# Systematic judgment aggregators: An algebraic connection between social and logical structure

Daniel Eckert[1], Frederik Herzberg[2]

[1] Institut für Finanzwissenschaft, Karl-Franzens-Universität Graz
A-8010, Graz, Universitätsstrasse 15/E4, Austria
daniel.eckert@uni-graz.at
[2] Institut für mathematische Wirtschaftsforschung, Universität Bielefeld
D-33615, Bielefeld, Universitätsstraße 25, Germany
fherzberg@uni-bielefeld.de

**Abstract.** We present several results that show that systematic (complete) judgment aggregators can be viewed as both (2-valued) Boolean homomorphisms and as syntatic versions of reduced (ultra)products. Thereby, Arrovian judgment aggregators link the Boolean algebraic structures of (i) the set of coalitions (ii) the agenda, and (iii) the set of truth values of collective judgments. Since filters arise naturally in the context of Boolean algebras, these findings provide an explanation for the extraordinary effectiveness of the filter method in abstract aggregation theory.

**Keywords.** Judgment aggregation; social structure; Boolean homomorphism; ultraproduct

## 1 Introduction

The relation between rationality and power is one of the oldest puzzles in philosophy. According to Habermas, power neutrality is even a precondition of collective rationality ([1]). Recent extensions of the social choice literature from the aggregation of preferences to judgement aggregation however suggest that rationality even in the weakest possible sense of logical consistency bears a close relation to power: In fact, the recent literature on judgment aggregation (for a survey see [2]) shows that the logical structure of the agenda of a collective decision process (given by the logical interconnections between the propositions) shapes the social structure (given by a distribution of decision power) and that this power structure can be as asymmetric as a dictatorship.

According to the social choice literature, the social structure is modelled by a partition of the power set of individuals in decisive and non-decisive coalitions.[1]

[1] As a referee rightly noted, this coarse distinction is by no ways the only reasonable approach to the modelling of social structure, and future work will have to address more complex social (inter)dependencies, for instance networks.

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For the analysis of the relation between the logical and the social structure of an aggregation problem, filters and ultrafilters have been proven particularly useful (see e.g. [3], [4], [5], [6]). As these concepts arise first and foremost in the context of Boolean algebras, it is natural to expect that the understanding of the relation between the logical and the social structure of an aggregation problem can be deepened through concepts from Boolean algebra ([7]).

Indeed, we prove that non-trivial systematic universal judgment aggregators are in canonical one-to-one correspondences with (a) Boolean algebra homomorphisms (see Theorem 8) and (b) propositional reduced products (see Theorem 10). Complete non-trivial universal systematic judgment aggregators are even in canonical one-to-one correspondences with (a) 2-valued Boolean algebra homomorphisms (see Theorem 5) and (b) propositional ultraproducts (see Theorem 11).

Thus, systematic judgment aggregators connect the Boolean algebra structures on (i) the set of coalitions, (ii) the agenda, and (iii) the set of truth values of individual and collective judgments. This analysis supports the intuition that the social structure of the population is shaped by its relation to the syntactic structure of the agenda and the semantic structure of the collective judgments.

For technical and expository reasons, we shall assume the strong independence condition of systematicity together with a (mild) agenda richness condition inspired by Lauwers and Van Liedekerke ([8]).

## 2 Framework

*Judgment sets* Consider a monotonic logic $L$, containing the connectives $\neg$ and $\wedge$. Let $\vdash$ be a provability relation for $L$.

Let $X$ be a set of sentences in the logic $L$. $X$ is called the *agenda*. We assume that $X$ is the union of proposition-negation pairs (i.e. there exists a non-empty set $X'$ of sentences such that $X = \bigcup_{p \in X'} \{p, \neg p\}$). For every $p \in X$ we denote by $\sim p$ an element $q$ of $X$ such that either $q = \neg p$ or $p = \neg q$.

Subsets of $X$ will be called *judgment sets* and we denote the power-set of $X$ by $\mathcal{P}(X)$.

For every judgment set $Y$, we define the following: $Y$ is *consistent* if and only if $Y \nvdash (p \wedge \neg p)$ for any sentence $p$. (In particular for every $p \in X$, we assume $\{p\}$ to be consistent.) $Y$ is *deductively closed (in $X$)* if and only if for all $p \in X$, if $Y \vdash p$, then $p \in Y$. $Y$ is *complete (in $X$)* if and only if for all $p \in X$, $p \notin Y$ implies $\sim p \in Y$. $Y$ is *algebraically consistent* if and only if for all $p \in X$, $\sim p \in Y$ implies $p \notin Y$.

We denote by $D$ the set of all consistent and complete subsets of $X$, by $D^*$ the set of all consistent and deductively closed subsets of $X$, by $D'$ the set of all deductively closed subsets of $X$, by $\bar{D}$ the set of all consistent subsets of $X$, by $D^{ac}$ the set of all algebraically consistent and complete subsets of $X$, and by $D^a$ the set of all algebraically consistent subsets of $X$.

Clearly $D \subseteq D^* \subseteq D'$, and $D \subseteq D^{ac} \subseteq D^a$.

A subset $Y \subseteq X$ is in $D^{ac}$ if and only if $p \notin Y \Leftrightarrow \sim p \in Y$ for all $p \in X$.

*Judgment aggregators* Let $N$ be a non-empty set of *individuals*, called the *population* set. We call subsets of $N$ *coalitions*. The power-set of $N$ is denoted by $\mathcal{P}(N)$.

A *judgment aggregator* is a mapping $f : \mathfrak{D}_f \to \mathcal{P}(X)$ with $\varnothing \neq \mathfrak{D}_f \subseteq \mathcal{P}(X)^N$. Elements of $\mathfrak{D}_f$, usually denoted $\underline{A} = \langle A_i \rangle_{i \in N}$, are called *profiles*, components $A_i$ of profiles are called *individual judgment sets*, elements of the range of $f$ will be called *collective judgment sets*.

We say that $f$ is *complete* (or *consistent*, or *deductively closed*, or *algebraically consistent*, respectively) if its range only consists of complete (or consistent, or deductively closed, or algebraically consistent, respectively) judgment sets.

$f$ is called *dictatorial* if and only if there exists some $i_f \in N$ such that $f(\underline{A}) = A_{i_f}$ for all $\underline{A} \in \mathfrak{D}_f$. $f$ is called *oligarchic* if and only if there exists some non-empty $M_f \subseteq N$ such that $f(\underline{A}) = \bigcap_{i \in M_f} A_i$ for all $\underline{A} \in \mathfrak{D}_f$.

*Coalitions* For all $p \in X$ and $\underline{A} \in \mathfrak{D}_f$, the *coalition supporting $p$ given $\underline{A}$* is

$$\underline{A}(p) := \{i \in N \ : \ p \in A_i\} \, .$$

We say that $\underline{A}(p)$ is *winning for $p$ given $\underline{A}$ under $f$* if and only if $p \in f(\underline{A})$.

We collect all *winning coalitions* in the set

$$\mathcal{F}_f := \{\underline{A}(p) \ : \ \underline{A} \in \mathfrak{D}_f, \quad p \in f(\underline{A})\} \, .$$

Given any $C, C' \subseteq N$, we shall write $C \sim_f C'$ (in words: $C$ and $C'$ *share the same part of a winning coalition*) if and only if there exists some $U \in \mathcal{F}_f$ such that $C \cap U = C' \cap U$. Note that the set of winning coalitions for $p$ is the same for each profile if and only if $f$ is *independent* in the sense that for every $p \in X$ and $\underline{A}, \underline{A}' \in \mathfrak{D}_f$,

$$\underline{A}(p) = \underline{A}'(p) \Rightarrow \big(p \in f(\underline{A}) \Leftrightarrow p \in f(\underline{A}')\big) \, .$$

As a notational device, we regard $f(\underline{A})$, for all $\underline{A} \in \mathfrak{D}_f$, as a function $f(\underline{A}) : X \to \{0, 1\}$, defined through

$$f(\underline{A})(p) = \begin{cases} 1, p \in f(\underline{A}) \\ 0, p \notin f(\underline{A}) \end{cases}$$

## 3 Axioms

In the spirit of Arrovian social choice theory, we introduce the following set of aggregator axioms. A judgment aggregator which satisfies the agenda richness and rationality axioms (A2-A5) will also be called *Arrovian* for the purposes of this paper.

**A1. Finite population.** $N$ is finite.

**A2. Agenda richness.** There are propositions $p, q \in X$ such that each of the propositions $p \wedge q, p \wedge \neg q, \neg p \wedge q$ is consistent and $\in X$.

**A3. Universality.** $\mathfrak{D}_f \supseteq D^N$.

**A4. Non-triviality.** $f$ is neither constantly $= \varnothing$ nor constantly $= X$.

**A5. Systematicity.** For all $p, q \in X$ and $\underline{A}, \underline{A}' \in \mathfrak{D}_f$: If $\underline{A}(p) = \underline{A}'(q)$, then $p \in f(\underline{A}) \Leftrightarrow q \in f(\underline{A}')$.

The axiom of non-triviality, which to the knowledge of the authors is new in the judgment-aggregation literature, is satisfied in two important special cases:

**Remark 1.** *$f$ satisfies (A4) whenever $f$ satisfies (A3) as well as* strict unanimity preservation *(that is, for all $p \in X$ and $\underline{A} \in \mathfrak{D}_f$, if $\underline{A}(p) = N$ then $p \in f(\underline{A})$, and if $\underline{A}(p) = \varnothing$ then $p \notin f(\underline{A})$). $f$ also satisfies (A4) if $f$ is both complete and consistent.*

*Proof.* Since $X$ is comprised of proposition-negation pairs, it contains some consistent proposition $p$ and some proposition $q$ which is not universally valid. Hence there must be some $\underline{A} \in D^N$ with $\underline{A}(p) = N$ and some $\underline{A}' \in D^N$ with $\underline{A}'(q) = \varnothing$. If $f$ satisfies both (A3) and strict unanimity preservation, then $p \in f(\underline{A})$ and $q \notin f(\underline{A}')$.

If $f$ is both complete and consistent, then $f(\underline{A}) \neq \varnothing$ (as $\varnothing$ is incomplete) and $f(\underline{A}) \neq X$ (as $X$ is inconsistent, being comprised of proposition-negation pairs) for all $\underline{A} \in \mathfrak{D}_f$. □

We end this section with a brief discussion of the apparently very strong axiom of systematicity; herein, we follow the presentation in Klamler and Eckert ([6]) where more details, including proofs and further references, can be found.

Clearly, if $f$ is systematic, then also independent. The converse is true if the agenda satisfies an additional condition known as *total blockedness* (see e.g. [9]) which asserts that any proposition in the agenda is related to any other proposition by a sequence of conditional entailments.

Given any distinct $p, q \in X$, we say that $p$ *entails* $q$ *conditionally* (denoted $p \vdash^* q$) if there exists a minimally inconsistent superset $S$ of $\{p, \neg q\}$. $X$ is called *totally blocked* if the transitive closure of the conditional entailment relation is all of $X \times X$.

Finally, we say that $f$ is *weakly unanimity-preserving* if and only if for all $p \in X$ and $\underline{A} \in \mathfrak{D}_f$, if $\underline{A}(p) = N$ then $p \in f(\underline{A})$.

**Lemma 2.** *Let $X$ be totally blocked and consider a unanimity-preserving judgment aggregator $f$. If $f$ satisfies (A3) and is both independent and unanimity-preserving, then $f$ also satisfies (A5).*

For systematic $f$, the set $\mathcal{F}_f$ of winning coalitions allows for a natural characterization (see e.g. Eckert and Herzberg [10]):

**Lemma 3.** *Suppose $f$ satisfies (A5). Then for all $\underline{A} \in \mathfrak{D}_f$ and $p \in X$, one has $\underline{A}(p) \in \mathcal{F}_f$ if and only if $p \in f(\underline{A})$.*

*Proof.* Let $\underline{A} \in \mathfrak{D}_f$ and $p \in X$. By definition, if $p \in f(\underline{A})$, then $\underline{A}(p) \in \mathcal{F}_f$.

Conversely, if $\underline{A}(p) \in \mathcal{F}_f$, then there must be some $q \in X$ and $\underline{A}' \in \mathfrak{D}_f$ such that $\underline{A}(p) = \underline{A}'(q)$ and $q \in f(\underline{A}')$. By (A5), this readily yields $p \in f(\underline{A})$. □

# 4 Results

## 4.1 Aggregators as homomorphisms: Translating coalitions into truth values

Both the set of coalitions $\mathcal{P}(N)$ and the set of truth values $2 := \{0, 1\}$ are canonically endowed with a Boolean algebraic structure: Both the power set algebra $\left\langle \mathcal{P}(N), \cap, \cup, \complement, \varnothing, N \right\rangle$ (wherein $\complement B := N \setminus B$ for all $B \subseteq N$) and the algebra 2 of truth values $\langle \{0, 1\}, \wedge, \vee, ^*, 0, 1 \rangle$ (wherein $0^* = 1$, $1^* = 0$) are Boolean algebras. For the following, we adopt standard terminology of Boolean algebra (cf. e.g. Bell and Slomson [11]). In particular, a *(Boolean algebra) homomorphism* is a map $\phi : B_1 \to B_2$ between two Boolean algebras $B_1, B_2$ which preserves the algebraic operations; the *shell* of such a homomorphism is $\phi^{-1}\{1_{B_2}\}$, the pre-image of the 1-element of the image algebra. We will first show that non-trivial universal systematic (complete) judgment aggregators are derived from (2-valued) Boolean algebra homomorphisms with domain $\mathcal{P}(N)$ and vice versa. The shell of these homomorphisms will be nothing else than the set of winning coalitions. *En passant*, we obtain very general impossibility results. The proofs of the main results have been deferred to the appendix. A detailed exposition of the proofs for Theorems 5 and 8 can also be found in Herzberg [7].

**Lemma 4.** *If $f$ satisfies (A2), (A3) and (A5), then there is a well-defined map*

$$\pi : \mathcal{P}(N) \to 2, \quad \underline{A}(p) \mapsto f(\underline{A})(p).$$

*If $f$ is also deductively closed, then $\pi^{-1}\{1\}$ equals $\mathcal{F}_f$ and is both closed under supersets and closed under intersections.*

Note that generically, $\pi$ does not have to be a lattice homomorphism, let alone a Boolean algebra homomorphism.

**Theorem 5.** *If $f$ satisfies (A2), (A3) and (A5) and is both consistent and complete, then $f$ also satisfies (A4) and $\pi$ is a homomorphism with shell $\mathcal{F}_f$.*

*Conversely, assuming (A2), if $\rho : \mathcal{P}(N) \to 2$ is a homomorphism, then the judgment aggregator*

$$f : D^N \to \mathcal{P}(X), \quad \underline{A} \mapsto \{p \in X \ : \ \rho\left(\underline{A}(p)\right) = 1\}$$

*satisfies (A2-A5) and is both algebraically consistent and complete.*

**Corollary 6.** *If $f$ satisfies (A2), (A3) and (A5) and is both consistent and complete, then $\mathcal{F}_f$ is an ultrafilter. If, in addition, (A1) holds, then $f$ is dictatorial.*

*Proof of Corollary 6.* Theorem 5 ensures that $\pi$ is a 2-valued homomorphism. However, every shell of a 2-valued homomorphism is an ultrafilter. Therefore, $\mathcal{F}_f = \pi^{-1}\{1\}$ is an ultrafilter on $N$. Now, every ultrafilter $\mathcal{F}$ on a finite set $N$ is principal. Hence, if (A1) is satisfied in addition, then there must be some $i_f \in N$ such that $\pi^{-1}\{1\} = \mathcal{F}_f = \{C \subseteq N \ : \ i_f \in C\}$, hence

$$p \in f(\underline{A}) \Leftrightarrow \pi\left(\underline{A}(p)\right) = 1 \Leftrightarrow \underline{A}(p) \in \mathcal{F}_f \Leftrightarrow i_f \in \underline{A}(p) \Leftrightarrow p \in A_{i_f}$$

for all $\underline{A} \in \mathfrak{D}_f$ and $p \in X$. □

A *congruence relation* is an equivalence relation on a Boolean algebra which respects the Boolean operations. Recall that two coalitions $C, C'$ stand in relation $\sim_f$ to each other if and only if they share the same part of some winning coalition.

**Lemma 7.** *If $f$ satisfies axioms (A2-A5) and is deductively closed, then $\sim_f$ is a congruence relation on the Boolean algebra $\mathcal{P}(N)$ and the Boolean operations on $\mathcal{P}(N)$ induce a Boolean algebra structure on $\mathcal{P}(N)/\sim_f$.*

Theorem 5 can be generalized as follows:

**Theorem 8.** *If $f$ satisfies (A2-A5) and is deductively closed, then the canonical surjection $\sigma : \mathcal{P}(N) \to \mathcal{P}(N)/\sim_f$ is a homomorphism with shell $\mathcal{F}_f$.*

*Conversely, assuming (A2), if $\tau : \mathcal{P}(N) \to B$ is a homomorphism for some Boolean algebra $B$, then the judgment aggregator*

$$f : D^N \to \mathcal{P}(X), \quad \underline{A} \mapsto \{p \in X \ : \ \tau\left(\underline{A}(p)\right) = 1_B\}$$

*satisfies (A2-A5) and is algebraically consistent.*

**Corollary 9.** *If $f$ satisfies axioms (A2-A5) and is deductively closed, then $\mathcal{F}_f$ is a filter. If, in addition, (A1) holds, then $f$ is oligarchic.*

*Proof of Corollary 9.* As the shell of a homomorphism, $\mathcal{F}_f$ is a filter.

For every filter $\mathcal{F}$ on a finite set $N$, there exists some $M \subseteq N$ such that $\mathcal{F} = \{C \subseteq N \ : \ M \subseteq C\}$. Hence, if (A1-A5) are satisfied, then there must be some $M_f \subseteq N$ such that $\pi^{-1}\{1\} = \mathcal{F}_f = \{C \subseteq N \ : \ M_f \subseteq C\} = \bigcap_{i \in M_f} \{C \subseteq N \ : \ i \in C\}$, so

$$p \in f(\underline{A}) \Leftrightarrow \pi\left(\underline{A}(p)\right) = 1 \Leftrightarrow \underline{A}(p) \in \mathcal{F}_f \Leftrightarrow \left(\forall i \in M_f \quad \underbrace{i \in \underline{A}(p)}_{\Leftrightarrow p \in A_i}\right),$$

hence $p \in f(\underline{A}) \Leftrightarrow p \in \bigcap_{i \in M_f} A_i$ for all $\underline{A} \in D^N$ and $p \in X$. □

### 4.2 Aggregators as reduced products: Translating profiles into proposition sets

If $\mathcal{G}$ is an ultrafilter on $N$, we define the *(propositional) ultraproduct* of $\underline{A} \in D^N$ by

$$\prod \underline{A}/\mathcal{G} := \{p \in X \ : \ \{i \in N \ : \ p \in A_i\} \in \mathcal{G}\}, \tag{1}$$

in other words,

$$p \in \prod \underline{A}/\mathcal{G} \Leftrightarrow \underline{A}(p) \in \mathcal{G} \tag{2}$$

for every $p \in X$.

One can show that $\prod \underline{A}/\mathcal{G} \in D$, so $\prod \underline{A}/\mathcal{G}$ is a maximally consistent subset of $X$, whence the propositional ultraproduct may be viewed as an interpretation of the propositional variables occurring in the propositions from $X$. Also, the definition of the propositional ultraproduct exhibits an obvious formal analogy to the definition of an ultraproduct in classical model theory. This is sufficient to justify the term "propositional ultraproduct".

In Equation (1), $\mathcal{G}$ could be an arbitrary filter rather than an ultrafilter, and the set $\prod \underline{A}/\mathcal{G}$ will still be well-defined, even for arbitrary $\underline{A} \in \mathcal{P}(X)^N$. It will be called, in analogy to the terminology of classical model theory, *reduced product*. (However, generically, reduced products are not maximally consistent set.)

Propositional reduced products are deductively closed Arrovian judgment aggregators:

**Theorem 10.** *If $f$ satisfies axioms (A2-A5) and is deductively closed, then $\mathcal{F}_f$ is a filter and $f(\underline{A}) = \prod \underline{A}/\mathcal{F}_f$ for all $\underline{A} \in D^N$.*

*Conversely, assuming (A2), if $\mathcal{G}$ is a filter on $N$, then the judgment aggregator*

$$f : \mathcal{P}(X)^N \to \mathcal{P}(X), \quad \underline{A} \mapsto \prod \underline{A}/\mathcal{G}$$

*satisfies axioms (A2-A5). Furthermore, $f \restriction (D')^N$ is deductively closed and $f \restriction \bar{D}^N$ is consistent, whence $f \restriction (D^*)^N$ is both consistent and deductively closed.*

Propositional ultraproducts are consistent complete Arrovian judgment aggregators:

**Theorem 11.** *If $f$ satisfies (A2), (A3) and (A5) and is both consistent and complete, then $\mathcal{F}_f$ is an ultrafilter and $f(\underline{A}) = \prod \underline{A}/\mathcal{F}_f$ for all $\underline{A} \in D^N$.*

*Conversely, assuming (A2), if $\mathcal{G}$ is an ultrafilter on $N$, then the judgment aggregator*

$$f : D^N \to \mathcal{P}(X), \quad \underline{A} \mapsto \prod \underline{A}/\mathcal{G}$$

*satisfies axioms (A2-A5) and is both consistent and complete.*

Theorems 11 and 10 are partially contained in Dietrich and Mongin [3]; more general versions of these theorems, with a somewhat different notation, can be found in papers by Herzberg ([5] and [12]) which were inspired by the work of Lauwers and Van Liedekerke [8] on the relationship between preference aggregation and first-order model theory.

## 5 Conclusion

This paper contains two results which provide a formal justification of the perception of aggregation as a link between social and logical structure.

Firstly, Boolean algebra provides a framework to interpret universal systematic judgment aggregators as homomorphisms which relate the coalition structure, viz. the power-set algebra of the population set, with the formal semantic structure of possible collective outcomes, viz. the truth-value algebra.

Secondly, universal (complete) systematic judgment aggregators can be viewed as the natural extension of reduced product (ultraproduct) constructions in the setting of propositional logic (or more general monotonic logics). Thereby, judgment aggregators relate the structure of the set of coalitions with the syntactic structure of the agenda.

## A Proofs

**Remark 12.** *Let $\kappa$ be finite or infinite. Let $N = \bigcup_{j\in\kappa} C_j$ be a disjoint decomposition of $N$ and let $\langle Y_j \rangle_{j\in\kappa}$ be a family of consistent subsets of $X$. Each $Y_j$ can be extended to a maximally consistent, thus consistent and complete subset $Z_j$ of $X$. Hence, there exists some profile $\underline{A} \in D^N$ such that $A_i = Z_j \supseteq Y_j$ for every $i \in C_j$ and $j \in \kappa$.*

**Remark 13.** *If (A2) is satisfied, then $\left\{\underline{A}(p) \;:\; \underline{A} \in D^N, \quad p \in X\right\} = \mathcal{P}(N)$.*

*Proof of Remark 13.* (A2) implies that $X$ contains a *contingent* sentence $p$ (i.e. both $\{p\}$ and $\{\neg p\}$ are consistent). Let $C \subseteq N$. By Remark 12, there exists a profile $\underline{A} \in D^N$ such that for all $i \in N$, if $i \in C$ then $p \in A_i$ and if $i \in N \setminus C$ then $\neg p \in A_i$, hence $p \notin A_i$ whenever $i \in N \setminus C$ since $A_i$ is consistent. Thus, $p \in A_i \Leftrightarrow i \in C$ for all $i \in N$, so $\underline{A}(p) = C$. Therefore, every coalition $C$ is of the form $\underline{A}(p)$ for some $\underline{A} \in D^N$ and $p \in X$. $\square$

*Proof of Lemma 4.* Suppose $f$ satisfies (A2), (A3) and (A5), and let $p, q$ denote the two sentences whose existence is postulated in (A2). We verify:

*$\pi$ is well-defined on $\mathcal{P}(N)$.* The map $\pi : \underline{A}(p) \mapsto f(\underline{A})(p)$ is well-defined on $\mathfrak{D}_\pi := \{\underline{A}(p) \;:\; \underline{A} \in \mathfrak{D}_f, \quad p \in X\}$ because of (A5). Also, $\mathfrak{D}_f \supseteq D^N$ by (A3). Therefore $\mathfrak{D}_\pi \supseteq \left\{\underline{A}(p) \;:\; \underline{A} \in D^N, \quad p \in X\right\}$, hence $\mathfrak{D}_\pi = \mathcal{P}(N)$ by Remark 13.

*$\pi^{-1}\{1\}$ equals $\mathcal{F}_f$.* For all $p \in X$ and $\underline{A} \in \mathfrak{D}_f$, one has $f(\underline{A})(p) = 1 \Leftrightarrow p \in f(\underline{A})$ by convention, therefore $\pi^{-1}\{1\} = \mathcal{F}_f$.

*$\pi^{-1}\{1\}$ is $\supseteq$-closed.* Let $C' \in \pi^{-1}\{1\}$ and $C \supseteq C'$. By (A2) and Remark 12 there exists a profile $\underline{A} \in D^N$ such that

$$\forall i \in C \setminus C' \quad p \wedge \neg q \in A_i, \quad \forall i \in N \setminus C \quad \neg p \wedge q \in A_i, \quad \forall i \in C' \quad p \wedge q \in A_i.$$

Then $\underline{A}(p \wedge q) = C' \in \pi^{-1}\{1\}$, whence $p \wedge q \in f(\underline{A})$ because $\pi$ is well-defined. However, $f(\underline{A})$ is deductively closed, therefore $p \in f(\underline{A})$, hence $\pi^{-1}\{1\} \ni \underline{A}(p) = (C \setminus C') \cup C' = C$.

$\pi^{-1}\{1\}$ *is* $\cap$*-closed.* Let $C', C'' \in \pi^{-1}\{1\}$. By (A2) and Remark 12 there exists a profile $\underline{A}' \in D^N$ such that

$$\forall i \in C'' \setminus C' \quad p \wedge \neg q \in A'_i, \quad \forall i \in N \setminus C'' \quad \neg p \wedge q \in A'_i, \quad \forall i \in C' \cap C'' \quad p \wedge q \in A'_i.$$

Then $\underline{A}'(p) = (C' \cap C'') \cup (C'' \setminus C') = C'' \in \pi^{-1}\{1\}$, so $p \in f(\underline{A}')$ since $\pi$ is well-defined. On the other hand, $\underline{A}'(q) = (C' \cap C'') \cup (N \setminus C'') \supseteq (C' \cap C'') \cup (C' \setminus C'') = C' \in \pi^{-1}\{1\}$. Hence, $\underline{A}'(q) \in \pi^{-1}\{1\}$ because we have already seen that $\pi^{-1}\{1\}$ is $\supseteq$-closed. Again, since $\pi$ is well-defined, $\underline{A}'(q) \in \pi^{-1}\{1\}$ implies $q \in f(\underline{A}')$. So, $p, q \in f(\underline{A}')$, whence $p \wedge q \in f(\underline{A}')$ because $f(\underline{A}')$ is deductively closed and $p \wedge q \in X$. It follows that $\pi^{-1}\{1\} \ni \underline{A}'(p \wedge q) = C' \cap C''$. □

*Proof of Theorem 5.* For the first part of the Theorem, suppose $f$ satisfies (A2), (A3) and (A5) and is consistent and complete. Remark 1 then teaches that $f$ satisfies (A4). From Lemma 4, we already know $\pi^{-1}\{1\} = \mathcal{F}_f$. It remains to verify that $\pi$ preserves algebraic operations.

$\pi$ *preserves meets.* Let $C, C' \subseteq N$. By Lemma 4, $\pi^{-1}\{1\}$ is both $\supseteq$-closed and $\cap$-closed, so

$$C \cap C' \in \pi^{-1}\{1\} \Leftrightarrow \left(C \in \pi^{-1}\{1\}, \quad C' \in \pi^{-1}\{1\}\right).$$

As $\pi$ is $\{0,1\}$-valued, this means

$$\pi\left(C \cap C'\right) = 1 \Leftrightarrow (\pi(C) = 1, \quad \pi(C') = 1) \Leftrightarrow \pi(C) \wedge \pi(C') = 1.$$

Thus $\pi\left(C \cap C'\right) = \pi(C) \wedge \pi(C')$.

$\pi$ *preserves complements.* Let $\underline{A} \in D^N$ and $p \in X$. For every $i \in N$, the set $A_i$ is consistent and complete, hence

$$p \in A_i \Leftrightarrow {\sim} p \notin A_i.$$

Therefore $\underline{A}(p) = N \setminus \underline{A}({\sim} p) = \complement \underline{A}({\sim} p)$ or equivalently

$$\complement \underline{A}(p) = \underline{A}({\sim} p). \tag{3}$$

But $f(\underline{A})$ is also assumed to be consistent and complete, whence ${\sim} p \in f(\underline{A})$ if and only if $p \notin f(\underline{A})$. Combining this:

$$\pi\left(\complement \underline{A}(p)\right) = 1 \Leftrightarrow \pi\left(\underline{A}({\sim} p)\right) = 1 \Leftrightarrow {\sim} p \in f(\underline{A}) \Leftrightarrow p \notin f(\underline{A}) \Leftrightarrow \pi\left(\underline{A}(p)\right) = 0,$$
$$\pi\left(\complement \underline{A}(p)\right) = 0 \Leftrightarrow \pi\left(\complement \underline{A}(p)\right) \neq 1 \Leftrightarrow \pi\left(\underline{A}(p)\right) \neq 0 \Leftrightarrow \pi\left(\underline{A}(p)\right) = 1.$$

$\pi$ *preserves joins.* Let $C, C' \subseteq N$. First, suppose $\pi(C) \vee \pi(C') = 1$. Then either $\pi(C) = 1$ or $\pi(C') = 1$, hence either $C \in \pi^{-1}\{1\}$ or $C' \in \pi^{-1}\{1\}$. This means that $C \cup C'$ will be the superset of an element of $\pi^{-1}\{1\}$, hence by $\supseteq$-closedness of $\pi^{-1}\{1\}$, we deduce $C \cup C' \in \pi^{-1}\{1\}$, so $\pi\left(C \cup C'\right) = 1$.

Now suppose $\pi(C) \vee \pi(C') = 0$, hence $\pi(C) = \pi(C') = 0$. We have already verified that $\pi$ preserves complements, hence we deduce that $\pi\left(\complement C\right) = \pi\left(\complement C'\right) = 1$. Since we have also already seen that $\pi$ preserves meets, we obtain

$\pi\left(\complement C \cap \complement C'\right) = 1 \wedge 1 = 1$. By de Morgan's law, $\pi\left(\complement(C \cup C')\right) = 1$, hence, again exploiting that $\pi$ preserves complements, we arrive at $\pi\left(C \cup C'\right) = 0$.

Thus, $\pi$ is a homomorphism and the first part of the Theorem established.

For the converse part of the Theorem, suppose $\rho : \mathcal{P}(N) \to 2$ is a homomorphism. We verify:

*Axiom (A2).* (A2) is satisfied by assumption.

*Axiom (A3).* (A3) holds by definition of $f$.

*Axiom (A5).* (A5) also holds by definition of $f$-

*Axiom (A4).* Since $\rho$ is a homomorphism, $\rho(\varnothing) = 0$ and $\rho(N) = 1$. By Remark 13, we can find $\underline{A}, \underline{A}' \in D^N$ and $p, q \in X$ such that $\underline{A}(p) = \varnothing$ and $\underline{A}'(q) = N$. Then, by construction of $f$, both $q \in f\left(\underline{A}'\right)$ and $p \notin f\left(\underline{A}\right)$, so $f\left(\underline{A}'\right) \neq \varnothing$ and $f\left(\underline{A}\right) \neq X$.

Finally, for every $\underline{A} \in D^N$ and $p \in X$, one has $\underline{A}(\sim p) = \complement \underline{A}(p)$ by Equation (3). Hence, using that $\rho$ is a homomorphism,

$$\begin{aligned} p \in f(\underline{A}) \Leftrightarrow \rho\left(\underline{A}(p)\right) = 1 \Leftrightarrow \rho\left(\complement \underline{A}(p)\right) = 0 \Leftrightarrow \rho\left(\underline{A}(\sim p)\right) = 0 \\ \Leftrightarrow \rho\left(\underline{A}(\sim p)\right) \neq 1 \Leftrightarrow \sim p \notin f(\underline{A}). \end{aligned}$$

So, $f(\underline{A})$ is complete and algebraically consistent for every $\underline{A} \in D^N$. □

*Proof of Lemma 7.* Suppose $f$ satisfies (A2-A5). Then $\mathcal{F}_f = \pi^{-1}\{1\}$ is non-empty by (A4) and $\cap$-closed by Lemma 4. Therefore, $\sim_f$ must be a congruence relation (cf. e.g. Bell and Slomson [11, Chapter 1, proof of Lemma 4.3, proof of Lemma 4.4]). For all $C \subseteq N$, denote by $|C|$ the equivalence class of $C$ with respect to $\sim_f$. Since $\sim_f$ is a congruence relation, the operations $\wedge, \vee, ^*$, introduced representative-wise via

$$|C| \wedge |C'| := |C \cap C'|\,, \quad |C| \vee |C'| := |C \cup C'|\,, \quad |C|^* := \left|\complement C\right|$$

for all $C, C' \subseteq N$, are well-defined. If we define, in addition,

$$0_{\sim_f} := |\varnothing|, \qquad 1_{\sim_f} := |N|$$

then through straightforward calculations one can check that $\left\langle \mathcal{P}(N)/\sim_f, \wedge, \vee, ^*, 0_{\sim_f}, 1_{\sim_f} \right\rangle$ is indeed a Boolean algebra. □

*Proof of Theorem 8.* Using the same notation as in the proof of Lemma 7, the map $\sigma : C \mapsto |C|$ trivially preserves the Boolean operations.

For every $C \subseteq N$, one has

$$\begin{aligned} & C \sim_f N \Leftrightarrow \exists U \in \mathcal{F}_f \quad (C \cap U = N \cap U) \Leftrightarrow \exists U \in \mathcal{F}_f \quad (C \cap U = U) \\ & \Leftrightarrow \exists U \in \mathcal{F}_f \quad U \subseteq C, \end{aligned}$$

hence, by the $\supseteq$-closedness of $\mathcal{F}_f$,

$$|C| = |N| \Leftrightarrow C \sim_f N \Leftrightarrow C \in \mathcal{F}_f.$$

Therefore $\sigma^{-1}\left\{1_{\sim_f}\right\} = \sigma^{-1}\left\{|N|\right\} = \mathcal{F}_f$.

Let us move to the converse part of the Theorem. As in the proof of Theorem 5, one can verify that $f$ satisfies axioms (A2-A5).

It remains to show that $f$ is algebraically consistent. Assume otherwise. Then there are $\underline{A} \in D^N$ and $p \in X$ such that both $\sim p \in f(\underline{A})$ and $p \in f(\underline{A})$. Therefore $\tau\left(\underline{A}(p)\right) = f(\underline{A})(p) = 1$ as well as $\tau\left(\underline{A}(\sim p)\right) = g(\underline{A})(\sim p) = 1$. However $\underline{A}(\sim p) = \complement \underline{A}(p)$ by Equation (3), so $\tau\left(\complement \underline{A}(p)\right) = 1$. On the other hand, since $\tau$ is a homomorphism and $\tau\left(\underline{A}(p)\right) = 1$, one has $\tau\left(\complement \underline{A}(p)\right) = 0$, a contradiction. □

*Proof of Theorem 10.* By Corollary 9, $\mathcal{F}_f$ is a filter. Combining Lemma 3 and Equation (2), one gets

$$p \in f(\underline{A}) \Leftrightarrow \underline{A}(p) \in \mathcal{F}_f \Leftrightarrow p \in \prod \underline{A}/\mathcal{F}_f$$

for all $\underline{A} \in D^N$ and $p \in X$.

For the converse part, we verify the properties stipulated in the Theorem:

*Axiom (A2).* By assumption.

*Axiom (A3).* By definition of the reduced product.

*Axiom (A4).* Since $N \in \mathcal{G}$ but $\varnothing \notin \mathcal{G}$, we have $p \in \prod \underline{A}/\mathcal{G} = f(\underline{A})$ if $\underline{A}(p) = N$ but $p \notin \prod \underline{A}/\mathcal{G} = f(\underline{A})$ if $\underline{A}(p) = \varnothing$ (for all $p \in X$ and $\underline{A} \in \mathcal{P}(X)^N$). Therefore, $f$ satisfies strict unanimity preservation. Since $f$ also satisfies (A3), Remark 1 yields that $f$ satisfies (A4).

*Axiom (A5).* Evident from Equation (2).

*Deductive closedness (in $X$) of $f \restriction (D')^N$.* Let $\underline{A} \in (D')^N$ and $q \in X$ with $f(\underline{A}) \vdash q$. Since proofs have finite length, there exists a finite set $Y \subseteq f(\underline{A})$ with $Y \vdash q$. By definition of $f$ as a reduced product, $\underline{A}(p) \in \mathcal{G}$ for all $p \in Y$. Since filters are closed under finite intersections, $\bigcap_{p \in Y} \underline{A}(p) \in \mathcal{G}$. Note that for all $i \in \bigcap_{p \in Y} \underline{A}(p)$, one has $Y \subseteq A_i$ and therefore $A_i \vdash q$, which readily means $q \in A_i$ (since $A_i$ is deductively closed, as $\underline{A} \in (D')^N$, and $q \in X$). Hence $\underline{A}(q) \supseteq \bigcap_{p \in Y} \underline{A}(p) \in \mathcal{G}$, so $\underline{A}(q) \in \mathcal{G}$ and thus $q \in \prod \underline{A}/\mathcal{G} = f(\underline{A})$.

*Consistency of $f \restriction \bar{D}^N$.* Let $\underline{A} \in \bar{D}^N$ and suppose $\prod \underline{A}/\mathcal{G}$ were inconsistent. Then, since proofs have finite length, there must be some inconsistent finite subset $Y \subseteq \prod \underline{A}/\mathcal{G}$. Then, $\underline{A}(p) \in \mathcal{G}$ for all $p \in Y$, hence $\bigcap_{p \in Y} \underline{A}(p) \in \mathcal{G}$ since filters are closed under finite intersections. But this means $\bigcap_{p \in Y} \underline{A}(p) \neq \varnothing$ since filters do not contain $\varnothing$, so there is some $i \in N$ such that $p \in A_i$ for all $p \in Y$, thus $Y \subseteq A_i$. $Y$ being inconsistent, $A_i$ is inconsistent, too, whence $A_i \notin \bar{D}$, a contradiction. □

*Proof of Theorem 11.* By Corollary 6, $\mathcal{F}_f$ is an ultrafilter, and by Theorem 5, $f$ also satisfies (A4). Thus, the first half of Theorem 10 may be applied and yields $f(\underline{A}) = \prod \underline{A}/\mathcal{F}_f$ for all $\underline{A} \in D^N$.

To prove converse part, note that the second half of Theorem 10 already ensures that $f$ satisfies (A2-A5) and that $f$ is consistent.

What remains to be shown is the completeness (in $X$) of $f$. Let $p \in X$. The maximality of the ultrafilter ensures that either $\{i \in N \ : \ p \in A_i\} \in \mathcal{G}$ or

$\{i \in N \ : \ p \notin A_i\} \in \mathcal{G}$. In the former case, already $p \in \prod \underline{A}/\mathcal{G}$. In the latter case, note that for every $i \in N$, one has $p \notin A_i$ if and only if $\sim p \in A_i$ (since $A_i \in D$), whence $\{i \in N \ : \sim p \in A_i\} \in \mathcal{G}$ and therefore $\sim p \in \prod \underline{A}/\mathcal{G}$. $\square$

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