Semantic structures for one-stage and iterated belief revision

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May 2007.

Abstract

Semantic structures for belief revision are proposed. We start with one-stage revision structures that generalize the notion of choice function from rational choice theory. A correspondence between these one-stage structures and AGM belief revision functions is established. We then add branching time and consider more general structures that accommodate iterated revision. AGM temporal revision structures are defined and a syntactic axiomatization is provided.

1 Introduction

We present "possible-worlds" semantic structures for one-stage and iterated belief revision. We begin, in Section 2, with a review of the AGM belief revision functions. In Section 3 we introduce one-stage revision structures that are related to the choice structures considered in the rational choice literature and in Section 4 we show that a sub-class of these structures corresponds to the set of AGM belief revision functions. In Section 5 we consider a generalization of the structures of Section 3, which we call temporal belief revision frames. These incorporate branching time and allow one to model iterated belief revision. We then define the sub-class of AGM temporal frames and provide, in Section 7, a modal axiomatization of this class.

*This paper was presented at the Workshop on *Formal models of belief change in rational agents*, Dagstuhl, August 2007 (http://kathrin.dagstuhl.de/07351/Materials2/).

Dagstuhl Seminar Proceedings 07351
Formal Models of Belief Change in Rational Agents
http://drops.dagstuhl.de/opus/volltexte/2007/1205

2 AGM belief revision functions

We begin by recalling the theory of belief revision due to Alchourrón, Gärdenfors and Makinson [1], known as the AGM theory (see also [8]). In their approach beliefs are modeled as sets of formulas in a given syntactic language and belief revision is construed as an operation that associates with every deductively closed set of formulas K (thought of as the initial beliefs) and formula ϕ (thought of as new information) a new set of formulas $K^{\circledast}(\phi)$ representing the revised beliefs (a more common notation in the literature is $K^{\circledast}_{\phi}$ but we prefer the to use the more explicit functional form $K^{\circledast}(\phi)$).

Let S be a countable set of atomic propositions and $\mathbb{L}_0$ the propositional language built on S. Thus the set Φ_0 of formulas of $\mathbb{L}_0$ is defined recursively as follows: if $p \in S$ then $p \in \Phi_0$ and if $\phi, \psi \in \Phi_0$ then $\neg\phi \in \Phi_0$ and $\phi \vee \psi \in \Phi_0$.

Given a subset $K \subseteq \Phi_0$, its PL-deductive closure $[K]^{PL}$ (where 'PL' stands for Propositional Logic) is defined as follows: $\psi \in [K]^{PL}$ if and only if there exist $\phi_1, ..., \phi_n \in K$ such that $(\phi_1 \wedge ... \wedge \phi_n) \to \psi$ is a tautology (that is, a theorem of Propositional Logic). A set $K \subseteq \Phi_0$ is *consistent* if $[K]^{PL} \neq \Phi_0$ (equivalently, if there is no formula ϕ such that both ϕ and $\neg\phi$ belong to $[K]^{PL}$). A set $K \subseteq \Phi_0$ is *deductively closed* if $K = [K]^{PL}$. A *belief set* is a set $K \subseteq \Phi_0$ which is deductively closed. The set of belief sets will be denoted by $\mathbb{K}$ and the set of consistent belief sets by $\mathbb{K}^{con}$.

Let $K \in \mathbb{K}^{con}$ be a consistent belief set representing the agent's initial beliefs. A *belief revision function for* K is a function

$$K^{\circledast} : \Phi_0 \to 2^{\Phi_0}$$

that associates with every formula $\phi \in \Phi_0$ (thought of as new information) a set $K^{\circledast}(\phi) \subseteq \Phi_0$ (thought of as the new belief). A belief revision function is called an *AGM revision function* if it satisfies the following properties, known as the AGM postulates: $\forall \phi, \psi \in \Phi_0$,

$(\circledast 1)$	$K^{\circledast}(\phi) \in \mathbb{K}$
$(\circledast 2)$	$\phi \in K^{\circledast}(\phi)$
$(\circledast 3)$	$K^{\circledast}(\phi) \subseteq [K \cup \{\phi\}]^{PL}$
$(\circledast 4)$	if $\neg\phi \notin K$, then $[K \cup \{\phi\}]^{PL} \subseteq K^{\circledast}(\phi)$
$(\circledast 5a)$	if ϕ is a contradiction then $K^{\circledast}(\phi) = \Phi_0$
$(\circledast 5b)$	if ϕ is not a contradiction then $K^{\circledast}(\phi) \neq \Phi_0$
$(\circledast 6)$	if $\phi \leftrightarrow \psi$ is a tautology then $K^{\circledast}(\phi) = K^{\circledast}(\psi)$
$(\circledast 7)$	$K^{\circledast}(\phi \wedge \psi) \subseteq [K^{\circledast}(\phi) \cup \{\psi\}]^{PL}$
$(\circledast 8)$	if $\neg\psi \notin K^{\circledast}(\phi)$, then $[K^{\circledast}(\phi) \cup \{\psi\}]^{PL} \subseteq K^{\circledast}(\phi \wedge \psi)$.

$(\circledast 1)$ requires the revised belief set to be deductively closed.

$(\circledast 2)$ requires that the information be believed.

$(\circledast 3)$ says that beliefs should be revised minimally, in the sense that no new formula should be added unless it can be deduced from the information received and the initial beliefs.

($\circledast$4) says that if the information received is compatible with the initial beliefs, then any formula that can be deduced from the information and the initial beliefs should be part of the revised beliefs.

($\circledast 5ab$) require the revised beliefs to be consistent, unless the information ϕ is contradictory (that is, $\neg\phi$ is a tautology).

($\circledast$6) requires that if ϕ is propositionally equivalent to ψ then the result of revising by ϕ be identical to the result of revising by ψ.

($\circledast$7) and ($\circledast$8) are a generalization of ($\circledast$3) and ($\circledast$4) that

> "applies to *iterated* changes of belief. The idea is that if $K^{\circledast}(\phi)$ is a revision of K [prompted by ϕ] and $K^{\circledast}(\phi)$ is to be changed by adding further sentences, such a change should be made by using expansions of $K^{\circledast}(\phi)$ whenever possible. More generally, the minimal change of K to include both ϕ and ψ (that is, $K^{\circledast}(\phi \wedge \psi)$) ought to be the same as the expansion of $K^{\circledast}(\phi)$ by ψ, so long as ψ does not contradict the beliefs in $K^{\circledast}(\phi)$" (Gärdenfors [8], p. 55).[1]

3 Choice structures and one-stage revision frames

We now turn to a semantic counterpart to the AGM belief revision functions, which is in the spirit of Grove's [10] system of spheres. The structures we will consider are known in rational choice theory as *choice functions* (see, for example, [15] and [16]).

Definition 1 *A* choice structure *is a quadruple* $\langle \Omega, \mathcal{E}, \mathbb{O}, \mathbf{R} \rangle$ *where*

- Ω *is a non-empty set of* states*; subsets of* Ω *are called* events*.*
- $\mathcal{E} \subseteq 2^{\Omega}$ *is a collection of events (*2^{Ω} *denotes the set of subsets of* Ω*).*
- $\mathbf{R} : \mathcal{E} \to 2^{\Omega}$ *is a function that associates with every event* $E \in \mathcal{E}$ *an event* $\mathbf{R}_E \subseteq \Omega$ *(we use the notation* $\mathbf{R}_E$ *rather than* $\mathbf{R}(E)$*).*
- $\mathbb{O} \in \mathcal{E}$ *is a distinguished element of* $\mathcal{E}$ *with* $\mathbb{O} \neq \varnothing$*.*

In rational choice theory a set $E \in \mathcal{E}$ is interpreted as a set of available alternatives and $\mathbf{R}_E$ is interpreted as the subset of E which consists of those alternatives that could be rationally chosen. In our case, we interpret the elements of $\mathcal{E}$ as possible items of information that the agent might receive and the interpretation of $\mathbf{R}_E$ is that, if informed that event E has occurred, the agent considers as possible all and only the states in $\mathbf{R}_E$. For the distinguished element $\mathbb{O}$, we interpret $\mathbf{R}_{\mathbb{O}}$ as the *original* or *initial* beliefs of the agent.[2]

Note that we do not impose the requirement that $\Omega \in \mathcal{E}$.

[1] The expansion of $K^{\circledast}(\phi)$ by ψ is $\left[K^{\circledast}(\phi) \cup \{\psi\}\right]^{PL}$.

[2] In the rational choice literature there is no counterpart to the distinguished set $\mathbb{O}$.

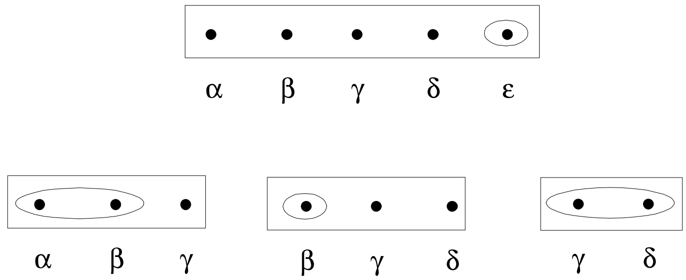

Figure 1

Definition 2 *A* one-stage revision frame *is a choice structure* $\langle \Omega, \mathcal{E}, \mathbb{O}, \mathbf{R} \rangle$ *that satisfies the following properties:* $\forall E, F \in \mathcal{E}$,

(BR1) $\mathbf{R}_E \subseteq E$,

(BR2) if $E \neq \varnothing$ *then* $\mathbf{R}_E \neq \varnothing$,

(BR3) if $E \subseteq F$ *and* $\mathbf{R}_F \cap E \neq \varnothing$ *then* $\mathbf{R}_E = \mathbf{R}_F \cap E$.

(BR4) if $\mathbf{R}_{\mathbb{O}} \cap E \neq \varnothing$ *then* $\mathbf{R}_E = \mathbf{R}_{\mathbb{O}} \cap E$.

In the rational choice literature, (BR1) and (BR2) are taken to be part of the definition of a choice function, while (BR3) is known as Arrow's axiom (see [16] p. 25). Property (BR4), which corresponds to our Qualitative Bayes Rule (see below), has not been investigated in that literature.

The following is an example of a one-stage belief revision frame: $\Omega = \{\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta\}$, $\mathcal{E} = \{\{\alpha, \beta, \gamma, \delta, \varepsilon\},\ \{\alpha, \beta, \gamma\},\ \{\beta, \gamma, \delta\},\ \{\gamma, \delta\}\}$, $\mathbb{O} = \{\alpha, \beta, \gamma, \delta, \varepsilon\}$, $\mathbf{R}_{\mathbb{O}} = \{\varepsilon\}$, $\mathbf{R}_{\{\alpha,\beta,\gamma\}} = \{\alpha, \beta\}$, $\mathbf{R}_{\{\beta,\gamma,\delta\}} = \{\beta\}$, $\mathbf{R}_{\{\gamma,\delta\}} = \{\gamma, \delta\}$. The frame is shown in Figure 1, where the elements of $\mathcal{E}$ are represented by rectangles and, for every $E \in \mathcal{E}$, the set $\mathbf{R}_E$ is represented by an oval inside the rectangle that represents E.

Let S be a set of sentence letters or atomic propositions and Φ_0 the set of propositional formulas based on S. A *one-stage revision model* is a quintuple $\langle \Omega, \mathcal{E}, \mathbb{O}, \mathbf{R}, V \rangle$ where $\langle \Omega, \mathcal{E}, \mathbb{O}, \mathbf{R} \rangle$ is a one-stage revision frame and $V : S \to 2^{\Omega}$ is a function (called a *valuation*) that associates with every atomic proposition p the set of states at which p is true. Truth of an arbitrary formula $\phi \in \Phi_0$ in a model is defined recursively as follows ($\omega \models \phi$ means that formula ϕ is true at state ω): (1) for $p \in S$, $\omega \models p$ if and only if $\omega \in V(p)$, (2) $\omega \models \neg\phi$ if and only if $\omega \not\models \phi$ and (3) $\omega \models \phi \vee \psi$ if and only if either $\omega \models \phi$ or $\omega \models \psi$ (or both). The truth set of a formula ϕ is denoted by $\|\phi\|$. Thus $\|\phi\| = \{\omega \in \Omega : \omega \models \phi\}$.

Given a one-stage revision model, we say that

(1) the agent *initially believes that* ϕ if and only if $\mathbf{R}_{\mathbb{O}} \subseteq \|\phi\|$,

(2) the agent *believes that* ϕ *upon learning that* ψ if and only if $\|\psi\| \in \mathcal{E}$ and $\mathbf{R}_{\|\psi\|} \subseteq \|\phi\|$.

Definition 3 *A one-stage revision model is* comprehensive *if for every formula* ϕ*,* $\|\phi\| \in \mathcal{E}$*. It is* rich *if, for every finite set* $P = \{p_1, ..., p_n, q_1, ..., q_m\}$ *of atomic propositions, there is a state* $\omega_P \in \Omega$ *such that* $\omega_P \models p_i$ *for every* $i = 1, ..., n$ *and* $\omega_P \models \neg q_j$ *for every* $j = 1, ..., m$ *.*

Thus in a comprehensive one-stage revision model every formula is a possible item of information. For example, a model based on a one-stage revision frame where $\mathcal{E} = 2^{\Omega}$ is comprehensive. In a rich model every formula consisting of a conjunction of atomic proposition or the negation of atomic propositions is true at some state.

4 Correspondence

The following propositions (proved in [5]) show that the set of AGM belief revision functions corresponds to the set of comprehensive and rich one-stage revision models, in the sense that
(1) given a comprehensive and rich one-stage revision model, we can associate with it a consistent belief set K and a corresponding AGM belief revision function $K^{\circledast}$, and
(2) given a consistent belief set K and an AGM belief revision function $K^{\circledast}$ there exists a comprehensive and rich one-stage revision model whose associated belief set and AGM belief revision function coincide with K and $K^{\circledast}$, respectively.

Proposition 1 *Let* $\langle \Omega, \mathcal{E}, \mathbb{O}, \mathbf{R}, V \rangle$ *be a comprehensive one-stage revision model. Define* $K = \{\psi \in \Phi_0 : \mathbf{R}_{\mathbb{O}} \subseteq \|\psi\|\}$*. Then* K *is a consistent belief set. For every* $\phi \in \Phi_0$ *define* $K^{\circledast}(\phi) = \{\psi \in \Phi_0 : \mathbf{R}_{\|\phi\|} \subseteq \|\psi\|\}$*. Then the function* $K^{\circledast} : \Phi_0 \to 2^{\Phi_0}$ *so defined satisfies AGM postulates (⊛1)-(⊛5a) and (⊛6)-(⊛8). If the model is rich then also (⊛5b) is satisfied.*

Proposition 2 *Let* $K \in \mathbb{K}$ *be a consistent belief set and* $K^{\circledast} : \Phi_0 \to 2^{\Phi_0}$ *be an AGM belief revision function (that is,* $K^{\circledast}$ *satisfies the AGM postulates (⊛1)-(⊛8)). Then there exists a comprehensive and rich one-stage revision model* $\langle \Omega, \mathcal{E}, \mathbb{O}, \mathbf{R}, V \rangle$ *such that* $K = \{\psi \in \Phi_0 : \mathbf{R}_{\mathbb{O}} \subseteq \|\psi\|\}$ *and, for every* $\phi \in \Phi_0$*,* $K^{\circledast}(\phi) = \{\psi \in \Phi_0 : \mathbf{R}_{\|\phi\|} \subseteq \|\psi\|\}$*.*

5 Temporal belief revision frames

In order to model iterated belief revision, we now turn to the richer structures introduced in [4], which are branching-time structures with the addition of a belief relation and an information relation for every instant t. We then show that these structures are a generalization of the one-stage belief revision structures considered above.

A *next-time branching frame* is a pair $\langle T, \rightarrowtail \rangle$ where T is a non-empty, countable set of instants and $\rightarrowtail$ is a binary relation on T satisfying the following properties: $\forall t_1, t_2, t_3 \in T$,

(1)	backward uniqueness	if $t_1 \rightarrowtail t_3$ and $t_2 \rightarrowtail t_3$ then $t_1 = t_2$
(2)	acyclicity	if $\langle t_1, ..., t_n \rangle$ is a sequence with $t_i \rightarrowtail t_{i+1}$ for every $i = 1, ..., n-1$, then $t_n \neq t_1$.

The interpretation of $t_1 \rightarrowtail t_2$ is that t_2 is an *immediate successor* of t_1 or t_1 is the *immediate predecessor* of t_2: every instant has at most a unique immediate predecessor but can have several immediate successors.

Definition 4 *A* temporal belief revision frame *is a tuple* $\langle T, \rightarrowtail, \Omega, \{\mathcal{B}_t, \mathcal{I}_t\}_{t \in T} \rangle$ *where* $\langle T, \rightarrowtail \rangle$ *is a next-time branching frame,* Ω *is a non-empty set of states (or possible worlds) and, for every* $t \in T$*,* $\mathcal{B}_t$ *and* $\mathcal{I}_t$ *are binary relations on* Ω*.*

The interpretation of $\omega \mathcal{B}_t \omega'$ is that at state ω and time t the individual considers state ω' possible (an alternative expression is "ω' is a doxastic alternative to ω at time t"), while the interpretation of $\omega \mathcal{I}_t \omega'$ is that at state ω and time t, according to the information received, it is possible that the true state is ω'. We shall use the following notation:

$$\mathcal{B}_t(\omega) = \{\omega' \in \Omega : \omega \mathcal{B}_t \omega'\} \text{ and, similarly, } \mathcal{I}_t(\omega) = \{\omega' \in \Omega : \omega \mathcal{I}_t \omega'\}.$$

Figure 2 illustrates a temporal belief revision frame. For simplicity, in all the figures we assume that the information relations $\mathcal{I}_t$ are equivalence relations (whose equivalence classes are denoted by rectangles) and the belief relations $\mathcal{B}_t$ are serial, transitive and euclidean[3] (we represent this fact by enclosing states in ovals and, within an equivalence class for $\mathcal{I}_t$, we have that - for every two states ω and ω' - $\omega' \in \mathcal{B}_t(\omega)$ if and only if ω' belongs to an oval).[4] For example, in Figure 2 we have that $\mathcal{I}_{t_1}(\gamma) = \{\alpha, \beta, \gamma\}$ and $\mathcal{B}_{t_1}(\gamma) = \{\alpha, \beta\}$.

[3] $\mathcal{B}_t$ is serial if, $\forall \omega \in \Omega$, $\mathcal{B}_t(\omega) \neq \varnothing$; it is transitive if $\omega' \in \mathcal{B}_t(\omega)$ implies that $\mathcal{B}_t(\omega') \subseteq \mathcal{B}_t(\omega)$; it is euclidean if $\omega' \in \mathcal{B}_t(\omega)$ implies that $\mathcal{B}_t(\omega) \subseteq \mathcal{B}_t(\omega')$.

[4] Note, however, that our results do *not* require $\mathcal{I}_t$ to be an equivalence relation, nor do they require $\mathcal{B}_t$ to be serial, transitive and euclidean.

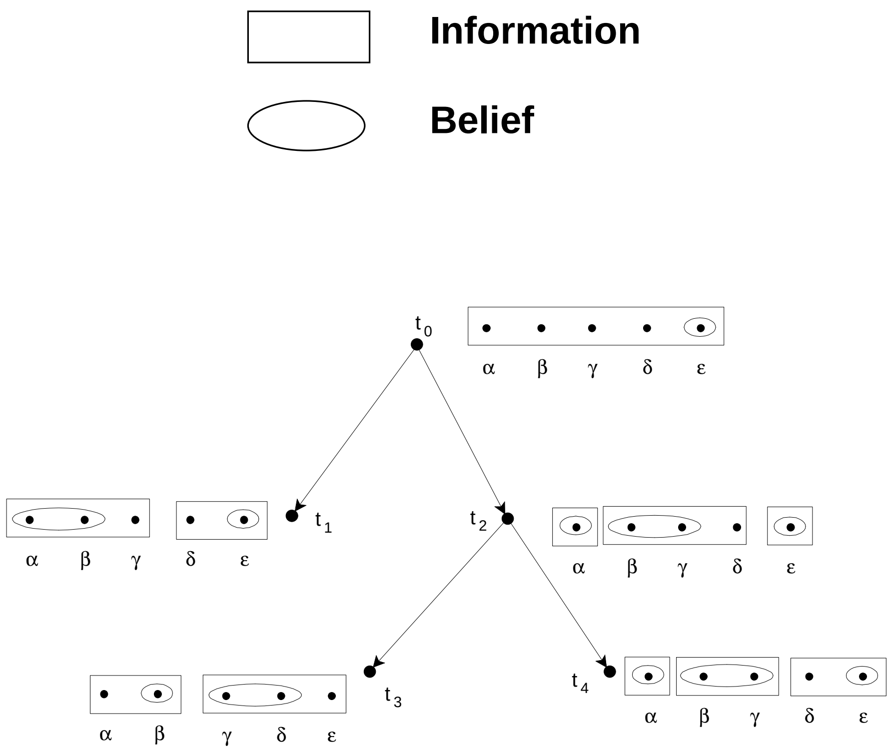
Information
Belief
t_0
α β γ δ ε
t_1
α β γ δ ε
t_2
α β γ δ ε
t_3
α β γ δ ε
t_4
α β γ δ ε

Figure 2

Temporal belief revision frames can be used to describe either a situation where the objective facts describing the world do not change − so that only the beliefs of the agent change over time − or a situation where both the facts and the doxastic state of the agent change. In the literature the first situation is called belief revision, while the latter is called belief update (see [12]). We shall focus on belief revision.

Definition 5 *An AGM-frame is a temporal belief revision frame that satisfies the following properties:* $\forall \omega \in \Omega$, $\forall t, t_1, t_2, t_3 \in T$,

(1) $\mathcal{B}_t(\omega) \subseteq \mathcal{I}_t(\omega)$,

(2) if $\mathcal{I}_t(\omega) \neq \varnothing$ *then* $\mathcal{B}_t(\omega) \neq \varnothing$.

(3) if $t_1 \rightarrowtail t_2$, $t_1 \rightarrowtail t_3$, $\mathcal{I}_{t_3}(\omega) \subseteq \mathcal{I}_{t_2}(\omega)$ *and* $\mathcal{I}_{t_3}(\omega) \cap \mathcal{B}_{t_2}(\omega) \neq \varnothing$ *then* $\mathcal{B}_{t_3}(\omega) = \mathcal{I}_{t_3}(\omega) \cap \mathcal{B}_{t_2}(\omega)$.

(4) if $t_1 \rightarrowtail t_2$ *and* $\mathcal{B}_{t_1}(\omega) \cap \mathcal{I}_{t_2}(\omega) \neq \varnothing$ *then* $\mathcal{B}_{t_2}(\omega) = \mathcal{B}_{t_1}(\omega) \cap \mathcal{I}_{t_2}(\omega)$

Property (4) is called the Qualitative Bayes Rule in [4]. The frame of Figure 2 is not an *AGM*-frame, because, although it satisfies properties (1) and (2), it fails the other two properties. Failure of property (3) can be seen from the fact that $t_2 \rightarrowtail t_3$, $t_2 \rightarrowtail t_4$, $\mathcal{I}_{t_4}(\delta) = \{\delta, \varepsilon\} \subseteq \mathcal{I}_{t_3}(\delta) = \{\gamma, \delta, \varepsilon\}$, $\mathcal{I}_{t_4}(\delta) \cap \mathcal{B}_{t_3}(\delta) = \{\delta\}$ (since $\mathcal{B}_{t_3}(\delta) = \{\gamma, \delta\}$) and yet $\mathcal{B}_{t_4}(\delta) = \{\varepsilon\} \neq \mathcal{I}_{t_4}(\delta) \cap \mathcal{B}_{t_3}(\delta) = \{\delta\}$. Failure of property (4) can be seen from the fact that $t_2 \rightarrowtail t_3$, $\mathcal{B}_{t_2}(\gamma) = \{\beta, \gamma\}$ and $\mathcal{I}_{t_3}(\gamma) = \{\gamma, \delta, \varepsilon\}$, so that $\mathcal{B}_{t_2}(\gamma) \cap \mathcal{I}_{t_3}(\gamma) = \{\gamma\}$ and yet $\mathcal{B}_{t_3}(\gamma) = \{\gamma, \delta\} \neq \mathcal{B}_{t_2}(\beta) \cap \mathcal{I}_{t_3}(\beta)$. On the other hand, the temporal frame of Figure 3 below is an *AGM*-frame.

The notion of *AGM*-frame is a generalization of that of one-stage revision frame (see Definition 2). In fact, given an *AGM*-frame $\langle T, \rightarrowtail, \Omega, \{\mathcal{B}_t, \mathcal{I}_t\}_{t \in T} \rangle$ we can associate with every state-instant pair (ω_0, t_0) a one-stage revision frame $\left\langle \Omega^0, \mathcal{E}^0, \mathbb{O}^0, \mathbf{R}^0 \right\rangle$ as follows. Let $\overset{\rightarrowtail}{t_0} = \{t \in T : t_0 \rightarrowtail t\}$, then

- $\Omega^0 = \Omega$,
- $\mathcal{E}^0 = \left\{ E \subseteq \Omega : E = \mathcal{I}_t(\omega_0) \text{ for some } t \in \overset{\rightarrowtail}{t_0} \right\}$,
- $\mathbb{O}^0 = \mathcal{I}_{t_0}(\omega_0)$,
- $\mathbf{R}_{\mathbb{O}^0} = \mathcal{B}_{t_0}(\omega_0)$
- for every $E \in \mathcal{E}$, if $E = \mathcal{I}_t(\omega_0)$ (for some $t \in \overset{\rightarrowtail}{t_0}$) then $\mathbf{R}^0_E = \mathcal{B}_t(\omega_0)$,

By Property (1) of *AGM*-frames the frame $\left\langle \Omega^0, \mathcal{E}^0, \mathbb{O}^0, \mathbf{R}^0 \right\rangle$ so defined satisfies property BR1 of the definition of one-stage revision frame, while Property (2) ensures that BR2 is satisfied, Property (3) ensures that BR3 is satisfied and Property (4) ensures that BR4 is satisfied.

Consider, for example, the *AGM*-frame of Figure 3 and the state-instant pair (γ, t_0). Then the associated one-stage revision frame is as follows:

$\mathcal{E} = \{\{\alpha, \beta, \gamma, \delta, \varepsilon\}, \{\alpha, \beta, \gamma\}, \{\beta, \gamma, \delta\}, \{\gamma, \delta\}\}$, $\mathbb{O} = \{\alpha, \beta, \gamma, \delta, \varepsilon\}$, $\mathbf{R}_{\mathbb{O}} = \{\varepsilon\}$, $\mathbf{R}_{\{\alpha,\beta,\gamma\}} = \{\alpha, \beta\}$, $\mathbf{R}_{\{\beta,\gamma,\delta\}} = \{\beta\}$ and $\mathbf{R}_{\{\gamma,\delta\}} = \{\gamma, \delta\}$.

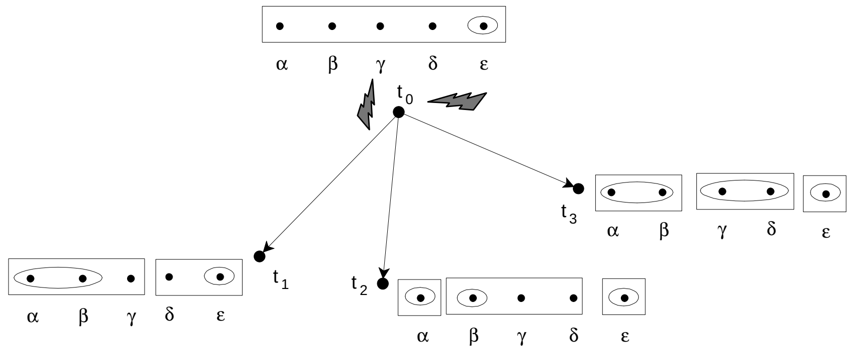

Figure 3

6 Syntactic characterization of AGM-frames

A syntactic characterization of AGM-frames is provided in [4] and [5] and is briefly reviewed here. We consider a propositional language with five modal operators: the next-time operator $\bigcirc$ and its inverse $\bigcirc^{-1}$, the belief operator B, the information operator I and the "all state" operator A. The intended interpretation is as follows:

$\bigcirc\phi$:	"at every next instant it will be the case that ϕ"
$\bigcirc^{-1}\phi$:	"at every previous instant it was the case that ϕ"
$B\phi$:	"the agent believes that ϕ"
$I\phi$:	"the agent is informed that ϕ"
$A\phi$:	"it is true at every state that ϕ".

The "all state" operator A is needed in order to capture the non-normality of the information operator I (see below). For a thorough discussion of the "all state" operator see [9].

Note that, while the other operators apply to arbitrary formulas, *we restrict the information operator to apply to Boolean formulas only*, that is, to formulas that do not contain modal operators. Boolean formulas are defined recursively as follows: (1) every atomic proposition is a Boolean formula, and (2) if ϕ and ψ are Boolean formulas then so are $\neg\phi$ and $(\phi \vee \psi)$. The set of Boolean formulas is denoted by Φ^B. Boolean formulas represent facts and, therefore, we restrict information to be about facts.

Given a temporal belief revision frame $\langle T, \rightarrowtail, \Omega, \{\mathcal{B}_t, \mathcal{I}_t\}_{t\in T}\rangle$ one obtains a *model based on it* by adding a function $V : S \to 2^\Omega$ (where S is the set of atomic propositions and 2^Ω denotes the set of subsets of Ω) that associates with every atomic proposition p the set of states at which p is true. Note that defining a valuation this way is what frames the problem as one of belief revision, since the truth value of an atomic proposition p depends only on the state and not on the time.[5] Given a model, a state ω, an instant t and a formula ϕ, we write $(\omega, t) \models \phi$ to denote that ϕ is true at state ω and time t. Let $\|\phi\|$ denote the truth set of ϕ, that is, $\|\phi\| = \{(\omega, t) \in \Omega \times T : (\omega, t) \models \phi\}$ and let $\lceil\phi\rceil_t \subseteq \Omega$ denote the set of states at which ϕ is true at time t, that is, $\lceil\phi\rceil_t = \{\omega \in \Omega : (\omega, t) \models \phi\}$. Truth of an arbitrary formula at a pair (ω, t) is defined recursively as follows:

if $p \in S$,	$(\omega, t) \models p$ if and only if $\omega \in V(p)$;
$(\omega, t) \models \neg\phi$	if and only if $(\omega, t) \nvDash \phi$;
$(\omega, t) \models \phi \vee \psi$	if and only if either $(\omega, t) \models \phi$ or $(\omega, t) \models \psi$ (or both);
$(\omega, t) \models \bigcirc\phi$	if and only if $(\omega, t') \models \phi$ for every t' such that $t \rightarrowtail t'$;
$(\omega, t) \models \bigcirc^{-1}\phi$	if and only if $(\omega, t'') \models \phi$ for every t'' such that $t'' \rightarrowtail t$;
$(\omega, t) \models B\phi$	if and only if $\mathcal{B}_t(\omega) \subseteq \lceil\phi\rceil_t$, that is, if $(\omega', t) \models \phi$ for all $\omega' \in \mathcal{B}_t(\omega)$;
$(\omega, t) \models I\phi$	if and only if $\mathcal{I}_t(\omega) = \lceil\phi\rceil_t$, that is, if (1) $(\omega', t) \models \phi$ for all $\omega' \in \mathcal{I}_t(\omega)$, and (2) if $(\omega', t) \models \phi$ then $\omega' \in \mathcal{I}_t(\omega)$;
$(\omega, t) \models A\phi$	if and only if $\lceil\phi\rceil_t = \Omega$, that is, if $(\omega', t) \models \phi$ for all $\omega' \in \Omega$.

Note that, while the truth condition for the operator B is the standard one, the truth condition for the operator I is non-standard: instead of simply requiring that $\mathcal{I}_t(\omega) \subseteq \lceil\phi\rceil_t$ we require equality: $\mathcal{I}_t(\omega) = \lceil\phi\rceil_t$. Thus our information operator is formally similar to the "all and only" operator introduced in [11] and the "only knowing" operator studied in [14], although the interpretation is different. It is also similar to the "assumption" operator used in [6].

Remark 1 *The truth value of a Boolean formula does not change over time: it is only a function of the state. That is, fix an arbitrary model and suppose that* $(\omega, t) \models \phi$ *where* $\phi \in \Phi^B$*; then, for every* $t' \in T$, $(\omega, t') \models \phi$.

A formula ϕ is *valid in a model* if $\|\phi\| = \Omega \times T$, that is, if ϕ is true at every state-instant pair (ω, t). A formula ϕ is *valid in a frame* if it is valid in every model based on it.

The formal language is built in the usual way (see [2]) from a countable set of atomic propositions, the connectives $\neg$ and $\vee$ (from which the connectives $\wedge$, $\rightarrow$ and $\leftrightarrow$ are defined as usual) and the modal operators $\bigcirc$, $\bigcirc^{-1}$, B, I and A, with the restriction that $I\phi$ is a well-formed formula if and only if ϕ is a Boolean formula. Let $\Diamond\phi \overset{def}{=} \neg\bigcirc\neg\phi$, and $\Diamond^{-1}\phi \overset{def}{=} \neg\bigcirc^{-1}\neg\phi$. Thus the interpretation of $\Diamond\phi$ is "at *some* next instant it will be the case that ϕ" while the interpretation of $\Diamond^{-1}\phi$ is "at some immediately preceding instant it was the case that ϕ".

We denote by $\mathbb{L}_0$ the basic logic defined by the following axioms and rules of inference.

AXIOMS:

1. All propositional tautologies.

[5]Belief update would require a valuation to be defined as a function $V : S \to 2^{\Omega \times T}$.

2. Axiom K for $\bigcirc$, $\bigcirc^{-1}$, B and A[6]: for $\Box \in \{\bigcirc, \bigcirc^{-1}, B, A\}$

$$(\Box\phi \wedge \Box(\phi \rightarrow \psi)) \rightarrow \Box\psi \quad \text{(K)}$$

3. Temporal axioms relating $\bigcirc$ and $\bigcirc^{-1}$:

$$\phi \rightarrow \bigcirc\Diamond^{-1}\phi \quad (\text{O}_1)$$
$$\phi \rightarrow \bigcirc^{-1}\Diamond\phi \quad (\text{O}_2)$$

4. Backward Uniqueness axiom:

$$\Diamond^{-1}\phi \rightarrow \bigcirc^{-1}\phi \quad \text{(BU)}$$

5. S5 axioms for A:

$$A\phi \rightarrow \phi \quad (\text{T}_A)$$
$$\neg A\phi \rightarrow A\neg A\phi \quad (5_A)$$

6. Inclusion axiom for B (note the absence of an analogous axiom for I):

$$A\phi \rightarrow B\phi \quad (\text{Incl}_B)$$

7. Axioms to capture the non-standard semantics for I: for $\phi, \psi \in \Phi^B$ (recall that Φ^B denotes the set of Boolean formulas),

$$(I\phi \wedge I\psi) \rightarrow A(\phi \leftrightarrow \psi) \quad (\text{I}_1)$$
$$A(\phi \leftrightarrow \psi) \rightarrow (I\phi \leftrightarrow I\psi) \quad (\text{I}_2)$$

RULES OF INFERENCE:

1. Modus Ponens: $\frac{\phi, \quad \phi\rightarrow\psi}{\psi}$ (MP)

2. Necessitation for A, $\bigcirc$ and $\bigcirc^{-1}$: for every $\Box \in \{A, \bigcirc, \bigcirc^{-1}\}$, $\frac{\phi}{\Box\phi}$ (Nec).

Note that from MP, Incl_B and Necessitation for A one can derive necessitation for B. On the other hand, necessitation for I is *not* a rule of inference of this logic (indeed it is not validity preserving).

Remark 2 *By MP, axiom K and Necessitation, the following is a derived rule of inference for the operators* $\bigcirc$, $\bigcirc^{-1}$, B *and* A: $\frac{\phi\rightarrow\psi}{\Box\phi\rightarrow\Box\psi}$ *for* $\Box \in \{\bigcirc, \bigcirc^{-1}, B, A\}$. *We call this rule RK. On the other hand, rule RK is not a valid rule of inference for the operator* I.

[6] Axiom K for I is superfluous, since it can be derived from axioms I_1 and I_2 below (see [3] , p. 204).

Our purpose is to model how the beliefs of an individual change over time in response to *factual* information. Thus *the axioms we introduce are restricted to Boolean formulas*, which are formulas that do not contain any modal operators.

Let $\mathbb{L}_{AGM}$ be the logic obtained by adding to $\mathbb{L}_0$ the following six axioms.

The first axiom requires the individual not to drop any of his current factual beliefs at any next instant at which he is informed of some fact that he currently considers possible: if ϕ and ψ are Boolean,

$$(\neg B\neg\phi \wedge B\psi) \to \bigcirc(I\phi \to B\psi). \tag{ND}$$

The second axiom requires that if the individual considers it possible that $(\phi \wedge \neg\psi)$ then at any next instant at which he is informed that ϕ he does not believe that ψ: if ϕ and ψ are Boolean,

$$\neg B\neg(\phi \wedge \neg\psi) \to \bigcirc(I\phi \to \neg B\psi). \tag{NA}$$

The third axiom states that information is believed:

$$I\phi \to B\phi. \tag{A}$$

The fourth axiom says that if there is a next instant where the individual is informed that $\phi \wedge \psi$ and believes that χ, then at every next instant it must be the case that if the individual is informed that ϕ then he must believe that $(\phi \wedge \psi) \to \chi$ (we call this axiom $K7$ because it corresponds to AGM postulate ⊛7): if ϕ, ψ and χ are Boolean formulas,

$$\Diamond(I(\phi \wedge \psi) \wedge B\chi) \to \bigcirc(I\phi \to B\left((\phi \wedge \psi) \to \chi\right)). \tag{$K7$}$$

The fifth axiom says that if there is a next instant where the individual is informed that ϕ, considers $\phi\wedge\psi$ possible and believes that $\psi \to \chi$, then at every next instant it must be the case that if the individual is informed that $\phi \wedge \psi$ then he believes that χ (we call this axiom $K8$ because it corresponds to AGM postulate ⊛8): if ϕ, ψ and χ are Boolean formulas,

$$\Diamond(I\phi \wedge \neg B\neg(\phi \wedge \psi) \wedge B(\psi \to \chi)) \to \bigcirc(I(\phi \wedge \psi) \to B\chi). \tag{$K8$}$$

The sixth axiom says that if the individual receives consistent information then his beliefs are consistent, in the sense that he does not simultaneously believe a formula and its negation ('WC' stands for 'Weak Consistency'): if ϕ is a Boolean formula,

$$(I\phi \wedge \neg A\neg\phi) \to (B\psi \to \neg B\neg\psi). \tag{WC}$$

A logic $\mathbb{L}$ is *characterized by* (or *characterizes*) a class $\mathbb{F}$ of frames if (1) every axiom (hence, every theorem) of $\mathbb{L}$ is valid in every frame in $\mathbb{F}$, and (2) for every frame not in $\mathbb{F}$ there is an axiom in $\mathbb{L}$ that is not valid in that frame.

The following result is proved in [5].

Proposition 3 *Logic $\mathbb{L}_{AGM}$ is characterized by the class of AGM-frames.*

References

[1] Alchourrón, C., P. Gärdenfors and D. Makinson, On the logic of theory change: partial meet contraction and revision functions, *The Journal of Symbolic Logic*, 1985, 50: 510-530.

[2] Blackburn, P., M. de Rijke and Y. Venema, *Modal logic,* Cambridge University Press, 2001.

[3] Bonanno, G., A simple modal logic for belief revision, *Synthese*, 2005, 147: 193-228 (and *Knowledge, Rationality and Action*, 5-40). Reprinted in: W. van der Hoek (editor), *Uncertainty, rationality and agency*, Springer, Dordrect, 2006, pp. 139-174.

[4] Bonanno, G., Axiomatic characterization of the AGM theory of belief revision in a temporal logic, *Artificial Intelligence*, 2007, 171: 144–160.

[5] Bonanno, G., Belief revision in a temporal framework, Working Paper, Department of Economics, University of California Davis, May 2007.

[6] Brandenburger, A. and J. Keisler, An impossibility theorem on beliefs in games, *Studia Logica,* 2006, 84: 211-240.

[7] Chellas, B., *Modal logic: an introduction*, Cambridge University Press, 1984.

[8] Gärdenfors, P., *Knowledge in flux: modeling the dynamics of epistemic states*, MIT Press, 1988.

[9] Goranko, V. and S. Passy, Using the universal modality: gains and questions, *Journal of Logic and Computation*, 1992, 2: 5-30.

[10] Grove, A., Two modellings for theory change, *Journal of Philosophical logic,* 1988, 17: 157-170.

[11] Humberstone, I. L., The modal logic of all and only, *Notre Dame Journal of Formal Logic*, 1987, 28: 177-188.

[12] Katsuno, H and A.O. Mendelzon, Propositional knowledge base revision and minimal change, *Artificial Intelligence*, 1991, 52: 263–294.

[13] Kripke, S., A semantical analysis of modal logic I: normal propositional calculi, *Zeitschrift für Mathematische Logik und Grundlagen der Mathematik*, 1963, 9: 67-96.

[14] Levesque, H. J., All I know: a study in autoepistemic logic, *Artificial Intelligence*, 1990, 5: 263-309.

[15] Rott, H., *Change, choice and inference*, Clarendon Press, Oxford, 2001.

[16] Suzumura, K., *Rational choice, collective decisions and social welfare*, Cambridge University Press, 1983.