Line Planning and Connectivity

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Abstract. The line planning problem in public transport deals with the construction of a system of lines that is both attractive for the passengers and of low costs for the operator. In general, the computed line system should be connected, i.e., for each two stations there have to be a path that is covered by the lines. This subproblem is a generalization of the well-known Steiner tree problem; we call it the Steiner connectivity problem. We show in this talk that important results on the Steiner tree polytope can be carried over to the SCP case. Furthermore, we generalize the famous relation between undirected and directed Steiner tree formulations.

Keywords. Steiner tree, generalization, paths

The *line planning* problem can be described as follows: We are given a public transportation network $G = (V, E)$, a set of (simple) line paths $\mathcal{P}$, and a passenger demand matrix $D \in \mathbb{N}^{V \times V}$, which gives the number of passengers who want to travel between different stations in the network. The edges of G have nonnegative travel times $\tau \in \mathbb{R}^E_+$, the paths have nonnegative costs $c \in \mathbb{R}^{\mathcal{P}}_+$ and capacities $\kappa \in \mathbb{R}^{\mathcal{P}}_+$. The problem is to find a set of line paths $\mathcal{P}' \subseteq \mathcal{P}$ with associated frequencies $f_p \in \mathbb{R}_+$, $p \in \mathcal{P}'$, and a passenger routing, such that the overall capacities $\sum_{p \in \mathcal{P}', e \in p} f_p \cdot \kappa_p$ on the edges suffice to transport all passengers. There are two possible objectives: to minimize the travel time, or to minimize the cost of the line paths. For a detailed description of the line planning problem see e.g. [1],[2],[3] and the references therein.

If we assume large enough capacities the requirement to transport all passengers can usually be replaced by requiring a set of line paths $\mathcal{P}'$ that connect all stations with positive supply and/or demand. More precisely, let (T, F) be the *demand graph* of the line planning problem, where $T = \{v \in V \mid \sum_u (d_{uv} + d_{vu}) > 0\}$ is the set of nodes with positive supply or demand, and $F = \big\{\{u, v\} \mid d_{uv} + d_{vu} > 0\big\}$ a set of *demand edges*. Then the following holds: If the demand graph is connected, then the set of line paths $\mathcal{P}'$ of a solution of the line planning problem should also connect all demand nodes. In other words, if we neglect travel times of the passengers, as well as capacities and frequencies of the lines, the line planning problem with connected demand graph reduces to a connectivity

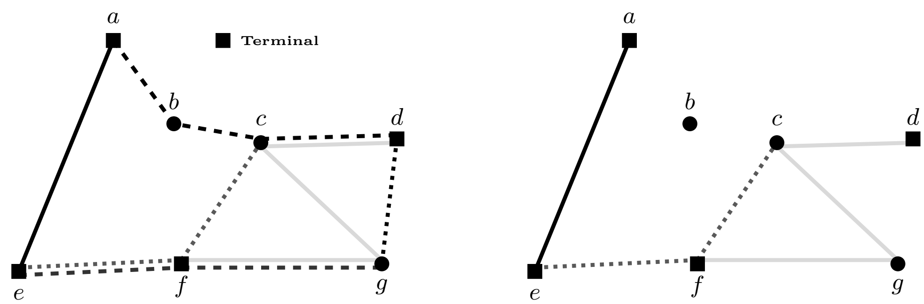

Fig. 1. Example of a Steiner connectivity problem. *Left:* A graph with four terminal nodes and six paths $\big(p_1 = (a, b, c, d),\ p_2 = (e, f, g),\ p_3 = (a, e),\ p_4 = (e, f, c),\ p_5 = (g, d),\ p_6 = (f, g, c, d)\big)$. *Right:* A feasible solution with three paths $(p_3,\ p_4,\ p_6)$.

problem, which we want to call the *Steiner connectivity problem* (SCP). The connection to line planning motivates a study of such a problem.

A general definition of the Steiner connectivity problem is the following. We are given an undirected graph $G = (V, E)$, a set of *terminal nodes* $T \subseteq V$, and a set of (simple) *paths* $\mathcal{P}$ in G. The paths have nonnegative costs $c \in \mathbb{R}^{\mathcal{P}}_{+}$. The problem is to find a set of paths $\mathcal{P}' \subseteq \mathcal{P}$ of minimal cost $\sum_{p \in \mathcal{P}'} c_p$ that connect the terminals, i.e., such that for each pair of distinct terminal nodes $t_1, t_2 \in T$ there exists a path from t_1 to t_2 in G that is completely covered by paths of $\mathcal{P}'$. We can assume w.l.o.g. that G does not contain edges that are not covered by any path of $\mathcal{P}$, i.e., for every $e \in E$ there is a $p \in \mathcal{P}$ such that $e \in p$. Figure 1 gives an example of a Steiner connectivity problem (left) and a feasible solution (right).

SCP is a generalization of the *Steiner tree problem* (STP), in which all paths contain exactly one edge. Similar to the STP with nonnegative costs, see [4,5,6] for an overview, there exists always an optimal solution of the SCP that is *minimally connected*, i.e., if we remove a path from the solution, there exist at least two terminals which are not connected. However, in contrast to the STP, there is not necessarily an optimal solution of the Steiner connectivity problem that forms a tree, see again the right of Figure 1.

A natural question is whether one can transfer structural results and algorithms from the Steiner tree problem to the Steiner connectivity problem. We show in this talk [7] and in [8] and [9] that this can indeed be done in many cases. In particular, an important result (see Chopra and Rao [10]) in the STP literature states that the undirected IP formulation of the STP, including all so-called Steiner partition inequalities, is dominated by a certain family of directed formulations. Using this connection, a super class of the Steiner partition inequalities can be separated in polynomial time. Similar results hold for the SCP as well. The directed formulation that we use, however, is constructed in a different way and must be strengthened by so-called flow-balance constraints.

Subtle differences also come up in the complexity analysis. For instance, the SCP is also solvable in polynomial time for a fixed number of terminals, but it is NP-hard in the case $T = V$.

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