Sharing Supermodular Costs

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Extended Abstract

Consider a situation in which a set of agents has the option of sharing the cost of their joint actions. For example, a group of retailers, instead of individually managing each of their own storage facilities, may decide to jointly participate in a centralized inventory management scheme with a common storage facility, and share the cost of optimally running this facility. In these situations, the agents may or may not be motivated to cooperate, depending on the structure of their costs. Cooperative game theory offers a mathematical framework to study the cooperative behavior between multiple agents. A *(transferable utility) cooperative game* is a pair (N, v) where $N = \{1, \ldots, n\}$ represents a set of agents, and $v : 2^N \rightarrow I\!R$ is a set function where for each $S \subseteq N$, $v(S)$ represents the cost to agents in S if they cooperate. By convention, $v(\emptyset) = 0$. A subset $S \subseteq N$ of agents is referred to as a *coalition*.

Cooperative games whose costs are determined by various problems in operations research and computer science have been studied before. A short and necessarily incomplete list of examples includes assignment games [1], linear production games [2], minimum-cost spanning tree games [3], traveling salesman games [4], scheduling-related games [5,6,7], facility location games [8], newsvendor games and inventory centralization games [9,10], and economic lot-sizing games [11,12].

In this work, we are concerned with situations in which agents face *supermodular* costs. A set function $v : 2^N \mapsto I\!R$ is supermodular if

$$v(S \cup \{j\}) - v(S) \leq v(S \cup \{j,k\}) - v(S \cup \{k\}) \qquad \text{for all } S \subseteq N \setminus \{j,k\}. \quad (1)$$

In words, supermodularity captures the notion of increasing marginal costs. We study cooperative games (N, v) where v is nonnegative, and supermodular. We call such games *supermodular cost cooperative games*. Supermodularity often naturally arises in situations in which the costs are intimately tied with congestion effects. It has been shown that several variants of the facility location problem

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have supermodular costs [13], and as we show in this work, various problems from scheduling and network design also exhibit supermodular costs.

Intuitively, cooperation amongst rational agents who face supermodular costs is unlikely: as the size of a coalition grows, the marginal cost associated with adding a particular agent increases, diminishing the appeal of cooperation. Various solution concepts from cooperative game theory help us formalize this intuition. Suppose $x \in I\!R^N$ is a cost allocation vector: for all $i \in N$, x_i represents the cost allocated to agent i. (For notational convenience, for any vector x we define $x(S) = \sum_{i \in S} x_i$ for any $S \subseteq N$.) The prominent solution concept for cooperative games is the *core* [14]. The core of a cooperative game (N, v) is the set of all cost allocations x such that

$$x(N) = v(N), \tag{2}$$

$$x(S) \leq v(S) \quad \text{for all } S \subseteq N. \tag{3}$$

The condition (2) requires that a cost allocation in the core is *efficient*: the total cost allocated to all agents, $x(N)$, is equal to the cost of all agents cooperating, $v(N)$. The conditions (3) guarantee that a cost allocation in the core is *stable*: no subset of agents, or coalition, would be better off by abandoning the rest of the agents and acting on its own. The existence of an efficient and stable cost allocation—in other words, a non-empty core—can be seen as a rudimentary indication that cooperation is attainable. It is well-known that when costs are submodular[1], the core is non-empty [15]. On the other hand, it is straightforward to see that for supermodular cost cooperative games, the core is in fact empty (as long as costs are not modular[2]).

In certain situations, the failure to cooperate may give rise to negative externalities. Consider the following example. A set of agents needs to process its jobs on a machine that generates an excessive amount of pollution. The agents have the opportunity to share the cost of processing their jobs on an existing single machine, but the cost of processing their jobs is such that it is cheaper for each agent to open their own machine, and as a result, generate more pollution. An authority may be interested in reducing such negative externalities. One approach would be to incorporate the cost of the pollution externalities directly into the processing costs; however, these externality costs may be hard to precisely define. Instead, one might ask, "How much do we need to charge for opening an additional machine in order to encourage all agents to share a single machine?" For an arbitrary cooperative game, the analogous question is, "How much do we need to penalize a coalition for acting independently in order to encourage all the agents to cooperate?" This notion is captured in the *least core value* of a cooperative game. The *least core* [16,17] of a cooperative game (N, v)

[1] A function v is submodular if $-v$ is supermodular.
[2] A set function is modular if it is both submodular and supermodular.

is the set of cost allocations x that are optimal solutions to the linear program

$$\begin{aligned} z^* = \text{minimize} \quad & z \\ \text{subject to} \quad & x(N) = v(N), \\ & x(S) \leq v(S) + z \quad \text{for all } S \subseteq N,\ S \neq \emptyset, N. \end{aligned} \tag{LC}$$

The optimal value z^* of (LC) is the least core value of the game (N, v). In words, the least core value z^* is the minimum penalty we need to charge a coalition for acting independently that ensures a basic prerequisite for cooperation is satisfied: the existence of an efficient and stable cost allocation. Note that the linear program (LC) is in fact equivalent to the optimization problem

$$z^* = \min_{x : x(N) = v(N)} \max_{\substack{S \subseteq N \\ S \neq \emptyset, N}} e(x, S),$$

where $e(x, S) = x(S) - v(S)$ for all $S \subseteq N$. The quantity $e(x, S)$ is the *dissatisfaction* of a coalition S under a cost allocation x: it is the extra cost that S pays when costs are allocated according to x.[3] A cost allocation in the least core therefore minimizes the maximum dissatisfaction of any coalition. Note that the least core is always well-defined and non-empty, regardless of whether the core is empty or non-empty[4].

A fair amount of attention has been devoted to the least core of various cooperative games in the economics and game theory literature [18,19,20,21,22, for example]. In addition, the computational complexity of computing a cost allocation in the least core has been studied previously in several contexts. Faigle, Kern and Paulusma [23] showed that computing a cost allocation in the least core of minimum-cost spanning tree games is NP-hard. Kern and Paulusma [24] presented a polynomial description of the linear program (LC) for cardinality matching games. Faigle, Kern and Kuipers [25] showed that by using the ellipsoid method, a so-called pre-kernel element in the least core of a cooperative game can be efficiently computed if the maximum dissatisfaction can be efficiently computed for any given efficient cost allocation. Properties of the least core value, on the other hand, seem to have been largely ignored. Deng [26] observed that polynomial-time algorithms for submodular function minimization can be used to compute the least core and least core value of submodular cost cooperative games in polynomial time.

Contributions of this work

In this work, we study the computational complexity and approximability of the least core value of supermodular cost cooperative games. We motivate the interest in supermodular cost cooperative games by providing a class of optimization

[3] This quantity is sometimes referred to as the *excess* of a coalition in the cooperative game theory literature.

[4] The linear program (LC) is clearly feasible. Adding the inequalities $x_i \leq v(\{i\}) + z$ for all $i \in N$ and using the equality $x(N) = v(N)$, we have that $z^* \geq \frac{1}{|N|}(v(N) - \sum_{i \in N} v(\{i\}))$. So as long as costs are finite, the optimal value of (LC) is finite and the least core value is well defined.

problems whose optimal costs are supermodular. This class of optimization problems includes a variety of classical scheduling and network design problems. We show that finding the least core value of supermodular cost cooperative games is strongly NP-hard, and design a $(3+\epsilon)$-approximation algorithm for computing the least core value of these games, using oracles that determine coalitions whose dissatisfaction is approximately maximum. As a by-product, we also show how to compute accompanying approximate least core cost allocations.

We apply our results to two subclasses of supermodular cost cooperative games: *scheduling games* and *matroid profit games*. Scheduling games are cooperative games in which the costs are derived from the minimum sum of weighted completion times on a single machine. We show that for these games, the Shapley value[5]—which is computable in polynomial time—is in the least core, while computing the least core value is NP-hard. We also give a fully polynomial time approximation scheme for computing the least core value of scheduling games. Matroid profit games are cooperative games in which the *profit* to a coalition arises from the maximum weight of an independent set of a matroid. Some scheduling and network design problems have been shown to be special cases of finding a maximum weight independent set of a matroid. Using our approximation framework with the appropriate natural modifications, we show that the least core value and a least core cost allocation of matroid profit games can be computed in polynomial time.

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[5] The Shapley value [27] of a cooperative game (N, v) is the cost allocation $\phi \in IR^N$, where

$$\phi_i = \sum_{S \subseteq N \setminus \{i\}} \tfrac{|S|!(|N|-|S|-1)!}{|N|!} \big(v(S \cup \{i\}) - v(S)\big) \quad \text{for all agents } i \in N.$$

In words, the Shapley value of each agent i reflects agent i's average marginal contribution to the coalition N. The Shapley value is one of the most important solution concepts in cooperative game theory; for example, see [28].

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