On Revenue Equivalence in Truthful Mechanisms

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Introduction. In distributed systems, where problem solutions have to be jointly derived by several selfish agents and where problem data is spread over the agents as private information, mechanism design is used to motivate agents to reveal their private information truthfully and to obtain a good overall solution for the system. As a simple example, consider single item auctions, where several bidders are asked to reveal their valuation for a certain good. Dependent on the bids, the mechanism allocates the good to one of the bidders and the price of the good is designed such that agents have an incentive to bid their true valuation. We consider *direct revelation mechanisms*, which consist of an allocation rule that selects an allocation depending on the agents' reports about their private information, and a payment scheme that assigns a payment to every agent. Allocation rules that give rise to a mechanism in which truth-telling is a dominant strategy for every agent are called *truthfully implementable.* Our concern is with the payment scheme that extends a truthfully implementable allocation rule to a truthful mechanism. The property of an allocation rule to have a unique payment scheme completing the allocation rule to a truthful mechanism is called *revenue equivalence.* We give a characterization for an allocation rule to satisfy revenue equivalence. In order to obtain this characterization, we prove a property on complete directed graphs and apply it to the so called *allocation graph*, which is defined by the allocation rule and the valuation function of an agent. The characterization holds for any (possibly infinite) outcome space. Furthermore, we give elementary and simple proofs for the uniqueness of the payment scheme in a truthful mechanism for the cases of finite and countably infinite outcome spaces under very weak assumptions. Many of the known results follow as immediate consequences of ours, e.g. results in Green and Laffont [2], Holmström [4], Krishna and Maenner [5], Milgrom and Segal [6], Suijs [10] and Chung and Olszewski [1]. For details and discussions, especially of the paper by Chung and Olszewski, we refer to the full version of

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the paper [11].

Setting and Basic Concepts. Let the set of *agents* be denoted by $\{1,\ldots,n\}$ and let A be the (possibly infinite) set of alternative allocations or *outcomes*. By t_i, we denote the *type* of agent $i \in \{1,\ldots,n\}$, which is an element of the *type space* $T_i \subseteq \mathbb{R}^{k_i}$ for some $k_i \in \mathbb{N}$. Agent i's preferences over outcomes are modeled by the *valuation function* $v_i \colon A \times T_i \to \mathbb{R}$, where $v_i(a,t_i)$ is the valuation of agent i for outcome a when he has type t_i. A *mechanism* (f,π) consists of an *allocation rule* $f\colon \times_{i=1}^n T_i \to A$ and a *payment scheme* $\pi\colon \times_{i=1}^n T_i \to \mathbb{R}^n$. In a *direct revelation mechanism*, the allocation rule chooses for a vector τ of aggregate type reports of all agents an outcome $f(\tau)$, whereas the payment scheme assigns a payment $\pi_i(\tau)$ to each agent i. Let the vector (t_{-i},t_i) denote the aggregate type report vector when i reports t_i and the other agents' reports are represented by t_{-i}. We assume *quasi-linear utilities*, that is, the utility of agent i when the aggregate report vector is (t_{-i},t_i) is $v_i(f(t_{-i},t_i),t_i) - \pi_i(t_{-i},t_i)$. In a *truthful* mechanism, truth-telling is a (weakly) dominant strategy for every agent:

Definition 1 (dominant strategy incentive compatible, truthful) *A direct revelation mechanism* (f,π) *is called* dominant strategy incentive compatible *or* truthful *if for every agent i, every type* $t_i \in T_i$*, all aggregate type vectors* t_{-i} *that the other agents could report and every type* $s_i \in T_i$ *that i could report instead of* t_i*:* $v_i(f(t_{-i},t_i),t_i) - \pi_i(t_{-i},t_i) \geq v_i(f(t_{-i},s_i),t_i) - \pi_i(t_{-i},s_i)$. *If for allocation rule f there exists a payment scheme* π *such that* (f,π) *is a truthful mechanism, then f is called* truthfully implementable.

Definition 2 (Revenue Equivalence) *A truthfully implementable allocation rule f satisfies the* revenue equivalence *property if for any two incentive compatible mechanisms* (f,π) *and* (f,π') *and any agent i there exists a function* h_i *that only depends on the reported types of the other agents* t_{-i} *such that* $\forall t_i \in T_i : \pi_i(t_i,t_{-i}) = \pi'_i(t_i,t_{-i}) + h_i(t_{-i})$.

Unique Node Potentials in Directed Graphs. Let $G=(V,E)$ be a complete directed graph with (possibly infinite) node set V and arc set E. We assume that G does not contain a negative cycle. By ℓ_{ab} we denote the (finite) length of the arc from node a to node b. Let $a,b \in V$ be two nodes and let P a (finite) path from a to b – or short (a,b)-path – in G. Denote its length by $length(P)$. For $a=b$, we regard the path without any edges as (a,b)-path as well and define its length as 0. Define $\mathcal{P}(a,b)$ as the set of all (a,b)-paths. Let $dist_G(a,b) = \inf_{P\in\mathcal{P}(a,b)} length(P)$. If V is a finite set, then $dist_G(a,b)$ simply equals the length of a shortest path from a to b in G. For infinite V, such a shortest path may not exist. Nevertheless, $dist_G(a,b)$ is finite, since we assume that G does not have any negative cycle and hence $length(P) + \ell_{ba} \geq 0$ holds for every (a,b)-path P. A node potential p is a function $p\colon V \to \mathbb{R}$ such that for all arcs $(i,j) \in E$ $p(j) \leq p(i) + \ell_{ij}$. It is well known that the existence of a node potential in a graph is equivalent to the non-existence of negative cycles in that graph. We prove the following characterization for its uniqueness.

Theorem 1 *Let $G = (V, E)$ be a complete directed graph that does not contain a negative cycle. Then the following statements are equivalent.*
(1) Any two node potentials in G differ only by a constant.
(2) For all $a, b \in V$, $dist_G(a, b) + dist_G(b, a) = 0$.

Next, we define a property of the graph G that is sufficient (though not necessary) for uniqueness of node potentials up to a constant.

Definition 3 (Two-Cycle Connected) *A graph with node set V and arc lengths ℓ is called* two-cycle connected *if for every partition $V_1 \cup V_2 = V$, $V_1 \cap V_2 = \emptyset$, $V_1, V_2 \neq \emptyset$, there are $a_1 \in V_1$ and $a_2 \in V_2$ with $\ell_{a_1 a_2} + \ell_{a_2 a_1} = 0$.*

Theorem 2 *Let G be a directed graph without negative cycle. If G is two-cycle connected then its node potential is uniquely defined up to a constant.*

Characterization of Revenue Equivalence. Fix agent i and let the reports of the other agents be fixed as well. For simplicity of notation we write T and v instead of T_i and v_i. We regard f and π as functions of i's type alone, i.e. $f : T \to A$ and $\pi : T \to \mathbb{R}$. Let f be truthfully implementable. Revenue equivalence asserts that any two payment schemes assigning a payment to each type of agent i differ by a constant. If (f, π) is truthful, it is easy to see that for any pair of types $s, t \in T$ such that $f(t) = f(s) = a$ for some $a \in A$, the payments must be equal, i.e. $\pi(t) = \pi(s) =: \pi_a$. A payment scheme for agent i is therefore completely defined if the numbers π_a are defined for all outcomes $a \in A$ such that $f^{-1}(a)$ is nonempty. Therefore, we may without loss of generality assume that f is onto.

As in Gui et al. [3] and Saks and Yu [9], let us define the complete directed and possibly infinite *allocation graph* G_f with node set A. Between any two nodes $a, b \in A$, there is a directed arc with length $\ell_{ab} = \inf_{t \in f^{-1}(b)}(v(b, t) - v(a, t))$. From Gui et al. [3] and Rochet [8] follows that f is truthfully implementable if and only if G_f does not have a finite cycle of negative length. Let us call this property the *nonnegative cycle property.* We observe that a payment scheme that complements an allocation rule f to a truthful mechanism can be interpreted as a node potential in G_f. Hence, a truthfully implementable allocation rule f satisfies revenue equivalence if and only if in G_f the node potential is uniquely defined up to a constant. Note that the existence of a node potential is already guaranteed by the nonnegative cycle property. The characterization of its uniqueness up to a constant follows immediately from Theorem 1.

Theorem 3 *Let f be a truthfully implementable allocation rule. Then f satisfies revenue equivalence iff $dist_{G_f}(a, b) + dist_{G_f}(b, a) = 0$ for all $a, b \in A$.*

For finite outcome spaces, we prove that revenue equivalence is satisfied under very weak conditions, which cannot be relaxed. More specifically, the following result follows from Theorem 2.

Theorem 4 *Let A be a finite outcome space. Let each agent $i \in \{1, \ldots, n\}$ have types from the (topologically) connected type space $T_i \subseteq \mathbb{R}^{k_i}$. Let each agent's* *valuation function $v_i(a, \cdot)$ be a continuous function in the type of the agent for every $a \in A$. Then, every truthfully implementable allocation rule f satisfies revenue equivalence.*

For countably infinite outcome spaces the following theorem follows from Theorem 1.

Theorem 5 *Let A be a countable outcome space. Let each agent $i \in \{1, \ldots, n\}$ have types from the (topologically) connected type space $T_i \subseteq \mathbb{R}^{k_i}$. Let for each agent the valuation functions $v_i(a, \cdot) : T_i \to \mathbb{R}$, $a \in A$ be equicontinuous functions. Then, every truthfully implementable allocation rule f satisfies revenue equivalence.*

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