Dynamic Context Logic and its Application to Norm Change

Guillaume Aucher[1], Davide Grossi[2], Andreas Herzig[3], Emiliano Lorini[3]
[1]University of Luxembourg
[2]University of Amsterdam
[3]Institut de Recherche en Informatique de Toulouse, France

Abstract

Building on a simple modal logic of context, the paper presents a dynamic logic characterizing operations of contraction and expansion on theories. We investigate the mathematical properties of the logic, and use it to develop an axiomatic and semantic analysis of norm change in normative systems. The proposed analysis advances the state of the art by providing a formal semantics of norm-change which, at the same time, takes into account several different aspects of the phenomenon, such as permission and obligation dynamics, as well as the dynamics of classificatory rules.

1 Introduction

Normative systems [4] have become a valuable abstraction for the design of multi-agent systems, and logic-based studies of norms have obtained increasing attention, in particular for their usefulness in providing computational models of norm-based interaction grounded on logical semantics (e.g. [1]). Taking up on pioneering work such as [3], the topic of how norms change over time has also become a topic of interest (e.g. [8]) given its relevance for understanding the ways social interaction evolves within multi-agent systems.

The aim of this work is to study norm change as a special instance of context change. Following [9] normative systems are, in a nutshell, logical theories concerning complex ways of classifying states of affairs as legal or illegal. As a consequence, each normative system specifies a context with respect to which rules of classification hold. Once such perspective is assumed, existing formal accounts of belief and knowledge dynamics can be transferred to study context change and, thus, norm change. We focus on dynamic epistemic logic (DEL) [17], and study two specific context change operations which can successfully account for norm change:

- **Context expansion** acounting for **norm promulgation**,
- **Context contraction** accounting for **norm derogation**.

Dagstuhl Seminar Proceedings 09121
Normative Multi-Agent Systems
http://drops.dagstuhl.de/opus/volltexte/2009/1900

In both cases, it is assumed that the authority of a normative system makes a proclamation in such a way that the norms of the normative system are modified. In the former case, the authority proclaims that from now on "a certain fact φ implies a violation", expanding the current set of obligations of the normative system. For example, the authority of a normative system might proclaim that from now on "driving faster than 110 km/h on a highway implies a violation". After this norm promulgation, it is obligatory to drive at most 110 km/h. In the latter case, the authority proclaims that "a certain fact φ does not imply a violation", contracting the current set of obligations of the normative system (and consequently making the normative system more 'permissive'). For example, the authority of a country might proclaim that from now on 'encrypting email does not imply a violation' by derogating the previous norm which forbade encryption in written communication. After this proclamation, it is permitted to encrypt email.

We start from the modal logic presented in [9]. This logic is based on a set of modal operators $[X]$ where X is a label denoting the context of a theory, i.e., in our case, the context of a normative system. A formula $[X]\varphi$ reads 'in the context of normative system X it is the case that φ'. Our aim in this paper is to extend this logic with two special kinds of events of the form $X{+}\psi$ and $X{-}\psi$, and corresponding modal operators $[X{+}\psi]$ and $[X{-}\psi]$. The former are similar to the operators for announcement studied in DEL [17]. Their function is to restrict the space of possible worlds accepted by the normative system X to the worlds where ψ is true. We use these operators to model norm promulgation. The function of modal operators of type $[X{-}\psi]$ is to add to the space of possible worlds accepted by the normative system X some worlds in which ψ is false. We use them to model norm derogation.

The paper is organized as follows. In Section 2 we will briefly present the modal logic of context of [9]. Section 3 is devoted to extend this logic with the two events $X{+}\psi$ and $X{-}\psi$ which allow to model context dynamics. Finally, in Section 4, we will apply our logical framework to norm change, *i.e.*norm promulgation and norm derogation.

2 A modal logic of context

The logic presented in this section is a simple modal logic designed to represent and reason about a localized notion of validity, that is, of validity with respect to all models in a given set. Such a given set is what is here called a *context*, in accord with much literature in artificial intelligence and linguistics on context theory (see, for instance, [14, 7]).

Let $\Phi = \{p, q, \ldots\}$ be a countable non-empty set of propositional letters, and let $\mathcal{C} = \{X, Y, \ldots\}$ be a countable set of contexts. $\mathcal{L}_{Prop}$ denotes the propositional language.

2.1 Models

Definition 1. *A context model ($\mathbf{Cxt}$-model) $\mathcal{M} = (W, R, \mathcal{I})$ is a tuple such that:*

- *W is a nonempty set of possible worlds;*

- $R : \mathcal{C} \longrightarrow 2^W$ *maps each context* X *to a subset of* W*;*
- $\mathcal{I} : \Phi \longrightarrow 2^W$ *is a valuation.*

We write R_X *for* $R(X)$ *and* $w \in \mathcal{M}$ *for* $w \in W$*. For* $w \in \mathcal{M}$*, the couple* $(\mathcal{M}, w)$ *is a pointed context model.*

A $\mathbf{Cxt}$-model represents a logical space together with some of its possible restrictions, i.e., the contexts. In our case, contexts are used to represent the restrictions to those sets of propositional models satisfying the rules stated by a given normative system [9]. Let us illustrate how they can be used to model normative systems.

Example 1. *Consider a normative system according to which: motorized vehicles must have a numberplate ; motorized vehicles must have an insurance; bikes should not have an insurance; bikes are classified as not being a motorized vehicle. Once a designated atom* V *is introduced in the language, which represents a notion of "violation" [5], the statements above obtain a simple representation:*

Rule 1: $(mt \wedge \neg pl) \rightarrow \mathsf{V}$

Rule 2: $(mt \wedge \neg in) \rightarrow \mathsf{V}$

Rule 3: $(bk \wedge in) \rightarrow \mathsf{V}$

Rule 4: $bk \rightarrow \neg mt$

A $\mathbf{Cxt}$*-model* $\mathcal{M} = (W, R, \mathcal{I})$ *where* $\mathcal{I}$ *maps atoms* mt*,* pl*,* in*,* bk *and* V *to subsets of* W *models the normative system above as a context* X *if* R_X *coincides with the subset of* W *where Rules 1-4 are true according to propositional logic.*

2.2 Logic

The logic $\mathbf{Cxt}$ is now presented which captures the notion of validity with respect to a context, thereby allowing to represent situations such as Example 1 in our language. To talk about $\mathbf{Cxt}$-models we use a modal language $\mathcal{L}_{\mathbf{Cxt}}$ containing modal operators $[X]$ for every $X \in \mathcal{C}$, plus the universal modal operator $[\mathsf{U}]$. The set of well-formed formulae of $\mathcal{L}_{\mathbf{Cxt}}$ is defined by the following BNF:

$$\mathcal{L}_{\mathbf{Cxt}} : \varphi ::= p \mid \neg\varphi \mid \varphi \wedge \varphi \mid [\mathsf{U}]\varphi \mid [X]\varphi$$

where p ranges over Φ and X over $\mathcal{C}$. The Boolean connectives $\top, \vee, \rightarrow, \leftrightarrow$ and the dual operators $\langle X \rangle$ are defined as usual within $\mathcal{L}_{\mathbf{Cxt}}$ as: $\langle X \rangle\varphi = \neg[X]\neg\varphi$, for $X \in \mathcal{C} \cup \{\mathsf{U}\}$.

We interpret formulas of $\mathcal{L}_{\mathbf{Cxt}}$ in a $\mathbf{Cxt}$-models as follows: the $[\mathsf{U}]$ operator is interpreted as the universal modality [6], and the $[X]$ operators model a restricted notion of validity.

Definition 2. *Let* $\mathcal{M}$ *be a* $\mathbf{Cxt}$*-model, and let* $w \in \mathcal{M}$*.*

$\mathcal{M}, w \models [X]\varphi$ *iff for all* $w' \in R_X$, $\mathcal{M}, w' \models \varphi$;
$\mathcal{M}, w \models [\mathsf{U}]\varphi$ *iff for all* $w' \in W$, $\mathcal{M}, w' \models \varphi$;
$\mathcal{M}, w \models p$ *iff* $w \in \mathcal{I}(p)$.

and as usual for the Boolean operators. Formula φ *is valid in* $\mathcal{M}$*, noted* $\mathcal{M} \models \varphi$*, iff* $\mathcal{M}, w \models \varphi$ *for all* $w \in \mathcal{M}$*.* φ *is* $\mathbf{Cxt}$*-valid, noted* $\models_{\mathbf{Cxt}} \varphi$*, iff* $\mathcal{M} \models \varphi$ *for all* $\mathbf{Cxt}$*-models* $\mathcal{M}$*.*

$\mathbf{Cxt}$-validity is axiomatized by the following schemas:

(P)	all propositional axiom schemas and rules
(4^{XY})	$[X]\varphi \to [Y][X]\varphi$
(5^{XY})	$\langle X\rangle\varphi \to [Y]\langle X\rangle\varphi$
$(\mathtt{T}^{\mathsf{U}})$	$[\mathsf{U}]\varphi \to \varphi$
$(\mathtt{K}^{X})$	$[X](\varphi \to \varphi') \to ([X]\varphi \to [X]\varphi')$
$(\mathtt{N}^{X})$	IF $\vdash \varphi$ THEN $\vdash [X]\varphi$

where $X, Y \in \mathcal{C} \cup \{\mathsf{U}\}$. The $[X]$ and $[Y]$ operators are $\mathbf{K45}$ modalities strengthened with the two inter-contextual interaction axioms 4^{XY} and 5^{XY}. $[\mathsf{U}]$ is an $\mathbf{S5}$ modality. Provability of a formula φ, noted $\vdash_{\mathbf{Cxt}} \varphi$, is defined as usual.

Logic $\mathbf{Cxt}$ is well-behaved from the point of view of both axiomatizability and complexity.

Theorem 1 ([9])**.** $\models_{\mathbf{Cxt}} \varphi$ *iff* $\vdash_{\mathbf{Cxt}} \varphi$.

Theorem 2. *Deciding* $\mathbf{Cxt}$*-validity is coNP-complete.*

Sketch of proof. Satisfiability of **S5** formulas is decidable in nondeterministic polynomial time [6]. Let $\mathcal{L}^{[\mathsf{U}]}$ be the language built from the set of atoms $\Phi \cup \mathcal{C}$ (supposing Φ and $\mathcal{C}$ are disjoint) and containing only one modal operator $[\mathsf{U}]$. That is:

$$\mathcal{L}^{[\mathsf{U}]} : \varphi ::= p \mid \neg\varphi \mid \varphi \wedge \varphi \mid [\mathsf{U}]\varphi$$

where p ranges over $\Phi\cup\mathcal{C}$. It gets a natural interpretation on context models where $[\mathsf{U}]$ is the global modality. Then one can show that the following is a satisfiability-preserving polytime reduction f of $\mathcal{L}_{\mathbf{Cxt}}$ to $\mathcal{L}^{[\mathsf{U}]}$: $f(p) = p$; $f(\neg\varphi) = \neg f(\varphi)$; $f(\varphi \wedge \varphi') = f(\varphi) \wedge f(\varphi')$; $f([\mathsf{U}]\varphi) = [\mathsf{U}]f(\varphi)$; $f([X]\varphi) = [\mathsf{U}](X \to f(\varphi))$. □

The same argument can be used to prove linear time complexity if the alphabet Φ is finite.

Another interesting property of $\mathbf{Cxt}$ is that every formula of $\mathcal{L}_{\mathbf{Cxt}}$ is provably equivalent to a formula without nested modalities, as the following proposition shows. We first formally define the language without nested modalities:

$$\mathcal{L}^1_{\mathbf{Cxt}} : \varphi ::= \alpha \mid [X]\alpha \mid [\mathsf{U}]\alpha \mid \neg\varphi \mid \varphi \wedge \varphi$$

where α ranges over $\mathcal{L}_{Prop}$ and X over $\mathcal{C}$.

Proposition 1. *For all* $\varphi \in \mathcal{L}_{\mathbf{Cxt}}$ *there is* $\varphi^1 \in \mathcal{L}^1_{\mathbf{Cxt}}$ *such that* $\vdash_{\mathbf{Cxt}} \varphi \leftrightarrow \varphi^1$.

Proof. By induction on φ. The Boolean cases clearly work. If φ is of the form $[X]\psi$ with $X \in \mathcal{C} \cup \{\mathsf{U}\}$ then by IH there are $\alpha_k, \alpha^i_j, \beta^i \in \mathcal{L}_{Prop}$ such that

$$\varphi \leftrightarrow [X] \bigwedge_{k \in \mathbb{N}_l} (\alpha_k \vee \bigvee_{i \in \mathbb{N}_{n_k}} ([X_i]\alpha^i_1 \vee \ldots \vee [X_i]\alpha^i_{n_i} \vee \langle X_i \rangle \beta^i))).$$

However, using (4^{XY}) and (5^{XY}), one can easily show that

$$\vdash_{\mathbf{Cxt}} [X](\alpha_k \vee \bigvee_{i \in \mathbb{N}_{n_k}} ([X_i]\alpha^i_1 \vee \ldots \vee [X_i]\alpha^i_{n_i} \vee \langle X_i \rangle \beta^i))) \leftrightarrow ([X]\alpha_k \vee \bigvee_{i \in \mathbb{N}_{n_k}} ([X_i]\alpha^i_1 \vee \ldots \vee [X_i]\alpha^i_{n_i} \vee \langle X_i \rangle \beta^i))).$$

□

We will use this result in the completeness proof of the dynamic extension of $\mathbf{Cxt}$ (Proposition 3).

2.3 Normative systems in $\mathbf{Cxt}$

We are ready to provide an object-level representation of Example 1. The contextual operators $[X]$ and the universal operator $[\mathsf{U}]$ can be used to define the concepts of *classificatory rule*, *obligation* and *permission* which are needed to model normative systems. Classificatory rules are of the form "φ counts as ψ in the normative system X" and their function in a normative systems is to specify classifications between different concepts [12]. For example, according to the classificatory rule "in the context of Europe, a piece of paper with a certain shape, color, *etc.*counts as a 5 Euro bill", in Europe a piece of paper with a certain shape, color, etc. should be classified as a 5 Euro bill. The concept of classificatory rule is expressed by the following abbreviation:

$$\varphi \Rightarrow_X \psi \stackrel{def}{=} [X](\varphi \rightarrow \psi)$$

where $\varphi \Rightarrow_X \psi$ reads 'φ counts as ψ in normative system X'. As done already in Example 1, by introducing the violation atom V we can obtain a reduction of deontic logic to logic $\mathbf{Cxt}$ along the lines first explored by Anderson [5]. As far as obligations are concerned, we introduce operators of the form $\mathbf{O}_X$ which are used to specify what is obligatory in the context of a certain normative system X:

$$\mathbf{O}_X\varphi \stackrel{def}{=} \neg\varphi \Rightarrow_X \mathsf{V}$$

According to this definition, 'φ is obligatory within context X' is identified with '$\neg\varphi$ counts as a violation in normative system X'. Note that we have the following $\mathbf{Cxt}$-theorem:

(1) $$\vdash_{\mathbf{Cxt}} ((\varphi \Rightarrow_X \psi) \wedge (\varphi \Rightarrow_X \neg\psi)) \rightarrow \mathbf{O}_X\neg\varphi$$

This will be of use in Section 4. Every $\mathbf{O}_X$ obeys axiom K and necessitation, and is therefore a normal modal operator.

(2) $$\vdash_{\mathbf{Cxt}} \quad \mathbf{O}_X(\varphi \rightarrow \psi) \rightarrow (\mathbf{O}_X\varphi \rightarrow \mathbf{O}_X\psi)$$

(3) $$\text{IF } \vdash_{\mathbf{Cxt}} \varphi \text{ THEN } \vdash_{\mathbf{Cxt}} \mathbf{O}_X\varphi$$

Note that the formula $\mathbf{O}_X\bot$ is consistent, hence our deontic operator does not satisfy the D axiom.

We define the permission operator in the standard way as the dual of the obligation operator: "φ is permitted within context X", noted $\mathbf{P}_X\varphi$. Formally:

$$\mathbf{P}_X\varphi \stackrel{def}{=} \neg\mathbf{O}_X\neg\varphi$$

$\mathbf{P}_{\mathsf{U}}\varphi$ should be read "φ is is deontically possible".

Example 2. *Consider again the normative system of Example 1. We can now express in* $\mathbf{Cxt}$ *that Rules 1-4 explicitly belong to context* X*:*

Rule 1′: $\mathbf{O}_X(mt \rightarrow pl)$

Rule 2′: $\mathbf{O}_X(mt \rightarrow in)$

Rule 3′: $\mathbf{O}_X(bk \rightarrow \neg in)$

Rule 4′: $bk \Rightarrow_X \neg mt$

Rules 1′-4′ explicitly localize the validity of Rules 1-4 of Example 1 to context X*. Logic* $\mathbf{Cxt}$ *is therefore enough expressive to represent several (possibly inconsistent) normative systems at the same time.*

The context representations enabled by $\mathbf{Cxt}$ are inherently static. The next section investigates context dynamics.

3 Dynamic context logic

3.1 Two relations on models

We first define the relations $\stackrel{X+\psi}{\longrightarrow}$ and $\stackrel{X-\psi}{\longrightarrow}$ on the set of pointed $\mathbf{Cxt}$-models.

Definition 3. *Let* $(\mathcal{M}, w) = (W, R, \mathcal{I}, w)$ *and* $(\mathcal{M}', w') = (W', R', \mathcal{I}', w')$ *be two pointed* $\mathbf{Cxt}$*-models, and let* $\varphi \in \mathcal{L}_{\mathbf{Cxt}}$ *and* $X \in \mathcal{C}$*.*
We set $(\mathcal{M}, w) \stackrel{X+\psi}{\longrightarrow} (\mathcal{M}', w')$ *iff* $W = W', w = w', \mathcal{I} = \mathcal{I}'$*, and*

- $R'_Y = R_Y$ *if* $Y \neq X$*;*
- $R'_X = R_X \cap ||\psi||_{\mathcal{M}}$.

We set $(\mathcal{M}, w) \stackrel{X-\psi}{\longrightarrow} (\mathcal{M}', w')$ *iff* $W = W', w = w', \mathcal{I} = \mathcal{I}'$*, and*

- $R'_Y = R_Y$ *if* $Y \neq X$*;*
- $R'_X = \begin{cases} R_X \text{ if } \mathcal{M}, w \models \neg[X]\psi \vee [\mathsf{U}]\psi \\ R_X \cup S \text{ otherwise, for some } \emptyset \neq S \subseteq ||\psi||_{\mathcal{M}} \end{cases}$

In case $(\mathcal{M}, w) \stackrel{X+\psi}{\longrightarrow} (\mathcal{M}', w')$ *(resp.* $(\mathcal{M}, w) \stackrel{X-\psi}{\longrightarrow} (\mathcal{M}', w')$*), we say that* $\mathcal{M}'$ *is a* (context) expansion *(resp.* contraction*) of* $\mathcal{M}$*.*

In the above definition, $||\psi||_{\mathcal{M}} = \{w \in \mathcal{M} : \mathcal{M}, w \models \psi\}$. So in both cases, it is only the context X which changes from $\mathcal{M}$ to $\mathcal{M}'$. In the first case, it is restricted to the worlds that satisfy ψ, and in the second case, it is enlarged with some worlds which satisfy $\neg\psi$, except if such worlds do not exist in the model ($[\mathsf{U}]\psi$) or if $\neg\varphi$ is already consistent with the context ($\neg[X]\psi$). Note that there might be several contractions of a given $\mathbf{Cxt}$-model but there is always a unique expansion. The relation $\stackrel{X-\psi}{\longrightarrow}$ thus defines implicitly a *family* of contraction operations. The following proposition shows that $\stackrel{X-\psi}{\longrightarrow}$ is essentially the converse relation of $\stackrel{X+\psi}{\longrightarrow}$.

Proposition 2. *Let $(\mathcal{M}, w)$ and $(\mathcal{M}', w')$ be two pointed $\mathbf{Cxt}$-models and $\psi \in \mathcal{L}_{\mathbf{Cxt}}$. Then $(\mathcal{M}, w) \stackrel{X+\psi}{\longrightarrow} (\mathcal{M}', w')$ iff*

$$(\mathcal{M}', w') \stackrel{X-\psi}{\longrightarrow} (\mathcal{M}, w) \text{ and } \mathcal{M}', w' \models [X]\psi.$$

3.2 Logic

The language of the logic $\mathbf{DCxt}$ is obtained by adding the dynamic operators $[X{+}\psi]$ and $[X{-}\psi]$ to the language $\mathcal{L}_{\mathbf{Cxt}}$:

$$\mathcal{L}_{\mathbf{DCxt}} : \varphi ::= p \mid \neg\varphi \mid \varphi \wedge \varphi \mid [X]\varphi \mid [\mathsf{U}]\varphi \mid [X{+}\psi]\varphi \mid [X{-}\psi]\varphi$$

where p ranges over Φ, X over $\mathcal{C}$ and ψ over $\mathcal{L}_{\mathbf{Cxt}}$. $[X{+}\psi]\varphi$ reads 'after the expansion of the context X by ψ, φ is true', and $[X{-}\psi]\varphi$ reads 'after *any* contraction of the context X by ψ, φ is true'.

Definition 4. *Let $\mathcal{M}$ be a $\mathbf{Cxt}$-model. The truth conditions for $\mathcal{L}_{\mathbf{DCxt}}$ in $\mathcal{M}$ are those of Definition 2, plus:*

$\mathcal{M}, w \models [X{+}\psi]\varphi$ *iff* $\mathcal{M}', w' \models \varphi$ *for all* $\mathbf{Cxt}$*-models* $(\mathcal{M}', w')$ *such that* $(\mathcal{M}, w) \stackrel{X+\psi}{\longrightarrow} (\mathcal{M}', w')$*;*
$\mathcal{M}, w \models [X{-}\psi]\varphi$ *iff* $\mathcal{M}', w' \models \varphi$ *for all* $\mathbf{Cxt}$*-models* $(\mathcal{M}', w')$ *such that* $(\mathcal{M}, w) \stackrel{X-\psi}{\longrightarrow} (\mathcal{M}', w')$.

As before, $\mathcal{M} \models \varphi$ iff $\mathcal{M}, w \models \varphi$ for all $w \in \mathcal{M}$, and φ is $\mathbf{DCxt}$-valid ($\models_{\mathbf{DCxt}} \varphi$) iff $\mathcal{M} \models \varphi$ for all $\mathbf{Cxt}$-models $\mathcal{M}$.

The operator $[X{-}\psi]$ is thus useful if we want to have general properties about our family of contractions or about a situation; for example, given some formulas $\psi_1, \ldots, \psi_n$, what would be true after any sequence of contractions and expansions by these formulas? Can we get an inconsistency with a specific choice of contractions?

In order to axiomatize the $\mathbf{DCxt}$-validities we define for every $X \in \mathcal{C}$ two auxiliary languages $\mathcal{L}_{\neq X}$ and $\mathcal{L}_{=X}$:

$$\mathcal{L}_{=X} : \varphi ::= [X]\alpha \mid \neg\varphi \mid \varphi \wedge \varphi$$
$$\mathcal{L}_{\neq X} : \varphi ::= \alpha \mid [Y]\alpha \mid \neg\varphi \mid \varphi \wedge \varphi$$

where α ranges over $\mathcal{L}_{Prop}$ and Y over $(\mathcal{C} \cup \{\mathsf{U}\}) - \{X\}$.

Logic $\mathbf{DCxt}$ is axiomatized by the following schemata:

(Cxt)	All axiom schemas and inference rules of $\mathbf{Cxt}$
(R+1)	$[X{+}\psi]\varphi_{\neq X} \leftrightarrow \varphi_{\neq X}$
(R+2)	$[X{+}\psi][X]\alpha \leftrightarrow [X](\psi \rightarrow \alpha)$
(R+3)	$[X{+}\psi]\neg\varphi \leftrightarrow \neg[X{+}\psi]\varphi$
(R–1)	$[X{-}\psi](\varphi_{\neq X} \vee \varphi_{=X}) \leftrightarrow (\varphi_{\neq X} \vee [X{-}\psi]\varphi_X)$
(R–2)	$\neg[X{-}\psi]\bot$
(R–3)	$[X{-}\psi]([X]\alpha_1 \vee \ldots \vee [X]\alpha_n \vee \langle X\rangle\alpha) \leftrightarrow$ $((\neg[X]\psi \vee [\mathsf{U}]\psi) \wedge ([X]\alpha_1 \vee \ldots \vee [X]\alpha_n \vee \langle X\rangle\alpha))$ $\vee (([X]\psi \wedge \neg[\mathsf{U}]\psi) \wedge$ $((\bigvee_i([X]\alpha_i \wedge [\mathsf{U}](\psi \vee \alpha_i))) \vee \langle X\rangle\alpha \vee [\mathsf{U}](\psi \vee \alpha)))$
(K^+)	$[X{+}\psi](\varphi \rightarrow \varphi') \rightarrow ([X{+}\psi]\varphi \rightarrow [X{+}\psi]\varphi')$
(K^-)	$[X{-}\psi](\varphi \rightarrow \varphi') \rightarrow ([X{-}\psi]\varphi \rightarrow [X{-}\psi]\varphi')$
(RRE)	Rule of replacement of proved equivalence

where $X \in \mathcal{C}$, $\varphi, \varphi' \in \mathcal{L}_{\mathbf{DCxt}}$, $\psi \in \mathcal{L}_{\mathbf{Cxt}}$, $\varphi_{=X} \in \mathcal{L}_{=X}$, $\varphi_{\neq X} \in \mathcal{L}_{\neq X}$, and $\alpha, \alpha_i \ldots \in \mathcal{L}_{Prop}$.

Note that from (R–1) and (R–2) one can deduce $[X{-}\psi]\varphi_{\neq X} \leftrightarrow \varphi_{\neq X}$. The above are reduction axioms:

Proposition 3. *For all $\varphi_{\mathbf{DCxt}} \in \mathcal{L}_{\mathbf{DCxt}}$ there is $\varphi_{\mathbf{Cxt}} \in \mathcal{L}_{\mathbf{Cxt}}$ such that $\vdash_{\mathbf{DCxt}} \varphi_{\mathbf{DCxt}} \leftrightarrow \varphi_{\mathbf{Cxt}}$.*

Sketch of proof. (By induction on the number of occurrences of dynamic operators.) Let $\varphi_{\mathbf{DCxt}} \in \mathcal{L}_{\mathbf{DCxt}}$ and $\varphi'_{\mathbf{DCxt}}$ be one of its sub-formulas of the form $[X{+}\psi]\varphi_{\mathbf{Cxt}}$ or $[X{-}\psi]\varphi_{\mathbf{Cxt}}$, with $\varphi_{\mathbf{Cxt}} \in \mathcal{L}_{\mathbf{Cxt}}$. By Proposition 1, there is $\varphi^1_{\mathbf{Cxt}} \in \mathcal{L}^1_{\mathbf{Cxt}}$ such that $\vdash_{\mathbf{Cxt}} \varphi_{\mathbf{Cxt}} \leftrightarrow \varphi^1_{\mathbf{Cxt}}$. So $\vdash_{\mathbf{DCxt}} [X{+}\psi]\varphi_{\mathbf{Cxt}} \leftrightarrow [X{+}\psi]\varphi^1_{\mathbf{Cxt}}$ by (REE) and (K^+). Now, thanks to axioms (R+1), (R+2) and (R+3) and because $\varphi^1_{\mathbf{Cxt}} \in \mathcal{L}^1_{\mathbf{Cxt}}$, one can easily show that there is $\psi_{\mathbf{Cxt}} \in \mathcal{L}_{\mathbf{Cxt}}$ such that $\vdash_{\mathbf{DCxt}} [X{+}\psi]\varphi^1_{\mathbf{Cxt}} \leftrightarrow \psi_{\mathbf{Cxt}}$. For the case $[X{-}\psi]\varphi_{\mathbf{Cxt}}$ we apply the same method using (R–1), (R–2) and (R–3). So $\vdash_{\mathbf{DCxt}} \varphi'_{\mathbf{DCxt}} \leftrightarrow \psi_{\mathbf{Cxt}}$. Now we replace $\varphi'_{\mathbf{DCxt}}$ by $\psi_{\mathbf{Cxt}}$ in $\varphi_{\mathbf{DCxt}}$. This yields an equivalent formula (thanks to (RRE)) with one dynamic operator less. We then apply to this formula the same process we applied to $\varphi_{\mathbf{Cxt}}$ until we get rid of all the dynamic operators. □

For example, $\vdash_{\mathbf{DCxt}} [X{-}\alpha]\neg[X]\alpha \leftrightarrow \langle\mathsf{U}\rangle\neg\alpha$. As in DEL, soundness and completeness follow from Proposition 3:

Theorem 3. $\models_{\mathbf{DCxt}} \varphi$ *iff* $\vdash_{\mathbf{DCxt}} \varphi$.

Theorem 4. *Deciding* $\mathbf{DCxt}$*-validity is decidable.*

Finally, we could perfectly enrich this formalism with specific contraction operators. For example we could add to $\mathcal{L}_{\mathbf{DCxt}}$ the contraction operator $[X \mathbin{\dot{-}} \psi]\varphi$ whose semantics would be defined as follows: for $\mathcal{M} = (W, R, \mathcal{I})$, $\mathcal{M}, w \models [X \mathbin{\dot{-}} \psi]\varphi$

iff $\mathcal{M}', w \models \varphi$, where $\mathcal{M}' = (W, R', \mathcal{I})$ with $R'_Y = R_Y$ for $Y \neq X$ and $R'_X = R_X \cup \{w \in W \mid \mathcal{M}, w \models \neg\psi\}$. To get a complete axiomatization, we just have to add to $\mathbf{DCxt}$ the following axiom schemas: (1) $[X \doteq \psi]\varphi_{\neq X} \leftrightarrow \varphi_{\neq X}$; (2) $[X \doteq \psi]\neg\varphi \leftrightarrow \neg[X \doteq \psi]\varphi$; (3) $[X \doteq \psi][X]\alpha \leftrightarrow [X]\alpha \wedge [\mathsf{U}](\neg\psi \to \alpha)$; and the distribution axiom ($\mathrm{K}^{\doteq}$). In fact this contraction $\doteq$ belongs to the family of contractions defined in Definition 3, and so we have $\vdash_{\mathbf{DCxt}} [X{-}\psi]\varphi \to [X \doteq \psi]\varphi$.

4 A logical account of norm change

Just as we defined the static notions of obligation and classificatory rules on the basis of $\mathbf{Cxt}$, we can in the same spirit define the dynamic notions of promulgation and derogation of obligation and classificatory rules on the basis of $\mathbf{DCxt}$:

$$
\begin{aligned}
+(\varphi \Rightarrow_X \psi) &\stackrel{def}{=} X{+}(\varphi \to \psi) \\
+\mathbf{O}_X\psi &\stackrel{def}{=} X{+}(\neg\psi \to \mathsf{V}) \\
-(\varphi \Rightarrow_X \psi) &\stackrel{def}{=} X{-}(\varphi \to \psi) \\
-\mathbf{O}_X\psi &\stackrel{def}{=} X{-}(\neg\psi \to \mathsf{V})
\end{aligned}
$$

$[+(\varphi \Rightarrow_X \psi)]\chi$ (resp. $[-(\varphi \Rightarrow_X \psi)]\chi$) should be read 'after the promulgation (resp. after *any* derogation) of the classificatory rule $\varphi \Rightarrow_X \psi$, χ is true'. Likewise, $[+\mathbf{O}_X\psi]\varphi$ (resp. $[-\mathbf{O}_X\psi]\varphi$) should be read 'after the promulgation (resp. after *any* derogation) within context X of the obligation ψ, χ is true'. Then we have the following intuitive $\mathbf{DCxt}$-theorems:

(4) $\vdash_{\mathbf{DCxt}} [+(\varphi \Rightarrow_X \psi)]\varphi \Rightarrow_X \psi$

(5) $\vdash_{\mathbf{DCxt}} [+\mathbf{O}_X\psi]\mathbf{O}_X\psi$

(6) $\vdash_{\mathbf{DCxt}} \mathbf{P}_{\mathsf{U}}\neg\psi \to [-\mathbf{O}_X\psi]\mathbf{P}_X\neg\psi$

In particular, $\mathbf{DCxt}$-theorem (6) says that "If $\neg\psi$ is deontically possible then after any derogation within context X of the obligation ψ, $\neg\psi$ is permitted".

Example 3. *In Example 2, after the legislator's proclamation that motorized vehicles having more than 50cc (mf) are obliged to have a numberplate (event $+\mathbf{O}_X((mt \wedge mf) \to pl)$ and that motorized vehicles having less than 50cc ($\neg mf$) are not obliged to have a numberplate (event $-\mathbf{O}_X((mt \wedge \neg mf) \to pl)$ we should expect that motorbikes having more than 50cc have the obligation to have a numberplate and motorbikes having less than 50cc have the permission not to have a numberplate. This is indeed the case:*

$$
\begin{gathered}
\vdash_{\mathbf{DCxt}} \mathbf{P}_{\mathsf{U}}(mt \wedge \neg mf \wedge \neg pl) \to ([+\mathbf{O}_X((mt \wedge mf) \to pl)] \\
[-\mathbf{O}_X((mt \wedge \neg mf) \to pl)]\mathbf{O}_X((mt \wedge mf) \to pl) \wedge \\
\mathbf{P}_X(mt \wedge \neg mf \wedge \neg pl)).
\end{gathered}
$$

We now consider two types of normative inconsistency, classificatory dilemma and normative dilemma, and show how they might arise from promulgation and derogation.

Classificatory dilemna By classificatory dilemma we mean that a certain fact φ is classified by a normative system both under ψ and under $\neg\psi$, i.e. $(\varphi \Rightarrow_X \psi) \wedge (\varphi \Rightarrow_X \neg\psi)$. An example of classificatory dilemma is the case of someone who finds an object in the sea and is classified by the normative system as the owner of the object. At the same time, someone who claims having lost the object and can prove this, is also classified as the owner of the object. Finally, according to the normative system, there is no more than one owner of an object. If a person finds an object in the sea and another person claims that she has lost this object and can prove that, we incur a classificatory dilemma: the former person is classified as the owner of the object and, at the same time, she is classified as not being the owner of it.

Example 4. *In Example 2, after the legislator's proclamation that bikes with an engine must be classified as a motorized vehicles (event* $+\!\!\!|((bk \wedge en) \Rightarrow_X mt)$*), bikes with an engine are classified as motorized vehicles and, at the same time, they are classified as not being motorized vehicles. This is a classificatory dilemma:*

$$[+\!\!\!|((bk \wedge en) \Rightarrow_X mt)](((bk \wedge en) \Rightarrow_X mt) \wedge$$

$$((bk \wedge en) \Rightarrow_X \neg mt)).$$

Example 4 illustrates the following $\mathbf{DCxt}$-theorem:

(7)
$$\begin{aligned} \vdash_{\mathbf{DCxt}} (\varphi \Rightarrow_X \psi) \rightarrow [+\!\!\!|(\varphi \Rightarrow_X \neg\psi)] \\ ((\varphi \Rightarrow_X \psi) \wedge (\varphi \Rightarrow_X \neg\psi)) \end{aligned}$$

The $\mathbf{Cxt}$-theorem (1) tells us that a classificatory dilemma implies $\mathbf{O}_X \neg\varphi$. It follows that if the normative system X is expanded with φ then $\bot$ becomes true in X, that is, the normative system becomes inconsistent:

(8)
$$\vdash_{\mathbf{DCxt}} ((\varphi \Rightarrow_X \psi) \wedge (\varphi \Rightarrow_X \neg\psi)) \rightarrow [X{+}\varphi][X]\bot$$

Thus, changes generating classificatory dilemmas can be considered as badly designed normative modifications.

Normative dilemna By normative dilemma we mean a situation in which a normative system prescribes that a certain fact ψ must be true under a certain condition φ and at the same time $\neg\psi$ must be true under the same condition, i.e. $\mathbf{O}_X(\varphi \rightarrow \psi) \wedge \mathbf{O}_X(\varphi \rightarrow \neg\psi)$. An example of normative dilemma is the case of a soldier having at the same time the obligation to kill his enemies during a war and the obligation for every person not to shoot other people. If a soldier is classified as a person and enemies are classified as people, we incur a normative dilemma: a soldier has the obligation to shoot his enemies and the obligation not to shoot his enemies. Note that $\mathbf{O}_X \neg\varphi$ implies $\mathbf{O}_X(\varphi \rightarrow \psi) \wedge \mathbf{O}_X(\varphi \rightarrow \neg\psi)$ for every ψ. So, to be more precise, we should exclude from the previous definition of normative dilemna the situation in which $\mathbf{O}_X \neg\varphi$ holds.

Example 5. *In Example 2, after the legislator's proclamation that every bike must have an insurance (event* $+\mathbf{O}_X(bk \rightarrow in)$ *), bikes have the obligation to have an insurance and the obligation not have it, which is a normative dilemma:*

$$[+\mathbf{O}_X(bk \rightarrow in)](\mathbf{O}_X(bk \rightarrow in) \wedge \mathbf{O}_X(bk \rightarrow \neg in)).$$

Example 5 illustrates the following $\mathbf{DCxt}$-theorem:

(9)
$$\vdash_{\mathbf{DCxt}} \mathbf{O}_X(\varphi \rightarrow \psi) \rightarrow [+\mathbf{O}_X(\varphi \rightarrow \neg\psi)] \\ (\mathbf{O}_X(\varphi \rightarrow \psi) \wedge \mathbf{O}_X(\varphi \rightarrow \neg\psi))$$

It is to be noted that, if a normative dilemma $\mathbf{O}_X(\varphi \rightarrow \psi) \wedge \mathbf{O}_X(\varphi \rightarrow \neg\psi)$ holds and the normative system is expanded with φ then every fact χ becomes obligatory in X:

(10)
$$\vdash_{\mathbf{DCxt}} (\mathbf{O}_X(\varphi \rightarrow \psi) \wedge \mathbf{O}_X(\varphi \rightarrow \neg\psi)) \rightarrow [X{+}\varphi]\mathbf{O}_X\bot$$

It is worth stressing the similarity between $\mathbf{DCxt}$-theorem (8) and $\mathbf{DCxt}$-theorem (10). While a classificatory dilemma results in an empty context ($\mathbf{DCxt}$-theorem (8)) under the assumption of the antecedent, a normative dilemma results in a context where legality is impossible ($\mathbf{DCxt}$-theorem (10)).

Finally, we have shown by $\mathbf{DCxt}$-theorems (7) and (9) that if we want to change a norm (a classificatory rule or an obligation) to a contrary norm by a sole act of norm promulgation we end up with a dilemma (either classificatory or normative). Thus, to avoid dilemmas, we must first derogate the old norm and then promulgate the contrary norm. This observation is formally expressed by the following $\mathbf{DCxt}$-theorems:

(11)
$$\vdash_{\mathbf{DCxt}} ((\varphi \Rightarrow_X \psi) \wedge \langle \mathsf{U} \rangle \neg(\varphi \rightarrow \psi)) \rightarrow \\ [-(\varphi \Rightarrow_X \psi)][+(\varphi \Rightarrow_X \neg\psi)] \\ \neg((\varphi \Rightarrow_X \neg\psi) \wedge (\varphi \Rightarrow_X \psi))$$

(12)
$$\vdash_{\mathbf{DCxt}} (\mathbf{O}_X(\varphi \rightarrow \psi) \wedge \mathbf{P}_{\mathsf{U}}\neg(\varphi \rightarrow \psi)) \rightarrow \\ [-\mathbf{O}_X(\varphi \rightarrow \psi)][+\mathbf{O}_X(\varphi \rightarrow \neg\psi)] \\ \neg(\mathbf{O}_X(\varphi \rightarrow \neg\psi) \wedge \mathbf{O}_X(\varphi \rightarrow \psi))$$

Note that by definition of $-$, these general results hold for *any* derogation (stemming from a contraction of Definition 3).

5 Related works

Formal models of norm change have been drawing attention since the seminal work of Alchourrón and Makinson on the logical structure of derogation in legal codes [3] which expanded into a more general investigation of the logic of theory change (alias belief change) [2]. AGM models are about the contraction of $\mathcal{L}_{Prop}$-theories, and focus on minimal change. In contrast, we here consider a modal language $\mathcal{L}_{\mathbf{Cxt}}$.[1] And our modal operator $-$ allows to express properties about a *family* of contractions, which actually do not necessarily satisfy the AGM criteria of minimal change. However, the validity $\neg[X]\psi \rightarrow (\varphi \leftrightarrow [X{-}\psi]\varphi)$ captures one of these minimality criteria. Another one is expressed by the valid formulas $\alpha \rightarrow [X{-}\psi][X{+}\psi]\alpha$ and $[Y]\alpha \rightarrow [X{-}\psi][X{+}\psi][Y]\alpha$, with $\alpha \in \mathcal{L}_{Prop}$, which correspond to the AGM principle of recovery. The invalid $\neg[X]p \rightarrow [X{-}p][X{+}p]\neg[X]p$ demonstrates that the above formula does not generalize to all α in $\mathcal{L}_{\mathbf{Cxt}}$.

[1] In fact, our formalism satisfies the same dynamic properties about Moore sentences as DEL [17].

Although formal analysis of norm change are available in the literature, the issue of a formal semantics for the dynamics of norms is relatively new. Indeed, most work in deontic logic is about defining formal semantics describing static deontic concepts. From this perspective, our research strategy is close in spirit to Segerberg's [13], who argued for an integration of AGM belief revision with Hintikka-like static logics of belief: we here do the same for deontic logic.

Among the few attempts to provide a formal semantics to norm change we here consider the approach proposed in [11]. There, an extension of the dynamic logic of permission (DLP) of [16] with operations of granting or revoking a permission was proposed. They call DLP$_{dyn}$ this DLP extension. Their operations are similar to our operations of norm promulgation and norm derogation. DLP is itself an extension of PDL (propositional dynamic logic) [10] where actions are used to label transitions from one state to another state in a model. The DLP$_{dyn}$ operation of granting a permission just augments the number of *permitted* transitions in a model, whereas the operation of revoking a permission reduces the number of *permitted* transitions. However there are important differences between our approach and Pucella & Weissman's. For us, normative systems are more basic than obligations and permissions, and the latter are defined from (and grounded on) the former. Moreover, dynamics of obligations and permissions are particular cases of normative system change (normative system expansion and contraction). Thus, we can safely argue that our approach is more general than Pucella & Weissman's in which only dynamics of permissions are considered. It is also to be noted that, while in our approach classificatory rules and their dynamics are crucial concepts in normative change, in DLP$_{dyn}$ they are not considered and even not expressible. In future work we will analyze the relationships between DLP$_{dyn}$ and our logic, and possibly a reduction of DLP$_{dyn}$ to our logic $\mathbf{DCxt}$.

While Pucella & Weissman's revocation of permissions corresponds to public announcements in DEL, no DEL approaches have proposed the counterpart of their operation of granting permissions, alias contractions (with the exception of [15], but in the framework of a logic of preference). Probably the reason for that is that it is difficult to define contraction operations both preserving standard properties of epistemic models such as transitivity and Euclidianity and allowing for reduction axioms. As we have shown, this is possible in our logic $\mathbf{DCxt}$ thanks to the intercontextual interaction axioms.

6 Conclusions

We have introduced a dynamic logic accounting for context change, and have analyzed several aspects of norm change, viz. the dynamics of permissions, obligations and classificatory rules. Although the logic has been applied here only to provide a formal analysis of norm-change, it is clear that its range of applications is much broader. Viewed in its generality, the logic is a logic of the dynamics of propositional theories, and as such, can be naturally applied to formal epistemology by studying theory-change, or to non-monotonic reasoning by studying how the context of an argumentation evolves during, for instance, a dialogue game. This kind of applications are future research. Another line of research would be to study the interaction between contexts, and so in a dynamic setting. Notice, in particular, that it would be straightforward to define a set algebra on contexts.

References

[1] T. Ågotnes, W. van der Hoek, J.A. Rodriguez-Aguilar, C. Sierra, and M. Wooldridge. On the logic of normative systems. In *Proc. of IJCAI'07*, pages 1181–1186. AAAI Press, 2007.

[2] C. Alchourrón, P. Gärdenfors, and D. Makinson. On the logic of theory change: Partial meet contraction and revision functions. *J. of Symbolic Logic*, 50:510–530, 1985.

[3] C. Alchourrón and D. Makinson. Hierarchies of regulations and their logic. In R. Hilpinen, editor, *Deontic Logic: Introductory and Systematic Readings*. D. Reidel, 1981.

[4] C. E. Alchourrón and E. Bulygin. *Normative Systems*. Springer-Verlag, 1971.

[5] A. Anderson. A reduction of deontic logic to alethic modal logic. *Mind*, 22:100–103, 1958.

[6] P. Blackburn, M. de Rijke, and Y. Venema. *Modal Logic*. Cambridge Univ. Press, 2001.

[7] C. Ghidini and F. Giunchiglia. Local models semantics, or contextual reasoning = locality + compatibility. *Artificial Intelligence*, 127(2):221–259, 2001.

[8] G. Governatori and A. Rotolo. Changing legal systems: abrogation and annulment (part I: revision of defeasible theories). In *Proc. of DEON'08*, LNAI, pages 3–18. Springer-Verlag, 2008.

[9] D. Grossi, J.-J.Ch. Meyer, and F. Dignum. The many faces of counts-as: A formal analysis of constitutive-rules. *J. of Applied Logic*, 6(2):192–217, 2008.

[10] D. Harel, D. Kozen, and J. Tiuryn. *Dynamic Logic*. MIT Press, Cambridge, 2000.

[11] R. Pucella and V Weissman. Reasoning about dynamic policies. In *Proc. FOSSACS'04*, LNCS, pages 453–467. Springer-Verlag, 2004.

[12] J. R. Searle. *Speech acts: An essay in the philosophy of language*. 1969.

[13] K. Segerberg. Two traditions in the logic of belief: bringing them together. In H. J. Ohlbach and U. Reyle, editors, *Logic, Language and Reasoning: essays in honour of Dov Gabbay*. Kluwer, 1999.

[14] R. Stalnaker. On the representation of context. *J. of Logic, Language, and Information*, 7:3–19, 1998.

[15] J. van Benthem and F. Liu. Dynamic logic of preference upgrade. *J. of Applied Non-Classical Logics*, 17(2):157–182, 2007.

[16] R. Van der Meyden. The dynamic logic of permission. *J. of Logic and Computation*, 6:465–479, 1996.

[17] H. van Ditmarsch, W. van der Hoek, and B. Kooi. *Dynamic Epistemic Logic*, volume 337 of *Synthese Library Series*. Springer, 2007.