Equilibrium Tracing in Bimatrix Games

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Abstract. We analyze the relations of the van den Elzen-Talman algorithm, the Lemke-Howson algorithm and the global Newton method introduced by Govindan and Wilson. It is known that the global Newton method encompasses the Lemke-Howson algorithm; we prove that it also comprises the van den Elzen-Talman algorithm, and more generally, the linear tracing procedure, as a special case. This will lead us to a discussion of traceability of equilibria of index +1. We answer negatively the open question of whether, generically, the van den Elzen-Talman algorithm is flexible enough to trace all equilibria of index +1.

Keywords. Bimatrix games, Equilibrium computation, Homotopy methods, Index

1 Introduction

In this paper we investigate several algorithms for the computation of Nash equilibria in bimatrix games. The Lemke-Howson and the van den Elzen-Talman algorithms are complementary pivoting methods; both have been studied extensively. The difference between the two methods is that while the Lemke-Howson method only allows for a restricted (finite) set of paths, the van den Elzen-Talman algorithm can start at any mixed strategy pair, called prior, and hence generates infinitely many paths. This implies that the van den Elzen-Talman algorithm is more flexible than the Lemke-Howson method. An even more versatile algorithm is the global Newton method [1]; it works for the more general case of finite normal form games.

We investigate the relations between those three algorithms: We show that the Lemke-Howson and van den Elzen-Talman algorithms differ substantially. However, both can be understood as special cases of the global Newton method. For the van den Elzen-Talman algorithm, this is a new result, which can be generalized to the statement that for N-player games, the global Newton method implements the linear tracing procedure introduced by Harsanyi [3].

As a special case of the global Newton method, the van den Elzen-Talman algorithm can generically find only equilibria of index +1. This leads us to the

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issue of traceability of equilibria. Following Hofbauer [6], we call an equilibrium traceable if it is found by the van den Elzen-Talman algorithm from an open set of priors. As explained above, the van den Elzen-Talman algorithm allows for much greater flexibility than the Lemke-Howson method. Hence one might hope that, unlike the Lemke-Howson algorithm, it is powerful enough to find all equilibria of index +1. This raises the until now open question if, generically, all equilibria of index +1 are traceable. We answer this question by analyzing traceability in coordination games.

If a 3×3 coordination game has a completely mixed equilibrium, this equilibrium is bound to have index +1. In addition, the game will have three pure equilibria, also of index +1, and three equilibria of support size two, which have index -1. Hofbauer [6] noted that in such a coordination game the completely mixed equilibrium is not traceable. This is correct in certain cases, for example when the payoff to each player is given by the identity matrix. However, we show that the qualitative equilibrium structure, as described above, is not sufficient to determine if the completely mixed equilibrium of a coordination game is traceable. More precisely, its traceability depends on the specific geometry of the best-reply regions.

We generalize this result to prove that there is an open set in the space of 3×3 bimatrix games, such that all of these have an untraceable equilibrium of index +1. This implies that the flexibility of the van den Elzen-Talman algorithm does not ensure generic traceability of all equilibria of index +1, which in turn has important consequences for the concept of sustainability. Myerson [9] suggested to call an equilibrium sustainable if it can be reached by Harsanyi's and Selten's tracing procedure from an open set of priors. Since the van den Elzen-Talman algorithm implements the tracing procedure, this notion of sustainability is equivalent to the concept of traceability. Hence the results of our paper imply that not all equilibria of index +1 will be sustainable.

The structure of our paper is as follows: In section 2 we give a short review of the van den Elzen-Talman method and analyze its relations to the Lemke-Howson algorithm. We assume the reader to be familiar with the latter method and abstain from outlining it. Detailed descriptions can be found in [8], [17] or [5]. In section 3, we give a brief introduction to the global Newton method, before showing that it encompasses the van den Elzen-Talman algorithm and, more generally, the linear tracing procedure, as a special case. Section 4 contains a discussion of traceability of equilibria.

2 Van den Elzen-Talman versus Lemke-Howson

2.1 The van den Elzen-Talman algorithm

The van den Elzen-Talman algorithm was introduced in [14]. It is a homotopy method that finds equilibria by starting at an arbitrary prior and adjusting the players' replies.

Let (A, B) be a non-degenerate $m \times n$ bimatrix game. Denote by Δ_m and Δ_n the strategy simplices of players one and two, respectively, and the strategy space by $\Delta := \Delta_m \times \Delta_n$. Take an arbitrary starting point $(\overline{x}, \overline{y}) \in \Delta$. The van den Elzen-Talman algorithm traces equilibria of the game (A, B) in the restricted strategy space

$$\Delta_t := (1-t)(\overline{x}, \overline{y}) + t \cdot \Delta$$

for $t \in [0, 1]$, where $t \cdot \Delta = \{tx \mid x \in \Delta\}$. The algorithm starts in $(\overline{x}, \overline{y})$ at $t = 0$ and reaches an equilibrium of (A, B) at $t = 1$. In general, degeneracies can occur along the path, a discussion on how to resolve these can be found in [16].

The van den Elzen-Talman algorithm can also be described as a complementary pivoting procedure: A point $(x, y) \in t \cdot \Delta$ yields an equilibrium in the restricted strategy space Δ_t if and only if there are suitable vectors w, z and real numbers u, v such that the following equations and inequalities hold:

$$\begin{aligned} A \cdot ((1-t)\overline{y} + y) + w &= u\mathbb{1} \\ B^T \cdot ((1-t)\overline{x} + x) + z &= v\mathbb{1} \\ x^T\mathbb{1} = t, y^T\mathbb{1} &= t \\ x^T w = 0, y^T z &= 0 \\ x, w, y, z &\geq 0 \end{aligned} \tag{1}$$

where $\mathbb{1}$ is a vector of 1's of suitable length. The vectors x and y indicate how much weight is put on each strategy in addition to that given by $(1-t)\overline{x}$ and $(1-t)\overline{y}$. The slack variables w and z show how far from being optimal a strategy is against the other player's strategy. The real numbers u and v track the equilibrium payoff during the computation.

The van den Elzen-Talman algorithm can also be understood geometrically as a completely labeled path in the strategy space Δ. Assume that the players' pure strategies are numbered $1, \ldots, m$ for player one and $m+1, \ldots, m+n$ for player two. Define the best reply region for a pure strategy j of player two to be

$$\mathrm{B}(j) := \{x \in \Delta_m \mid j \text{ is a best reply to } x\}$$

Now, for a point $p := (1-t) \cdot \overline{x} + t \cdot x \in \Delta_m$ define its labels at time t to be

$$\{j \mid p \in \mathrm{B}(j)\} \cup \{i \mid x_i = 0\}$$

and similarly for the other player. Then a point in the restricted strategy space Δ_t is an equilibrium of the game (A, B) restricted to Δ_t if and only if it is completely labeled at time t. This follows directly from the description of the algorithm in (1). The pivoting steps of the algorithm occur where one of the players picks up a new label, which then the other player can drop. An analogous description of the Lemke-Howson algorithm can be found in [17]. For further details on the van den Elzen-Talman algorithm we refer the reader to [5], [17] or the original papers [14] and [15].

2.2 A comparison of the Lemke-Howson algorithm and the van den Elzen-Talman method

What happens in the van den Elzen-Talman algorithm if we take the prior $\overline{x}$ to be any pure strategy vector and $\overline{y}$ its unique best reply? This would correspond to a starting point of the Lemke-Howson algorithm, and one might expect the two algorithms to find the same equilibrium.

However, this is not true. An example where the van den Elzen-Talman and Lemke-Howson paths lead to different equilibria is given by the 3×3 bimatrix game

$$\begin{pmatrix} 4 & 4 & 4 \\ 0 & 0 & 6 \\ 5 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 6 & 12 & 0 \\ 0 & 4 & 0 \\ 8 & 0 & 13 \end{pmatrix} \tag{2}$$

and starting points $\overline{x} = (0, 0, 1), \overline{y} = (0, 0, 1)$. The Lemke-Howson algorithm from this starting point (i.e. with missing label 3) finds the equilibrium $(5/11, 0, 6/11)$, $(4/5, 0, 1/5)$, whereas the van den Elzen-Talman algorithm finds the pure strategy equilibrium $(1, 0, 0), (0, 1, 0)$. A graphical description of the van den Elzen-Talman path for our example can be found in Figure 1. A further discussion of the relations between the two algorithms will be provided at the end of the next section.

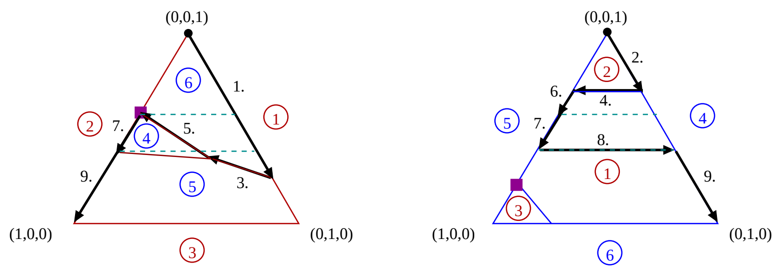

Fig. 1. The van den Elzen-Talman-path for example (2). The left simplex is player one's, the right one player two's. The labels in the simplex mark the players' best reply regions, the labels outside mark the edges of the simplex where the corresponding strategy is unplayed. The square dot is the equilibrium that is found by the Lemke-Howson algorithm. The black arrows give the path of the van den Elzen-Talman algorithm starting at $(0, 0, 1), (0, 0, 1)$, and are numbered in the order in which they occur. The dotted lines trace the restricted strategy space Δ_t after step 5 (upper line) and step 7 (lower line).

3 Relations to the global Newton method

3.1 A short review of the global Newton method

The global Newton method was introduced by Govindan and Wilson [1]; it is a homotopy method for the computation of Nash equilibria in finite normal form games. However, for simplicity we will keep our description of the algorithm to the case of non-degenerate bimatrix games.

First we need to introduce a procedure of creating new games from old ones that goes back to [7]: Starting from an $m \times n$ bimatrix game (A, B) and directional (column) vectors $a \in \mathbb{R}^m, b \in \mathbb{R}^n$, define a new game $(A, B) \oplus (a, b)$ by adding the vector a to each column of A, and the vector b^T to each row of B. Hence the game $(A, B) \oplus (a, b)$ is given by the matrices

$$A + \begin{pmatrix} a_1 & \dots & a_1 \\ \vdots & & \vdots \\ a_m & \dots & a_m \end{pmatrix}, \; B + \begin{pmatrix} b_1 & \dots & b_n \\ & \dots & \\ b_1 & \dots & b_n \end{pmatrix} \tag{3}$$

Note that in general this procedure changes the equilibria of the game.

The idea of the global Newton method is as follows: Assume we would like to calculate an equilibrium of a non-degenerate bimatrix game (A, B). For any pair of directional vectors (a, b) as above, consider the ray $\{(A, B) \oplus \lambda \cdot (a, b) \mid \lambda \geq 0\}$ in the space of games. Take the graph of the equilibrium correspondence over that ray, i.e. the correspondence that maps each game to the set of its equilibria. The structure theorem of Kohlberg and Mertens [7] implies that for (a, b) outside a lower-dimensional set, this graph will be a semi-algebraic one-dimensional manifold with boundary (where boundary points are equilibria of the game (A, B)). If we can find an equilibrium somewhere "far out" and trace it along that manifold, we arrive at an equilibrium of the original game.

Although the idea is conceptually straightforward, its implementation is technically demanding. Govindan and Wilson take advantage of the differentiable structure which is implicit in the structure theorem. They convert the problem of tracing equilibria over a ray to one of calculating zeros of piecewise differentiable functions, and for this they use an approach due to [13]. For further details we refer the reader to the original paper [1].

For our purpose, all we need to know is that for a non-degenerate bimatrix game (A, B) and a pair of directional vectors (a, b) in suitable Euclidean space, the global Newton method traces equilibria along the graph of the equilibrium correspondence over the ray $\{(A, B) \oplus \lambda \cdot (a, b) \mid \lambda \geq 0\}$. In other words, for $\mathcal{E}$ the graph of the equilibrium correspondence, the global Newton method traces equilibria along the set

$$\{((A, B) \oplus \lambda \cdot (a, b), (x, y)) \in \mathcal{E} \mid \lambda \geq 0\}$$

A crucial condition for the algorithm to work is that this set is non-degenerate, in the sense that its one-point compactification is a one-dimensional manifold without branch points. Generically, however, this is the case.

3.2 The van den Elzen-Talman algorithm as a special case of the global Newton method

We would now like to prove that the van den Elzen-Talman algorithm is a special case of the global Newton method. Let (A, B) be a non-degenerate $m \times n$ bimatrix game. Choose a starting point $(\overline{x}, \overline{y}) \in \Delta_m \times \Delta_n$. The van den Elzen-Talman algorithm traces the set

$$P_{ET}((A, B), (\overline{x}, \overline{y})) := \left\{ \begin{matrix} (t, (x, y)) \in [0, 1] \times \Delta_m \times \Delta_n \mid (x, y) \in \Delta_t \text{ and } (x, y) \\ \text{is an equilibrium for the game } (A, B) \text{ restricted to } \Delta_t \end{matrix} \right\}$$

for Δ_t the restricted strategy space defined in section 2.1.

For $\lambda \in \mathbb{R}$, define the game

$$(A,B)^{\overline{x},\overline{y}}(\lambda) := (A,B) \oplus \lambda \cdot \big(A\overline{y}, B^T\overline{x}\big)$$

where $\oplus$ is defined as in (3). Let $\mathcal{E}$ be the graph of the equilibrium correspondence over the space of games $\mathbb{R}^{m\times n} \times \mathbb{R}^{m\times n}$, and

$$P_{GNM}((A,B),(\overline{x},\overline{y})) := \big\{((A,B)^{\overline{x},\overline{y}}(\lambda),(x,y)) \in \mathcal{E} \mid \lambda \geq 0\big\}$$

the set of equilibria over the ray of games $\{(A,B)^{\overline{x},\overline{y}}(\lambda) \mid \lambda \geq 0\}$. This is the set traced by the global Newton method, when choosing as directional vector $\big(A\overline{y}, B^T\overline{x}\big)$. The following theorem states that it is homeomorphic to $P_{ET}\left((A,B),(\overline{x},\overline{y})\right)$, after removing the starting point $(0,(\overline{x},\overline{y}))$ from the latter. This is the central result of this section; it establishes the van den Elzen-Talman algorithm as a special case of the global Newton method.

Theorem 1. *Let (A,B) be a non-degenerate $m \times n$ bimatrix game. Choose a starting point $(\overline{x},\overline{y}) \in \Delta_m \times \Delta_n$. Let $\boldsymbol{\lambda} : (0,1] \to \mathbb{R}^{\geq 0}$, $t \mapsto \frac{1}{t} - 1$. Then the map*

$$\begin{array}{rcl} P_{ET}\left((A,B),(\overline{x},\overline{y})\right) \setminus \{(0,(\overline{x},\overline{y}))\} & \to & P_{GNM}\left((A,B),(\overline{x},\overline{y})\right) \\ \Big(t,(1-t)\overline{x}+tx,(1-t)\overline{y}+ty\Big) & \mapsto & \Big((A,B)^{\overline{x},\overline{y}}(\boldsymbol{\lambda}(t)),(x,y)\Big) \end{array}$$

is a homeomorphism.

Proof. In the game (A,B), the payoff vector for player one against a strategy $(1-t)\cdot\overline{y} + t\cdot y$ for $y \in \Delta_n$ is

$$(1-t)A\overline{y} + tAy = \big((1-t)(A\overline{y},\ldots,A\overline{y}) + tA\big)y$$

where we exploit the fact that $y^T\mathbb{1} = 1$. Similarly the payoff vector for player two against a strategy $(1-t)\cdot\overline{x} + t\cdot x \in \Delta_m$ is

$$\begin{aligned} (1-t)B^T\overline{x} + tB^Tx &= \big((1-t)(B^T\overline{x},\ldots,B^T\overline{x}) + tB^T\big)x \\ &= \left((1-t)\begin{pmatrix}\overline{x}^T\cdot B \\ \vdots \\ \overline{x}^T\cdot B\end{pmatrix} + tB\right)^T x. \end{aligned}$$

Hence $((1-t)\cdot\overline{x}+t\cdot x,(1-t)\cdot\overline{y}+t\cdot y) \in \Delta_t$ is an equilibrium of (A,B) restricted to Δ_t if and only if (x,y) is an equilibrium of $t\cdot(A,B)\oplus(1-t)\cdot\big(A\overline{y},B^T\overline{x}\big)$.

Since the equilibria of a game remain unchanged by multiplication of the payoffs by a positive constant, we get that the set $P_{ET}((A,B),(\overline{x},\overline{y}))$ is given by

$$\{(0,(\overline{x},\overline{y}))\} \quad \cup \quad \left\{\begin{array}{l}(t,(1-t)\cdot\overline{x}+t\cdot x,(1-t)\cdot\overline{y}+t\cdot y) \mid t \in (0,1],(x,y) \text{ is} \\ \text{an equilibrium of the game } (A,B)\oplus(\frac{1}{t}-1)\cdot\big(A\overline{y},B^T\overline{x}\big)\end{array}\right\},$$

which ensures that our map maps indeed to $P_{GNM}((A,B),(\overline{x},\overline{y}))$. Since it is obviously continuous, we just need to find a continuous inverse. This can be easily done by taking the inverse map to $\boldsymbol{\lambda}$ and taking the corresponding continuous map $P_{GNM}((A,B),(\overline{x},\overline{y})) \to P_{ET}((A,B),(\overline{x},\overline{y}))\setminus\{(0,(\overline{x},\overline{y}))\}$, which yields the inverse. $\square$

The map from Theorem 1 can easily be extended to the point $(0,(\overline{x},\overline{y}))$ by taking the one-point-compactification of $P_{GNM}((A,B),(\overline{x},\overline{y}))$. As an immediate consequence we get that a van den Elzen-Talman path as in Theorem 1 is a one-dimensional manifold without branch points if and only if the corresponding (compactified) path of the global Newton method is. If this is the case, both algorithms have non-degenerate paths and will find the same equilibrium.

Theorem 1 can be generalized to N-player games as follows: It has been proved in [15] that the van den Elzen-Talman algorithm implements the linear tracing procedure, which was introduced in [3]. The linear tracing procedure is a method for selecting a Nash equilibrium in an N-player game; it plays a key role in the equilibrium selection theory developed by Harsanyi and Selten [4]. For any prior (i.e. mixed strategy combination), the linear tracing procedure traces equilibria over a set of games whose payoffs are given as a convex combination of the original payoffs and payoffs against the prior. To make this more precise, choose an N-player normal form game $\mathit{\Gamma}$ and a prior p, and denote by $\mathit{\Gamma}_n(\sigma)$ the payoff of player n against a mixed strategy combination σ. For $0 \leq t \leq 1$, define a game $\mathit{\Gamma}^t$, which has the same sets of players and strategies as $\mathit{\Gamma}$, but the payoff in $\mathit{\Gamma}^t$ to player n from a strategy combination σ is defined as

$$\mathit{\Gamma}_n^t(\sigma) = t\mathit{\Gamma}_n(\sigma) + (1-t)\mathit{\Gamma}_n(\sigma_n, p_{-n}) \tag{4}$$

where (σ_n, p_{-n}) is the strategy combination that results from p by replacing player n's strategy p_n by σ_n. The linear tracing procedure traces the graph of the equilibrium correspondence over the set of games $\{\mathit{\Gamma}_t \mid t \in [0,1]\}$, which in almost all cases will be a one-dimensional manifold. For $t > 0$ we can divide the payoffs given in (4) by t without changing the equilibria of the game, and as in the proof of Theorem 1 we can conclude that the global Newton method implements the linear tracing procedure.[1] Since there is nothing new to the line of argument we omit the details.

It has been proved in [2] that the global Newton method also comprises the Lemke-Howson algorithm. Theorem 1 then raises the question of how the latter algorithm as a special case of the global Newton method differs from the van den Elzen-Talman algorithm. If we take the ith unit vector e_i for some pure strategy i of player one, the global Newton method for the ray $(A,B) \oplus \lambda \cdot (e_i, 0)$ corresponds to the Lemke-Howson algorithm with missing label i. An analogous statement holds for missing labels of player two; for further details we refer the reader to [2]. So the Lemke-Howson algorithm corresponds to taking unit vectors as directional vectors for the global Newton method, whereas the van den Elzen-Talman algorithm is based on directional vectors $(A\overline{y}, B^T\overline{x})$. Further differences

[1] We have only outlined the global Newton method for bimatrix games. However, the definition for N-player games works analogously; details can be found in [1].

between the two algorithms will emerge in the analysis of coordination games in the next section: We will see that in this type of game, the Lemke-Howson algorithm only finds the pure strategy equilibria, whereas for certain coordination games, the van den Elzen-Talman method can also find the completely mixed equilibrium.

4 Traceability and the index of equilibria

4.1 Traceability

In this section we would like to discuss which equilibria can be traced by the van den Elzen-Talman algorithm. Of course every equilibrium can be found by taking it as starting point; however we are only interested in those that are found generically. As suggested by Hofbauer [6], we call an equilibrium of a non-degenerate bimatrix game traceable if it can be reached by the van den Elzen-Talman algorithm from an open set of priors. As explained in the introduction, traceability in this sense corresponds to a notion of sustainability suggested by Myerson [9]. Govindan and Wilson [1] state that, generically, every equilibrium found by the global Newton method has index +1. Theorem 1 then implies that only equilibria of index +1 are traceable.

The reverse question is if, generically, every equilibrium of index +1 is traceable. This question has been discussed in [6] in the context of sustainability. We answer it by giving an analysis of coordination games. Following Hofbauer [6], we define a coordination game to be a symmetric game (A, A^T), where the matrix A has 1's on the diagonal and entries strictly smaller than 1 outside the diagonal. Results from [10] imply that a non-degenerate 3×3 coordination game can have up to seven equilibria. However, we are only interested in those games that have exactly seven equilibria, in which case there are three pure strategy equilibria, which have index +1, three equilibria with two strategies as support, which have index -1, and one completely mixed equilibrium, which in turn has index +1. From now on, whenever we use the term coordination game we mean a non-degenerate 3×3 coordination game with seven equilibria.

It is straightforward that in such a game, the Lemke-Howson algorithm only finds the pure strategy equilibria. Those equilibria are traced by the van den Elzen-Talman method as well, by starting from any prior nearby. However, compared to the Lemke-Howson method, the van den Elzen-Talman algorithm allows for a vast variety of starting points. The question is if this increased flexibility suffices to make the completely mixed equilibrium traceable as well. Hofbauer [6] answered this question negatively, which is correct in certain cases, for example when the payoff to each player is given by the identity matrix. However, in the next section, we show that the traceability of the completely mixed equilibrium depends on the type of coordination game at hand. On the one hand, we prove that there are coordination games for which the completely mixed equilibrium can indeed be traced. Hence for this class of games, the van den Elzen-Talman algorithm is stronger than the Lemke-Howson method, in the sense that the equilibria found by the latter method are a proper subset of the traceable equilibria.

On the other hand, we also give a class of coordination games for which the completely mixed equilibrium is not traceable. An immediate consequence of our analysis is that there is an open set in the space of 3×3 games, such that every game in that set has an equilibrium of index $+1$ that is not traceable. This implies that the flexibility of the van den Elzen-Talman algorithm does not ensure generic traceability of all equilibria of index $+1$.

We are not going to give an introduction to the index since the literature on this topic is very rich. The definition goes back to [12]; an overview can be found in [11]. All we need in the course of this paper is the fact that in a coordination game, the completely mixed equilibrium has index $+1$.

4.2 Traceability for coordination games

For the coordination game given by

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = B^T \tag{5}$$

it is easy to see that the completely mixed equilibrium is not traceable. The van den Elzen-Talman paths in this example are quite simple; as soon as a path arrives in the "same" best reply regions for both players, the corresponding pure strategy equilibrium is found, as depicted in Figure 2.

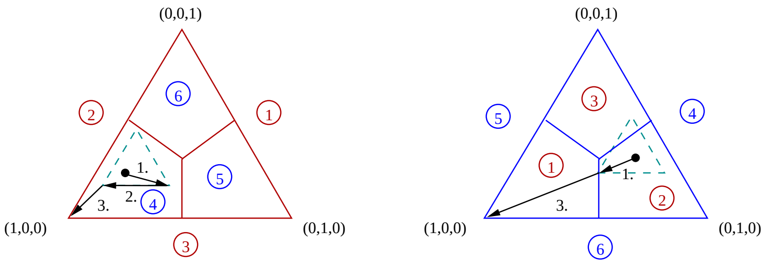

Fig. 2. A van den Elzen-Talman-path for example (5). The black arrows give the path of the algorithm; the dotted triangles trace the value of the restricted strategy space Δ_t.

A very similar, if less degenerate version of the coordination game is given by

$$A = \begin{pmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} = B^T \tag{6}$$

The edges between best reply regions for this game are given by the points $(0,1/3,2/3)$, $(2/3,0,1/3)$ and $(1/3,2/3,0)$, each connected to $(1/3,1/3,1/3)$. Hence those edges are each parallel to one of the edges of the strategy simplex. Again for this game, the completely mixed equilibrium is not traceable.

Consider a (possibly non-symmetric) perturbation of the coordination game (5), such that the perturbed game still has a completely mixed equilibrium. We say that this perturbed game has a nasty best reply structure if the following two conditions hold:

– The completely mixed equilibrium may be anywhere in the strategy space, as long as the first player's ith strategy is still a best reply to the second player's ith strategy, and vice versa.
– We restrict the slopes of the edges between best reply regions: We only allow slopes that are between the slopes in game (5) and game (6). More precisely, connect the completely mixed equilibrium vertically to the edges of the simplex. Then the allowed range of slopes is given by rotating any of those lines counter-clockwise, until the angle between the rotated line and the edge of the simplex becomes 60°. This is illustrated in Figure 3.

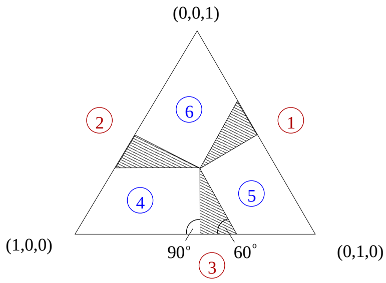

Fig. 3. The shaded areas depict the allowed range for edges between best reply regions in the definition of nasty best reply structures.

Theorem 2. *For any perturbation of game (5) that has a nasty best reply structure, the completely mixed equilibrium is not traceable.*

Proof. We only give the proof for the case where the slopes between best reply regions are strictly between those in game (5) and game (6). The borderline cases can be proved similarly. We have to analyze the different cases that can happen; our proof is best understood by following the different paths geometrically. For illustration we have done this for the last case in Figure 4. Recall that $\mathrm{B}(i)$ denotes the ith best reply region. We can assume without loss of generality that $\overline{x} \in \mathrm{B}(4)$. As always in a coordination game, if $\overline{y} \in \mathrm{B}(1)$, the equilibrium $(1,0,0),(1,0,0)$ is found straight away. Next, let us assume that $\overline{y} \in \mathrm{B}(2)$, and look at the different cases that may happen. The case of $\overline{y} \in \mathrm{B}(3)$ is symmetric, hence there is no need to discuss it.

The first part of the van den Elzen-Talman path is given by $((1-t)\cdot\overline{x} + (0,t,0), (1-t)\cdot\overline{y} + (t,0,0))$. This path is followed until it hits another best reply region.

- If it hits B(5) or B(1) first, the corresponding pure equilibrium is found straight away.
- If it hits B(6) first (this might be possible if the mixed equilibrium is no longer $(1/3, 1/3, 1/3)$), player two starts putting weight on his third strategy until
  (i) the path hits B(3) first. In this case, the homotopy parameter t starts shrinking until either of the upper vertices of the small triangle Δ_t hits the boundary between the relevant best reply regions. Then the other player leaves the corresponding best reply region and walks towards his upper vertex, while the homotopy parameter stalls. In any case, the equilibrium $(0,0,1),(0,0,1)$ is found.
  (ii) the path stays in B(2)[2], until player two plays $(1-t)\cdot\overline{y} + (0,0,t)$. The homotopy parameter starts growing again, until either B(3) or B(5) are hit. In the first case, the equilibrium $(0,0,1),(0,0,1)$ is found, in the second case $(0,1,0),(0,1,0)$.
- The most complex case is when the path hits B(3) first. What happens then is that player one starts putting more weight on his third strategy, while the homotopy parameter t remains constant. Due to the structure of the best reply regions, the path cannot hit B(5) during this process. If the path hits B(6), then the equilibrium $(0,0,1),(0,0,1)$ is found: The homotopy parameter t starts shrinking, until either of the players reaches the upper vertex of the small triangle Δ_t. In either case, the other player can leave the boundary between the best reply regions, and walks towards the upper vertex of his small triangle (while the homotopy parameter stalls). From there $(0,0,1),(0,0,1)$ is found straight away.
  The only remaining case is for the first player's path to remain in B(4) until he reaches the upper vertex $(1-t)\cdot\overline{x} + (0,0,t)$ of his small triangle. Then the homotopy parameter starts growing again until one of the following cases occur:
  (i) If the path hits B(1) first, then the equilibrium $(1,0,0),(1,0,0)$ is found.
  (ii) If the path hits B(6) first, then the equilibrium $(0,0,1),(0,0,1)$ is found.
  (iii) If the path hits B(5) first, then the homotopy parameter stalls while player two puts more weight on his second strategy. At some point he arrives at B(2) again. The homotopy parameter starts shrinking until player one reaches the right vertex of his small triangle.[3] Player two

[2] This is the only other case that can occur; due to the structure of the best reply regions the path cannot hit B(1) first.

[3] This is bound to happen before player two reaches the left vertex of his small triangle: Due to the history of the algorithm we can see that player two's relevant vertex is further away (in terms of the homotopy parameter) from the relevant boundary between best reply regions, than player one's.

can leave B(3), and $(0,1,0),(0,1,0)$ is found. For visualization, we have provided a graphic description of the last case in Figure 4. $\square$

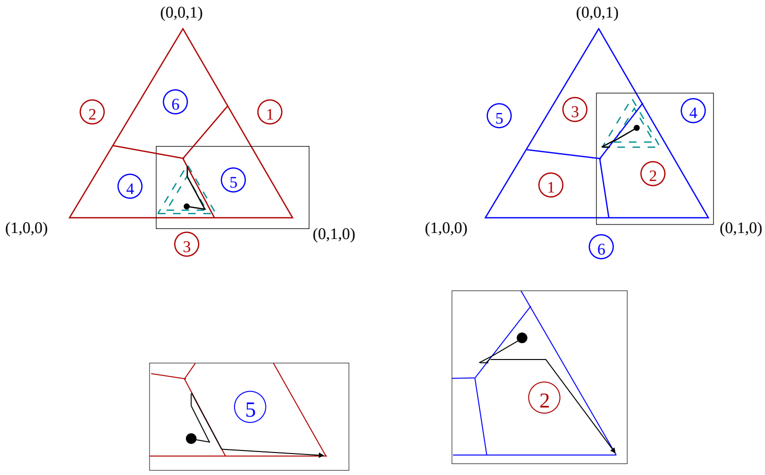

Fig. 4. A van den Elzen-Talman-path as in the proof of Theorem 2. The black arrows give the path of the algorithm, the dotted triangles trace the restricted strategy space Δ_t. The upper figure contains the first four steps of the algorithm, the lower one traces the whole path in greater detail.

If we perturb game (5) slightly towards (6), any game in a neighborhood of that perturbation will have a nasty best reply structure. As an immediate consequence, we get the central result of this section:

Corollary 1. *There is an open set in the space of* 3×3 *bimatrix games, such that every game in that set has an equilibrium of index +1 that is not traceable.*

We would like to conclude this section by proving that the mixed equilibrium of a coordination game is traceable as soon as one of the edges between best reply regions becomes steeper[4] than assumed in Theorem 2. The difference is that where t used to shrink before, now it starts growing, which enables us to find the completely mixed equilibrium. We only give the proof for the following example; however a general proof can easily be derived.

Proposition 1. *Take the coordination game*

$$A = \begin{pmatrix} 1 & 0 & -2 \\ -2 & 1 & 0 \\ 0 & -2 & 1 \end{pmatrix} = B^T$$

[4] By steeper we mean that in Figure 3, the corresponding angle becomes strictly smaller than 60°.

The completely mixed equilibrium in this game is traceable.

Proof. Starting from $\overline{x} = (45/100, 35/100, 20/100), \overline{y} = (15/100, 40/100, 45/100)$ we find the completely mixed equilibrium. Careful inspection of the path shows that the same holds for any prior nearby, which gives us an open set from which the completely mixed equilibrium is reached. □

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