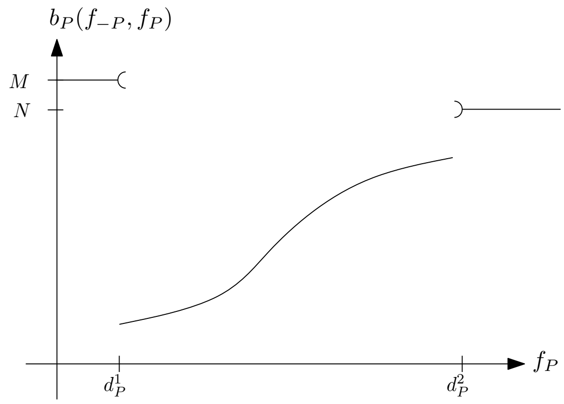

Fig. 1. Payoff of line P for a fixed frequency vector f_{-P}

continuous intervals. The left part is described by the (penalty) payment M in

case of not satisfying the minimal frequency f^{min}, the right part by the (penalty) payment N in case of exceeding a maximal frequency. The middle part is given by the sum of costs on the edges belonging to P. It is nondecreasing, if we have nondecreasing cost functions $c_e(f_e)$ on the respective edges of the path. The values that mark the boundaries of the intervals will be important later:
The *lower decision limit* of player P is given by

$$d_P^1(f_{-P}) = f^{min} - \sum_{P_k \in \mathcal{P} \setminus \{P\}} f_{P_k} \quad . \tag{4}$$

The *upper decision limit* of player P is denoted by

$$d_P^2(f_{-P}) = \min_{e \in P} \{ f_e^{max} - \sum_{P_k \in \mathcal{P} \setminus \{P\}} f_{P_k} \} \quad . \tag{5}$$

If no confusion regarding the strategies f_{-P} of the other players arises, we denote the lower and the upper decision limit by d_P^1 and d_P^2, respectively. We obtain that

$$b_p(f) = c_P(f) \text{ if and only if } d_P^1 \leq f_P \leq d_P^2,$$

i.e. whenever $f_P \in [d_P^1, d_P^2]$, the constraints (2) and (3) are satisfied. In this case, f is feasible, if it is nonnegative. Note that it may happen that $[d_P^1, d_P^2] \cap \mathbb{R}_+$ is empty for a player P, even if $\mathbb{F}^{LPG}$ is nonempty.

As usual in game theory, we are interested in finding the equilibria of the game, which in our case represent line plans with equally distributed probability for delays. In a line planning game, a frequency vector f^* is an *equilibrium* if and only if for all lines $P \in \mathcal{P}$ and for all $f_P \geq 0$ it holds that

$$b_P(f^*_{-P}, f^*_P) \leq b_P(f^*_{-P}, f_P) \quad ,$$

i.e. no player $P \in \mathcal{P}$ is able to improve its payoff by changing only his strategy. Equilibria in line planning games may be feasible or infeasible, which can be observed in the following example. As we are interested in implementable solutions, we analyze feasible frequencies in the following.

Example 1. We consider a line planning game with a line pool containing two lines. Let f_1 and f_2 be the frequencies of these lines. The minimal frequency $f^{min} = 1$ has to be covered from s to t. The game network consists of three edges, as illustrated in Figure 2. The maximal frequencies of the edges are given by $f_{e_1}^{max} = f_{e_2}^{max} = 2$ and $f_{e_3}^{max} = 3$. Furthermore, the following costs are assigned to the edges: $c_{e_1}(x) = x$, $c_{e_2}(x) = 2x$ and $c_{e_3}(x) = x^2$. Thus, we obtain payoffs: $c_1(f) = f_1 + (f_1 + f_2)^2$ for the first player and $c_2(f) = 2f_2 + (f_1 + f_2)^2$ for the second player. See Figure 3 for an illustration of the set of feasible frequencies $\mathbb{F}^{LPG}$. This line planning game provides multiple equilibria. Feasible equilibria are e.g. $f^1 = (1, 0)$ and $f^2 = (0, 1)$, with payoffs $b(f^1) = (2, 1)$ and $b(f^2) = (1, 3)$. There are also infeasible equilibria, e.g. $f^3 = (4, 4)$, where no player is able to receive a smaller payoff than N. The frequency vector $f^4 = (3, 3)$ is no equilibrium, although no player is able to reach the set of feasible frequencies within one step. It is a property of line planning games that outside the feasible region not necessarily each player gets punished. Here, e.g. player 1 could change his frequency to zero. The resulting frequency vector $\bar{f}^4 = (0, 3)$ is still infeasible, but player 1 is able to improve his payoff from $b_1(f^4) = N$ to $b_1(\bar{f}^4) = 9$.

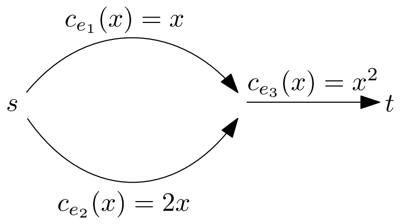

Fig. 2. Game network of Ex.1

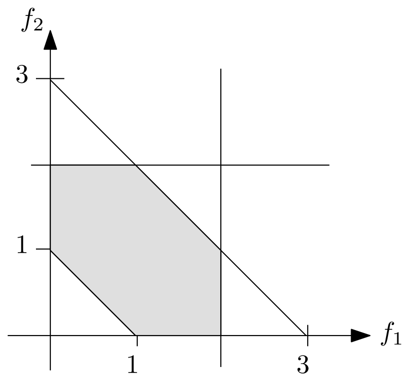

Fig. 3. Set of feasible frequencies $\mathbb{F}^{LPG}$

The above example illustrates that in line planning games, there may be areas of infeasible frequency vectors, where some players violate constraints and others do not. See Figure 4, the illustration of the two players game of Example 1 and consider the four infeasible regions A, B, C, and D. For frequency vectors that

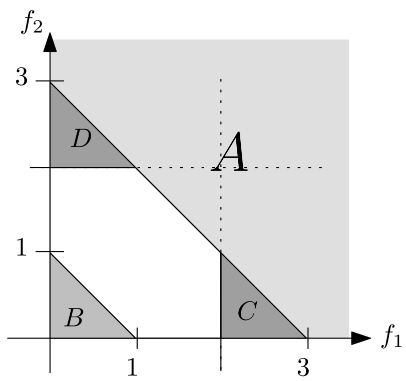

Fig. 4. Infeasible regions A, B, C, and D

lie in region A, both players get punished with payoff N, while in region B, both receive the payoff M. In region C, only player 1 gets punished with payoff N, while player 2 receives a payoff $c_2(f)$, as player 2 is satisfying the maximal frequencies f_e^{max} on his edges e_2 and e_3, but player 1 is violating $f_{e_1}^{max} = 2$. In region D the reverse situation occurs: player 2 gets punished and player 1 does not. Situations like in regions C and D happen since the players are not sharing the same set of constraints. A systematic investigation of such areas is a topic of future research (e.g. for for standard networks $G(n)$, see [7], which are a basic concept to represent all networks of a line planning game with n players, such that the set of equilibria remain unchanged).

3 Line Planning Games as Games on Polyhedra

Since we are not interested in solutions not satisfying the constraints (2) and (3) we now concentrate on feasible strategies f. First of all, note that the feasible region F^{LPG} of an LPG (see (2)) is a polyhedron. It can be represented as $S(A, b) = \{f : Af \leq b\}$ where the $(1 + |E| + |\mathcal{P}|) \times |\mathcal{P}|$-matrix A and the right-hand side vector $b \in \mathbb{R}^{(1+|E|+|\mathcal{P}|)}$ are given as

$$A = \begin{pmatrix} -\mathbb{1}_{|\mathcal{P}|} \\ \hline H \\ \hline -\mathbb{I}_{|\mathcal{P}|} \end{pmatrix} \qquad b = \begin{pmatrix} -f^{min} \\ \hline f^{max} \\ \hline \mathbb{0}_{|\mathcal{P}|} \end{pmatrix} .$$

In this formula, the $|E| \times |\mathcal{P}|$ matrix H is the edge-path incidence matrix of the underlying network with entries

$$h_{e,P} = \begin{cases} 1 & if \quad e \in P \\ 0 & else \end{cases} , \tag{6}$$

$\mathbb{1}_n = (1, \ldots, 1)$ is the vector containing n-times the entry 1, and $\mathbb{0}_n = (0, \ldots, 0)^T$ is the vector containing n-times the entry 0. Note that the polyhedron $S(A, b)$ is compact.

Example 2. In the line planning game of Example 1 the corresponding matrix H is given by

$$H = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix} .$$

The polyhedron $S(A, b)$ is described by

$$A = \begin{pmatrix} -1 & -1 \\ 1 & 0 \\ 0 & 1 \\ 1 & 1 \\ -1 & 0 \\ 0 & -1 \end{pmatrix} , \qquad b = \begin{pmatrix} -1 \\ 2 \\ 2 \\ 3 \\ 0 \\ 0 \end{pmatrix} .$$

Hence we can model LPG as a game on a polyhedron, as defined in [8]: In a game on a polyhedron with n players, each player chooses a coordinate x_n such that $(x_1, \ldots, x_n)$ lies within a given feasible set P, which is a polyhedron. Each of the players $i = 1, \ldots, n$ has a payoff function $\tilde{c}_i : P \to \mathbb{R}$ where the payoff $\tilde{c}_i(x)$ of player i depends on the complete vector x and hence also on the strategies of the other players. Since not only the payoff but also the feasibility of a strategy of player i depends on the decisions of the other players, games on polyhedra belong to the class of *generalized Nash equilibrium (GNE) games* (see [9]).
Consequently, if we are only looking for feasible solutions $f \in \mathbb{F}^{LPG}$, we will call the line planning game *generalized line planning game* to stress that the feasible strategy set of each player depends on the strategies the other players. As usual for GNE games, we now require that the equilibria are feasible, i.e. in generalized line planning games, a feasible frequency vector f^* is a *generalized equilibrium* if and only if for all lines $P \in \mathcal{P}$ and for all

$$f_P \in [d_P^1(f^*_{-P}), d_P^2(f^*_{-P})] \ \cap \ \mathbb{R}_+ \ ,$$

it holds that

$$b_P(f^*_{-P}, f^*_P) \le b_P(f^*_{-P}, f_P) \ .$$

Since only feasible solutions are considered, the payoff in the generalized line planning game is hence given by $b_P(f) = c_P(f)$. Furthermore, since the line planning game is a game on a polyhedron, we can transfer results from this type of games. One important property is the existence of a potential function.

A function $\Pi : f \to \mathbb{R}$ is an *exact restricted potential function* for a generalized line planning game Γ if for every $P \in \mathcal{P}$, for every f_{-P} with a nonempty set $[d_P^1(f_{-P}), d_P^2(f_{-P})] \cap \mathbb{R}_+$ and for every $x,\ z \in [d_P^1(f_{-P}), d_P^2(f_{-P})]$ it holds:

$$b_P(f_{-P}, x) - b_P(f_{-P}, z) = \Pi(f_{-P}, x) - \Pi(f_{-P}, z) \ . \tag{7}$$

A line planning game Γ is called an *exact restricted potential game* if it admits an exact restricted potential.

Exact potential functions have been introduced in [10]. The modification to a restricted version enables the investigation of GNE games and has been introduced in [7]. Although exact restricted potential functions do not exist in general for games on polyhedra, it can be shown that they exist if the cost structure of the game originates from a network. This property is called *path player game property* and has been introduced in [7]. To satisfy this property, it has to be possible to define for each subset of the set of players $\mathcal{P}$ a standard function that is dependent on the subset such that any player's payoff can be decomposed into these standard functions. In [7,8] it was shown that games on polyhedra have an exact restricted potential function whenever the path player game property holds. Fortunately, the line planning game has this property (see again [7]) such that the following holds:

Theorem 1. *A generalized line planning game is a game on a polyhedron with PPG-property. Hence, it has an exact restricted potential function.*

An exact restricted potential function is given as

$$\Pi(f) = \sum_{e \in E} [c_e(f_e) - c_e(0)].$$

To alternatively prove this result, consider the two feasible frequency vectors $f^x = (f_{-P}, x)$ and $f^z = (f_{-P}, z)$ and verify that equation (7) does hold.

$$\begin{aligned} \Pi(f^x) - \Pi(f^z) &= \sum_{e \in E} [c_e(f_e^x) - c_e(0)] - \sum_{e \in E} [c_e(f_e^z) - c_e(0)] \\ &= \sum_{e \in E} [c_e(f_e^x) - c_e(f_e^z)] = \sum_{e \in P} [c_e(f_e^x) - c_e(f_e^z)] \\ &= c_P(f^x) - c_P(f^z) = b_P(f^x) - b_P(f^z) \ . \end{aligned} \tag{8}$$

Equation (8) is true as f^x and f^z are different only with respect to line P. Since it is a general result that for infinite potential games with continuous payoffs on compact feasible strategy sets, equilibria exist (see [10]) we directly obtain the following corollary.

Corollary 1. *In the line planning game, equilibria exist.*

Using the shape of the potential function, we furthermore obtain:

Theorem 2. *For a generalized line planning game with feasible region $\mathbb{F}^{LPG}$, a generalized equilibrium is given by an optimal solution of the following problem:*

$$\min \sum_{e \in E} c_e(f_e) \quad \textit{subject to} \quad f \in \mathbb{F}^{LPG} \ .$$

Theorem 2 provides a method for calculating equilibria in the line planning game, namely by solving the optimization problem mentioned. This method is valid for all types of continuous cost functions $c_e(f_e)$, which is the strength of this approach. On the other hand, not necessarily all equilibria are found by using Theorem 2. Other approaches which determine all equilibria for line planning games with linear costs or strictly increasing costs are presented in [7].

Example 3. Consider the line planning game analyzed in Example 1. By Theorem 2 an equilibrium can be found by solving the following problem:

$$\min f_1 + 2f_2 + (f_1 + f_2)^2 \quad \text{subject to} \quad f \in \mathbb{F}^{LPG} \ . \tag{9}$$

The solution of the optimization problem, and thus an equilibrium is given by $f^* = (1, 0)$ with $b(f^*) = (2, 1)$. Note that f^* is the unique solution of (9), but not the unique equilibrium.

4 Multiple Origin-Destination-Pairs

In this section we consider a network $G = (V, E)$ with Q multiple origin-destination(OD)-pairs $\{s_q, t_q\}$, $q = 1, \ldots, Q$. For the q^{th} OD-pair, the pool of lines connecting s_q and t_q is given by $\mathcal{P}_q$. The paths are given as pairwise disjoint sets:

$$\mathcal{P}_{q_1} \cap \mathcal{P}_{q_2} = \emptyset \quad \forall\, q_1, q_2 = 1, \ldots, Q, q_1 \neq q_2 \ .$$

With $q(P)$ we denote the index of the OD-pair $\{s_q, t_q\}$ such that $P \in \mathcal{P}_q$. Since each line P is assigned to exactly one OD-pair, $q(P)$ is well-defined. Furthermore, the minimal frequency for the q^{th} OD-pair is given by f_q^{min}. We denote:

$$\mathcal{P} = \bigcup_{q=1,\ldots,Q} \mathcal{P}_q \quad \text{and} \quad f^{min} = \left(f_q^{min}\right)_{q=1,\ldots,Q} \ .$$

The maximal frequencies on edges f_e^{max} and the cost $c_e(f_e)$ assigned to the edges are defined as in the single origin-destination case. We call such a game *line planning game with multiple OD-pairs.*

The *payoff* for player $P \in \mathcal{P}_q$ and a nonnegative frequency vector f in an LPG with multiple OD-pairs is given by

$$b_P(f) = \begin{cases} c_P(f) & \textit{if } \sum_{P_k \in \mathcal{P}_{q(P)}} f_{P_k} \geq f_{q(P)}^{min} \wedge \forall\, e \in P : f_e \leq f_e^{max} \\ N & \textit{if } \sum_{P_k \in \mathcal{P}_{q(P)}} f_{P_k} \geq f_{q(P)}^{min} \wedge \exists\, e \in P : f_e > f_e^{max} \\ M & \textit{if } \sum_{P_k \in \mathcal{P}_{q(P)}} f_{P_k} < f_{q(P)}^{min} \end{cases} .$$

Like in the single OD-pair case a frequency vector f is called *feasible* if the bounds $f_q^{min}, q = 1, \ldots Q$ and $f_e^{max}, e \in E$ are satisfied. The *set of feasible frequencies for line planning games with multiple OD-pairs* is given by

$$\mathbb{F}^{LPGMOD} = \left\{ f \in \mathbb{R}_+^{|\mathcal{P}|} : \sum_{P \in \mathcal{P}_q} f_P \geq f_q^{min} \ \forall\, q \in Q \ \wedge \sum_{P : e \in P} f_P \leq f_e^{max} \ \forall\, e \in E \right\} .$$

Finally, for a player $P \in \mathcal{P}_q$ we have to adjust the definition of the lower decision limit presented in (4):

$$d_P^1(f_{-P}) = f_{q(P)}^{min} - \sum_{\substack{P_k \in \mathcal{P}_q \\ P_k \neq P}} f_{P_k} \ ,$$

while the upper decision limit stays the same as in (5). If we just consider feasible frequencies $f \in \mathbb{F}^{LPGMOD}$, we obtain a *generalized line planning game with multiple OD-pairs.* Generalized equilibria are defined in such games similar to the single OD-pair case.

Recall the definition of the edge path incidence matrix H. A line planning game with multiple OD-pairs is represented by a game on a polyhedron S(A,b) with:

$$A = \begin{pmatrix} -\mathbb{1}_{|\mathcal{P}_1|} & 0 & \dots & 0 & 0 \\ 0 & -\mathbb{1}_{|\mathcal{P}_2|} & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & -\mathbb{1}_{|\mathcal{P}_{m-1}|} & 0 \\ 0 & 0 & \dots & 0 & -\mathbb{1}_{|\mathcal{P}_m|} \\ \hline & & H & & \\ \hline & & -\mathbb{I}_{|\mathcal{P}|} & & \end{pmatrix} \qquad b = \begin{pmatrix} -f_1^{min} \\ -f_2^{min} \\ \vdots \\ -f_m^{min} \\ \hline f^{max} \\ \hline \mathbb{0}_{|\mathcal{P}|} \end{pmatrix} .$$

Example 4. We consider a line planning game with four OD-pairs as illustrated in Figure 5. Let $f_q^{min} = 1 \; \forall \; q = 1, \dots, Q$ and $f_e^{max} = 4 \; \forall \; e \in E$. We denote the

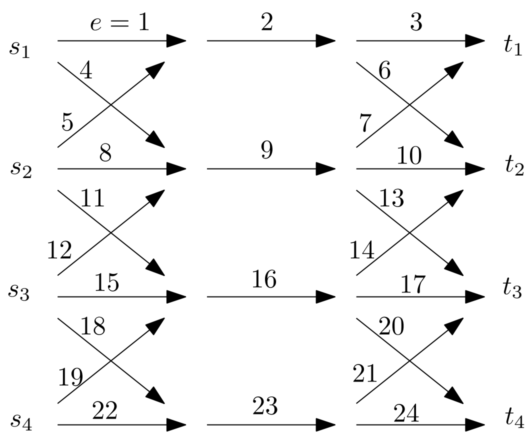

Fig. 5. Game network of Example 4

edges $e = 1, \dots, 24$ and the lines with $P^1, \dots, P^{10}$. The frequency of the lines is given by $f_1, \dots, f_{10}$. The line pools are given by

$$\begin{aligned} \mathcal{P}_1 &= \{P^1, P^2\} &&= \{(1,2,3), (4,9,7)\} \; , \\ \mathcal{P}_2 &= \{P^3, P^4, P^5\} &&= \{(5,2,6), (8,9,10), (11,16,14)\} \; , \\ \mathcal{P}_3 &= \{P^6, P^7, P^8\} &&= \{(12,9,13), (15,16,17), (18,23,21)\} \; , \\ \mathcal{P}_4 &= \{P^9, P^{10}\} &&= \{(19,16,20), (22,23,24)\} \; . \end{aligned}$$

We introduce cost functions $c_e(f_e) = f_e$ for all edges e in E. We apply Theorem 2 and solve

$$\begin{aligned}\min \sum_{e \in E} c_e(f_e) &= \min \sum_{e \in E} c_e \left(\sum_{P \in \mathcal{P}} h_{e,P} f_P \right) \\ &= \min \left(3f_1 + 3f_2 + 3f_3 + 3f_4 + 3f_5 + 3f_6 + 3f_7 + 3f_8 + 3f_9 + 3f_{10}\right) \\ &\quad \text{subject to} \quad f \in S(A,b) \ .\end{aligned}$$

As each frequency f_P has exactly the same coefficient in the objective function, each frequency that satisfies $\sum_{P \in \mathcal{P}_q} f_P = f_q^{min} = 1$, e.g. $f^1 = (1,0,1,0,0,1,0,0,1,0)$, is an optimal solution and thus also an equilibrium. The objective value is 4 for all these solutions. The payoff for f^1 is given by $b(f^1) = (4,1,4,1,1,3,1,0,3,0)$, while for $f^2 = (1,0,0,1,0,0,1,0,0,1)$ we have a payoff $b(f^2) = (3,1,1,3,1,1,3,1,1,3)$.

We can use this approach also for nonlinear cost functions. Set e.g. $c_e(f_e) = f_e^2$ for all edges e in E. The objective function $\min \sum_{e \in E} c_e(f_e)$ yields the optimal solution

$$f^3 = (0.538, 0.462, 0.385, 0.308, 0.308, 0.308, 0.308, 0.385, 0.462, 0.538)$$

with an objective value of 7.385. Solving this problem as an integer problem yields $f^4 = (0,1,1,0,0,0,0,1,1,0)$, with objective value 12.

5 Line Planning for Interregional Trains in Germany

We implemented our approach of Theorem 2 using real-world data, related to the German railway system of *Deutsche Bahn AG*. In particular we consider train stations connected by interregional trains, such as *InterCityExpress (ICE)*, *InterCity (IC)*, and *EuroCity (EC)*. The following studies are meant to test the possibility of implementing our method with realistic data and to obtain equilibria based on larger databases. Our numerical study is interesting due to the following two reasons:

- Although the investigation of line planning games is still in an early stadium, and the results are hence not ready for practical use yet, the study illustrates that further research in this field is worthwhile.
- Second, the numerical behavior of the method for finding equilibria in the line planning game is demonstrated.

The following data is at our disposal:

- OD-matrix describing 319 train stations and the minimal frequency f_q^{min} given for the OD-pairs.
- Three line databases of different size, containing 132 (S - small), 688 (M - medium) and 2770 (L - large) lines.

The line databases are not in a form suitable for our model. We will discuss later how line pools are created from this data. From theoretically $319 \times 318 = 101\,442$ OD-pairs, still 56 646 have a positive minimal frequency f_q^{min} and have to be considered. Thus, we have a line planning game with multiple OD-pairs. For those OD-pairs, $f_q^{min} \in [1, 4831]$ hold. Note that the values of f^{min} are to be interpreted as weights dependent on the number of passengers. From these weights, frequencies are obtained by a linear transformation.

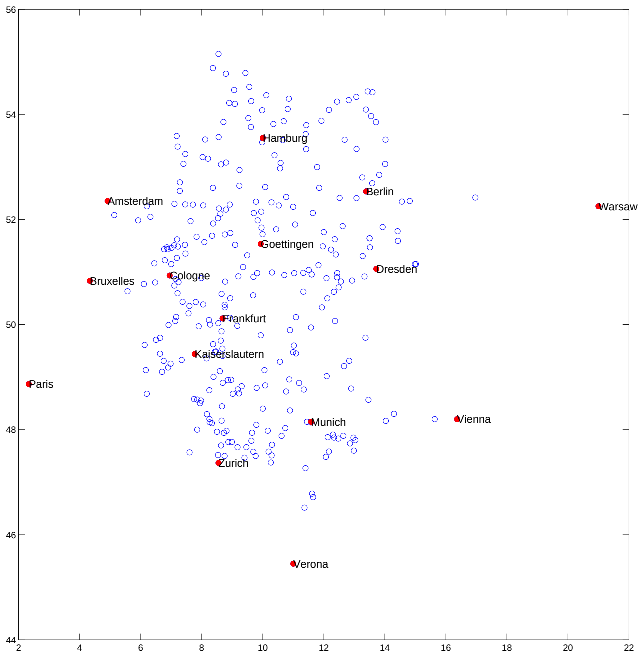

Fig. 6. Train stations under consideration

The train stations under consideration are located in Germany and neighboring countries. Figure 6 illustrates the locations of all 319 stations. The following information are needed for the line planning game, but are not given in the data. We do not have available the maximal frequency f_e^{max} on the edges, as well as we do not know the costs $c_e(f_e)$ assigned to the edges. Thus, we have to make assumptions for the implementation of our model. As there is no maximal frequency on the edges, we choose the value sufficiently large for each edge, such that the maximal frequency is satisfied for our problems. In particular, we set $f_e^{max} = 100\,000$. Regarding the cost function $c_e(f_e)$, Theorem 2 allows to use any continuous function. We implement the strictly increasing function $c_e(f_e) = f_e$

on all edges $e \in E$.

As the line planning game model considers only direct connections between stations, we neglect all OD-pairs where no direct connection exists in the line pool. For the future design of line databases, this should be taken into consideration. Furthermore, we introduce the bound U_q and consider only OD-pairs where $f_q^{min} > U_q$ does hold. This bound is used to consider just "important" OD-pairs with high minimal frequencies for our computations and it is a tool to control the size of the problem.

Furthermore, we have to construct a line pool from the line databases according to the definitions in our model. As we reduced the number of OD-pairs, we have to analyze only lines that are relevant for the OD-pairs under consideration. Thus, we generate the line pool by using these lines. On the other hand, one line may offer a direct connection for more than one OD-pair. In our model, we assume disjoint line pools, i.e. one line has to be assigned to exactly one OD-pair. According to this, we duplicate lines that provide a direct connection for more than one OD-pair. The lines have to be given such that we obtain a line pool $\mathcal{P} = \bigcup_{q=1,\ldots,Q} \mathcal{P}_q$ consisting of disjoint subsets $\mathcal{P}_q$. Note that the frequencies of the original lines from the databases S, M and L are then given by the sum over the frequencies of its duplicates.

We study five scenarios with a different number of OD-pairs and use different line databases. In Studies 1,2 and 3, we consider the same set of OD-pairs, namely for $f_q^{min} > 599$, but we change the size of the line database. In Studies 2,4 and 5, the line database is invariant (we choose the medium sized one), but the set of OD-pairs is changed.

	$f_q^{min} > 999$	$f_q^{min} > 599$	$f_q^{min} > 399$
small		Study 1	
medium	Study 4	Study 2	Study 5
large		Study 3	

Table 1 contains the computational results. We present a short explanation of the content in the following list:

Column 3 Number of OD-pairs which satisfy $f_q^{min} > U_q$
Column 5 Size of line databases
Column 6 Number of OD-pairs with direct connections and which satisfy $f_q^{min} > U_q$
Column 7 Size of line pool constructed from line database, including duplicates of lines
Column 8 Number of lines with positive frequency, i.e. that are established for the PTN (including duplicates)
Column 9 Objective function value of the optimization problem solved with Method 1
Column 10 Reference to Figure of PTN
Columns 12 − 14 Copied from the first part of the table, for easier reading
Columns 15 − 19 Statistical information about length of each line (number of stations)
Columns 20 − 24 Statistical information about number of lines (including duplicates) serving each train station

It can be observed from Studies 1 − 3 that for a larger database, more direct connections are available and thus more OD-pairs can be served. The number

Table 1. Computational results

1	2	3	4	5	6	7	8	9
Study	U_q	# OD-pairs	Line data-base	Size data base	# OD-pairs with direct connections	Size line pool	Lines with positive frequency	$c \times x$
1	599	251	S	132	87	262	88	1 402 494.001
2	599	251	M	688	117	1287	156	2 151 352.000
3	599	251	L	2770	157	5544	244	2 636 404.000
4	999	113	M	688	53	493	68	1 456 873, 000
5	399	499	M	688	132	2610	299	2 971 507.012

11	12	13	14	15	16	17	18	19	20	21	22	23
				# of stations per line					# of lines per station			
Study	Line data-base	# OD-pairs with direct connections	# of chosen lines	min	max	mean	var	histogram Figure	min	max	mean	var
1	S	87	88	6	33	15.15	25.94	7	1	43	4.18	52.76
2	M	117	156	9	20	15.06	10.87	8	1	73	7.36	131.97
3	L	157	244	6	37	14.18	23.83	9	1	121	10.84	323.17
4	M	53	68	9	20	15.21	11.66	10	1	34	3.24	30.59
5	M	132	299	9	20	15.34	10.44	11	1	129	14.38	435.38

of chosen lines hence also increases. Nevertheless, it is not growing in the same speed as the line pool, but about the same speed as the number of considered OD-pairs. Thus, in our examples, the number of established lines is not so much influenced by the size, but by the structure of the line pool, i.e. how much OD-pairs are served by the line pool. This should be taken into consideration for the design of line pools that are to be used for line planning games. In Studies 2,4 and 5, the lower bound U_q on minimal frequencies is small, hence we obtain a larger number of OD-pairs. It can also be observed that with increasing number of OD-pairs, the number of established lines is increasing, although the line data base is unchanged.

Considering the number of stations contained in the established lines, this example shows a relation to the chosen line database S, M or L. It can be observed that the results in the three scenarios using database M are similar.

In terms of lines per station, the station served by the highest number of lines in each study is Frankfurt(Main) Süd.

Histogram: Number of Stations per Line

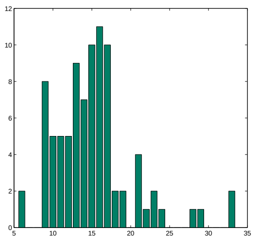

Fig. 7. $U_q = 599$, S

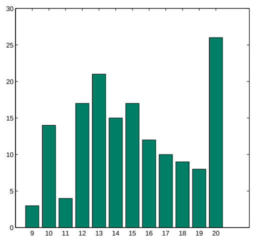

Fig. 8. $U_q = 599$, M

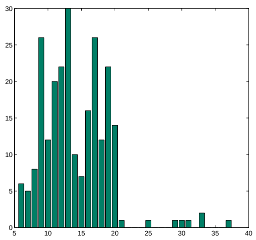

Fig. 9. $U_q = 599$, L

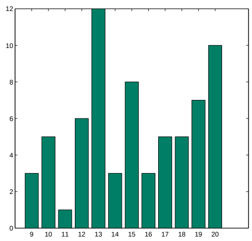

Fig. 10. $U_q = 999$, M

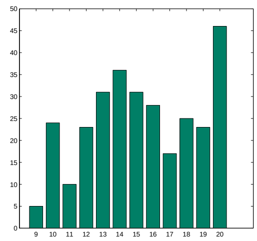

Fig. 11. $U_q = 399$, M

6 Conclusion and further research

The line planning game is a new model for analyzing line planning problems with game theoretical means. In particular it is a special case of a game on polyhedra in which an exact potential function exists. This result is the basis for an algorithm to calculate equilibria of the game. Numerical results have been presented.

Other methods for finding *all* equilibria in the path player games for linear or strictly increasing functions have been developed in [7]. The implementation of the second of these approaches, which seems to be realistic in line planning, is under research.

Although the resulting line plans seem to be suitable for practical applications, other aspects of line planning have been neglected and are topics for future research. Among these are setup costs for installing the lines (or fixed costs for operating a line) which can be approximated by bounding the maximal number of lines which may be installed. From the passengers' point of view, the travel time of their journeys should be considered; a first measure can be the length of the lines. Another drawback of the basic model presented in this work is that frequencies f_P are real numbers, while in practice, only $f_P \in \mathbb{N}_0$ make sense. A first extension to an integer LPG has been considered in [7], where also an algorithmic approach has been developed to find integer equilibria. Finally, it will be a reasonable extension of the current model to take into account that passengers may want to change lines. More results and an implementation of this topic is under research.

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