Characterizations and linear-time recognition of probe cographs

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Cographs are those graphs without induced path on four vertices. A graph G is a *probe cograph* if its vertex set can be partitioned into two sets, N (non-probes) and P (probes) where N is independent and G can be extended to a cograph by adding edges between certain non-probes. A *partitioned* probe cograph is a probe cograph with a given partition in N and P.

For convenience, we will use the following notion. For a graph class $\mathcal{C}$ and a graph $G = (V, E)$, a partition $V = P \cup N$ with an independent set N is called a *valid partition* (with respect to $\mathcal{C}$) if there exists $E' \subseteq \binom{N}{2}$ such that $G' = (V, E \cup E')$ is a member of $\mathcal{C}$. Such a graph G' is then called a *valid extension* for G. Thus, G is a probe graph of $\mathcal{C}$ if and only if G has a valid partition, and $G = (P, N, E)$ is a partitioned probe graph of $\mathcal{C}$ if and only if $V(G) = P \cup N$ is a valid partition.

Probe cographs versus singular cograph contractions Recall that cographs (or complement-reducible graphs) [4] are those graphs that can be constructed from a single vertex by repeated applications of complement and disjoint union. They are precisely the P_4-free graphs and are known under many different names and definition; see for example [6].

Cographs were generalized to cograph contractions as follows. A graph G is called a *cograph contraction* if there exists a cograph H together with some pairwise disjoint independent sets $S_1, \dots, S_t$ in H such that G is obtained from H by contracting each S_i to a single vertex s_i and then making the vertices $s_1, \dots, s_t$ to a clique. Cograph contractions have been introduced and investigated in [5] in connection to graph precoloring extensions and perfection, and have been characterised in [7].

A special case of cograph contractions where all independent sets $S_1, \dots, S_t$ are one-element sets has a close relationship to probe cographs. We formulate this situation in a more general setting.

Definition 1. *Let $\mathcal{C}$ be a class of graphs. A graph G is called a* singular $\mathcal{C}$ contraction *if there exists a graph $H \in \mathcal{C}$ together with a (not necessarily independent) set S of vertices of H such that G is obtained from H by making S to a clique.*

Let co-$\mathcal{C}$ be the class consisting of the complements of all graphs in $\mathcal{C}$. $\mathcal{C}$ is *self-complementary* if $\mathcal{C}$ and co-$\mathcal{C}$ coincide. Singular $\mathcal{C}$ contractions are related to probe graphs of $\mathcal{C}$ as follows.

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Proposition 1. *A graph G is a probe graph of $\mathcal{C}$ if and only if $\overline{G}$ is a singular co-$\mathcal{C}$ contraction. In particular, if $\mathcal{C}$ is self-complementary, then probe graphs of $\mathcal{C}$ coincide with the complements of singular $\mathcal{C}$ contractions.*

Corollary 1. *Probe cographs are exactly complements of singular cograph contractions. Also, probe perfect graphs are exactly complements of singular perfect contractions.*

P_n denotes the chordless path on n vertices. The vertices of degree one are called the *endpoints* and the others the *midpoints* of the path. C_n denotes the chordless cycle on n vertices. A *bull* consists of five vertices, a, b, c, d and e, and a, b, c form a triangle and d is adjacent exactly to a, e is adjacent exactly to b.

Singular cograph contractions, hence (complements of) probe cographs, can be characterised as follows.

Theorem 1. *A graph G is a singular cograph contraction if and only if G has a clique Q such that every P_4 has two vertices in Q, every $\overline{P_5}$ has its triangle in Q, and every bull has a vertex of degree one or a vertex of degree two in Q.*

Characterizing probe cographs Unpartitioned probe cographs can be characterized in several ways as follows.

First, for a given graph $G = (V, E)$, a particular 2-Sat instance $F(G)$ is created as follows.

- The boolean variables are the vertices of G,
- for each edge ab of G, $(\overline{a} \vee \overline{b})$ is a clause, the *edge-clause* for ab,
- for each $P_4 = abcd$ of G, $(a \vee b)$ and $(c \vee d)$ are two clauses, the P_4*-clauses* for that P_4,
- for each $P_5 = abcde$ of G, $(\overline{b})$ and $(\overline{d})$ are two clauses, the P_5*-clauses* for that P_5,
- for each bull of G with the triangle abc where b is the degree-2 vertex of the bull, $(a \vee b)$ and $(c \vee b)$ are two clauses, the *bull-clauses* for that bull.

The formula $F(G)$ is the conjunction of all edge-clauses, all P_4-clauses, all P_5-clauses, and all bull-clauses. We will see that G is a probe cograph if and only if $F(G)$ is satisfiable.

Let N be an independent set in $G = (V, E)$. Following [2] we call two vertices x, y in G *twins with respect to* N if both x, y are in N or outside N and x, y are twins in G, or $x \in V \setminus N$, $y \in N$ and $N(y) - x = N(x) \setminus N$.

In a bull, vertices belonging to the triangle of the bull are called *mid-points* of the bull.

Theorem 2. *The following statements are equivalent for any graph G.*

(i) *G is a probe cograph;*
(ii) *$F(G)$ is satisfiable;*

(iii) *G admits an independent set N meeting every P_4 in two vertices, every P_5 in three vertices, and every bull in a mid-point;*
(iv) *$\overline{G}$ is a singular cograph contraction;*
(v) *G admits an independent set N such that, for each induced subgraph H of G with at least two vertices, H has two twins with respect to $N \cap V(H)$.*

At this time we are unable to find a complete list of (minimal) forbidden induced subgraphs for probe cographs. In the partitioned case, however, we have a characterisation in terms of five forbidden induced subgraphs.

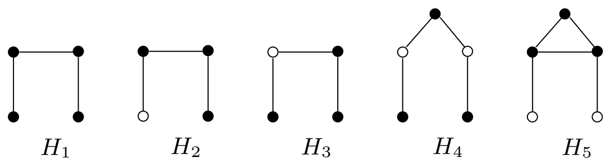

Fig. 1. Forbidden subgraphs for partitioned probe cographs (black vertices probes, white vertices non-probes).

Theorem 3. *Let $G = (P, N, E)$ be a partitioned graph with an independent set N. Then the following statements are equivalent.*

(i) *G is a partitioned probe cograph;*
(ii) *$F(G)$ is satisfied by the assigment $\flat(v) = \texttt{true}$ for $v \in N$ and $\flat(v) = \texttt{false}$ for $v \in P$;*
(iii) *In G, every P_4 has two vertices, every P_5 has three vertices, and every bull has at least a mid-point in N;*
(iv) *None of the graphs $H_1, \dots, H_5$ depicted in Fig. 1 is an induced subgraph in G;*
(v) *$\overline{G}$ is obtained from a cograph on the same vertex set by making N to a clique;*
(vi) *Each induced subgraph H of G with at least two vertices has two twins with respect to $N \cap V(H)$.*

Recognizing probe cographs In [3], Chang et al. give an $O(n^3)$ recognition for partitioned probe cographs and an $O(n^5)$ recognition for probe cographs (n is the number of vertices of the input graph). Chandler et al. [2] improved this to $O(n^2)$ by showing that the recognition of unpartitioned probe cographs can be reduced to the partitioned case in linear time and reducing recognition of partitioned probe cographs to the recognition of partitioned probe distance hereditary graphs, for which they give an $O(n^2)$ algorithm.

Using Theorem 3 (iv) and by modifying the recognition algorithm for cographs due to Corneil at al. [4], we give a linear time recognition algorithm for partitioned probe cographs.

As the unpartitioned case can be reduced to the partitioned case in linear time([1, 2]), probe cographs therefore can be recognized in linear time.

Theorem 4. *Partitioned and unpartitioned probe cographs can be recognized in linear time. Moreover, a valid extension of a given probe graph can be determined in linear time, too.*

Moreover, in the partitioned case, our recognition algorithm produces a certificate for membership that can be checked in linear time (the co-tree of a valid extension) and a certificate for non-membership that can be checked in sublinear time (one of the forbidden subgraphs in Fig. 1).

References

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