TIME–FREQUENCY ANALYSIS AND PDE'S

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ABSTRACT. We study the action on modulation spaces of Fourier multipliers with symbols $e^{i\mu(\xi)}$, for real-valued functions μ having unbounded second derivatives. We show that if μ satisfies the usual symbol estimates of order $\alpha \geq 2$, or if μ is a positively homogeneous function of degree α, the corresponding Fourier multiplier is bounded as an operator between the weighted modulation spaces $\mathcal{M}^{p,q}_\delta$ and $\mathcal{M}^{p,q}$, for every $1 \leq p, q \leq \infty$ and $\delta \geq d(\alpha-2)|\frac{1}{p}-\frac{1}{2}|$. Here δ represents the loss of derivatives. The above threshold is shown to be sharp for *all* homogeneous functions μ whose Hessian matrix is non-degenerate at some point.

1. INTRODUCTION AND STATEMENT OF THE RESULTS

The results presented here are part of a joint work with Fabio Nicola and Silvia Rivetti [9].

A Fourier multiplier is formally an operator of the type

$$\sigma(D)f(x) = \int_{\mathbb{R}^d} e^{2\pi i x\xi}\sigma(\xi)\hat{f}(\xi)\,d\xi,$$

where $\hat{f}(\xi) = \mathcal{F}f(\xi) = \int_{\mathbb{R}^d} e^{-2\pi i x\xi} f(x)\,dx$ is the Fourier transform. The function σ is called *symbol* of the multiplier. Whereas the action of these operators on $L^2(\mathbb{R}^d)$ is clear (by Parseval's formula), their study in L^p, $p \neq 2$, for several classes of symbols is a fundamental topic in Harmonic Analysis, with important applications to partial differential equations.

In particular, unimodular Fourier multipliers are defined by symbols of the type $\sigma(\xi) = e^{i\mu(\xi)}$, for real-valued functions μ. They arise when solving the Cauchy problem for dispersive equations. For example, for the solution $u(t,x)$ of the Cauchy problem

$$\begin{cases} i\partial_t u + |\Delta|^{\frac{\alpha}{2}} u = 0 \\ u(0,x) = u_0(x), \end{cases} \tag{1}$$

$(t,x) \in \mathbb{R}\times\mathbb{R}^d$, we have the formula $u(t,x) = \left(e^{it|2\pi D|^\alpha} u_0\right)(x)$. The cases $\alpha = 1, 2, 3$ are of particular interest because they correspond to the (half-)wave equation, the Schrödinger equation and (essentially) the Airy equation, respectively.

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Unimodular Fourier multiplier generally do not preserve any Lebesgue space L^p, except for $p = 2$. It is then natural to study boundedness properties on other function spaces arising in Fourier analysis. This was recently done in [1] for the modulation spaces $\mathcal{M}^{p,q}$, $1 \leq p, q \leq \infty$. These spaces were introduced by H. Feichtinger in 1980 (see [7]) and since then have found many applications in Time-frequency analysis, see e.g. Gröchenig's book [8] where the precise definition can be found. Here it suffices to observe that, for heuristic purposes, distributions in $\mathcal{M}^{p,q}$ may be regarded as functions which locally have the same regularity as a function in $\mathcal{F}L^q$ (the space of distributions whose Fourier transform is in L^q), but at infinity decay like a function in L^p.

Now, it was shown in [1], among other things, that symbols of the type $\sigma(\xi) = e^{i|\xi|^\alpha}$, with $0 \leq \alpha \leq 2$, give rise to bounded operators on all $\mathcal{M}^{p,q}$, $1 \leq p, q \leq \infty$. This can be rephrased by saying that the obstruction to the boundedness on L^p is just local in nature. Indeed if we keep the L^p decay but we measure the local regularity by any Fourier-Lebesgue space $\mathcal{F}L^q$ (which is of course preserved by unimodular Fourier multipliers) instead of L^p, boundedness is recaptured. Moreover, the conclusion extends to symbols $\sigma(\xi) = e^{i\mu(\xi)}$ where μ is a positively homogeneous function of degree $\alpha \in [0, 2]$, smooth away from the origin, or even a smooth functions on $\mathbb{R}^d$ whose derivatives of order ≥ 2 are bounded.

More generally, similar results also hold, when $p = q$, for a class of Fourier integral operators whose phases have bounded derivatives of order ≥ 2, see [3, 6]. However for $p \neq q$ a loss of regularity or decay may then occur; see [5] for an analysis of this phenomenon.

Now, we fix the attention on multipliers with symbols $e^{i|\xi|^\alpha}$, with $\alpha > 2$. In this case one still expects boundedness, but with a loss of regularity, namely from $\mathcal{M}^{p,q}_\delta$ to $\mathcal{M}^{p,q}$, for any $\delta \geq \delta(p, q)$ sufficiently large (δ represents the loss of derivatives). Here $\mathcal{M}^{p,q}_\delta = \{f \in \mathcal{S}'(\mathbb{R}^d) : (1 - \Delta)^{\delta/2} f \in \mathcal{M}^{p,q}\}$ is in fact a Sobolev-like space based on $\mathcal{M}^{p,q}$. Since, as we already observed, Fourier-Lebesgue spaces are trivially preserved by unimodular Fourier multipliers, the obstruction to the boundedness on $\mathcal{M}^{p,q}$ should be global in nature. As a consequence, the optimal threshold should depend on p only. In fact in [1, Theorem 16(b)] it was already proved that the multiplier $e^{i|D|^\alpha}$ is bounded from $\mathcal{M}^{p,q}_\delta$ to $\mathcal{M}^{p,q}$ for every $\delta > d\alpha \left|\frac{1}{2} - \frac{1}{p}\right|$. The proof relied on fine classical results about boundedness of wave multipliers on L^p, with loss of derivatives.

The main result we report is a refinement of [1, Theorem 16(b)], with a lower threshold, and can be stated as follows (we refer to [9] for more details and proofs). Let $\langle\xi\rangle = (1 + |\xi|^2)^{1/2}$, for $\xi \in \mathbb{R}^d$.

Theorem 1.1. *Consider a function $\mu \in \mathcal{C}^\infty(\mathbb{R}^d)$, real-valued, satisfying*

$$|\partial^\gamma \mu(\xi)| \leq C_\gamma \langle\xi\rangle^{\alpha-2}, \qquad \forall |\gamma| \geq 2,\ \xi \in \mathbb{R}^d, \tag{2}$$

for some $\alpha \geq 2$. Then the multiplier

$$e^{i\mu(D)}f(x) := \int_{\mathbb{R}^d} e^{2\pi i x\xi} e^{i\mu(\xi)} \hat{f}(\xi) d\xi$$

is bounded as an operator from $\mathcal{M}^{p,q}_\delta$ to $\mathcal{M}^{p,q}$ for

$$\delta \geq d(\alpha - 2) \left| \frac{1}{p} - \frac{1}{2} \right|, \tag{3}$$

and every $1 \leq p, q \leq \infty$.
The same conclusion holds true if $\mu(\xi)$ is smooth for $\xi \neq 0$ only, and positively homogeneous of degree α.

In particular, for $\alpha = 2$, the threshold in (3) vanishes, and we recapture the above result about boundedness without loss of derivatives. Actually, the proof of Theorem 1.1 makes use of the known result for $\alpha = 2$, combined with a Littlewood-Paley decomposition of the frequency domain and the dilation properties of modulation spaces [10].

We also prove that the threshold in (3) is generally sharp. Most interesting, it is sharp for *all* homogeneous functions μ whose Hessian matrix is non-degenerate at some point. This highlights that the unboundedness on $\mathcal{M}^{p,q}$ is due to the presence of some curvature of the graph of μ. Also, this suggests an investigation of the optimal threshold in terms of the number of principal curvatures which are identically zero. More precisely, if at every point the Hessian matrix of μ has rank at most r, we expect the threshold to be $r(\alpha - 2) \left| \frac{1}{p} - \frac{1}{2} \right|$. We plan to study these issues in greater details in future.

Notice that the above negative result shows that the Cauchy problem (1) is not locally wellposed in any $\mathcal{M}^{p,q}$, if $p \neq 2$ and $\alpha > 2$. For positive results in this connection we refer to [1, 2, 4, 11] and the references therein.

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