

## RESTRICTED SPACE ALGORITHMS FOR ISOMORPHISM ON BOUNDED TREEWIDTH GRAPHS

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**ABSTRACT.** The Graph Isomorphism problem restricted to graphs of bounded treewidth or bounded tree distance width are known to be solvable in polynomial time [2],[19]. We give restricted space algorithms for these problems proving the following results:

- Isomorphism for bounded tree distance width graphs is in L and thus complete for the class. We also show that for this kind of graphs a canon can be computed within logspace.
- For bounded treewidth graphs, when both input graphs are given together with a tree decomposition, the problem of whether there is an isomorphism which respects the decompositions (i.e. considering only isomorphisms mapping bags in one decomposition blockwise onto bags in the other decomposition) is in L.
- For bounded treewidth graphs, when one of the input graphs is given with a tree decomposition the isomorphism problem is in LogCFL.
- As a corollary the isomorphism problem for bounded treewidth graphs is in LogCFL. This improves the known  $TC^1$  upper bound for the problem given by Grohe and Verbitsky [8].

### 1. Introduction

The Graph Isomorphism problem consists in deciding whether two given graphs are isomorphic, or in other words, whether there exists a bijection between the vertices of both graphs preserving the edge relation. Graph Isomorphism is a well studied problem in NP because of its many applications and also because it is one of the few natural problems in this class not known to be solvable in polynomial time nor known to be NP-complete. Although for the case of general graphs no efficient algorithm for the problem is known, the situation is much better when certain parameters in the input graphs are bounded by a constant. For

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example the isomorphism problem for graphs of bounded degree [13], bounded genus [15], bounded color classes [14], or bounded treewidth [2] is known to be in P. Recently some of these upper bounds have been improved with the development of space efficient techniques, most notably Reingold's deterministic logspace algorithm for connectivity in undirected graphs [16]. In some cases logspace algorithms have been obtained. For example graph isomorphism for trees [12], planar graphs [5] or  $k$ -trees [10]. In other cases the problem has been classified in some other small complexity classes below P. The isomorphism problem for graphs of bounded treewidth is known to be in  $TC^1$  [8] and the problem restricted to graphs of bounded color classes is known to be in the #L hierarchy [1].

In this paper we address the question of whether the isomorphism problem restricted to graphs of bounded treewidth and bounded tree distance width can be solved in logspace. Intuitively speaking, the treewidth of a graph measures how much it differs from a tree. This concept has been used very successfully in algorithmics and fixed-parameter tractability (see e.g. [3, 4]). For many complex problems, efficient algorithms have been found for the cases when the input structures have bounded treewidth. As mentioned above Bodlaender showed in [2] that Graph Isomorphism can be solved in polynomial time when restricted to graphs of bounded treewidth. More recently Grohe and Verbitsky [8] improved this upper bound to  $TC^1$ . In this paper we improve this result showing that the isomorphism problem for bounded treewidth graphs lies in LogCFL, the class of problems logarithmic space reducible to a context free language. LogCFL can be alternatively characterized as the class of problems computable by a uniform family of polynomial size and logarithmic depth circuits with bounded AND and unbounded OR gates, and is therefore a subclass of  $TC^1$ . LogCFL is also the best known upper bound for computing a tree decomposition of bounded treewidth graphs [18, 7], which is one bottleneck in our isomorphism algorithm. We prove that if tree decompositions of both graphs are given as part of the input, the question of whether there is an isomorphism respecting the vertex partition defined by the decompositions can be solved in logarithmic space. Our proof techniques are based on methods from recent isomorphism results [5, 6] and are very different from those in [8].

The notion of tree distance width, a stronger version of the treewidth concept, was introduced in [19]. There it is shown that for graphs with bounded tree distance width the isomorphism problem is fixed parameter tractable, something that is not known to hold for the more general class of bounded treewidth graphs. We prove that for graphs of bounded tree distance width it is possible to obtain a tree distance decomposition within logspace. Using this result we show that graph isomorphism for bounded tree distance width graphs can also be solved in logarithmic space. Since it is known that the question is also hard for the class L under  $AC^0$  reductions [9], this exactly characterizes the complexity of the problem. We show that in fact a canon for graphs of bounded tree distance width, i.e. a fixed representative of the isomorphism equivalence class, can be computed in logspace. Due to space reasons, some proofs are omitted and will be provided in the full version of the paper.

## 2. Preliminaries

We introduce the complexity classes used in this paper. L is the class of decision problems computable by deterministic logarithmic space Turing machines. LogCFL consists of all decision problems that can be Turing reduced in logarithmic space to a context free language. There are several alternative more intuitive characterizations of LogCFL. Problems

in this class can be computed by uniform families of polynomial size and logarithmic depth circuits over bounded fan-in AND gates and unbounded fan-in OR gates. We will also use the characterization of LogCFL as the class of decisional problems computable by non-deterministic auxiliary pushdown machines (NAuxPDA). These are Turing machines with a logarithmic space work tape, an additional pushdown and a polynomial time bound [17]. The class  $TC^1$  contains the problems computable by uniform families of polynomial size and logarithmic depth threshold circuits. The known relationships among these classes are:

$$L \subseteq \text{LogCFL} \subseteq TC^1.$$

In this paper we consider undirected simple graphs with no self loops. For a graph  $G = (V, E)$  and two vertices  $u, v \in V$ ,  $d_G(u, v)$  denotes the distance between  $u$  and  $v$  in  $G$  (number of edges in the shortest path between  $u$  and  $v$  in  $G$ ). For a set  $S \subseteq V$ , and a vertex  $u \in V$ ,  $d_G(S, u)$  denotes  $\min_{v \in S} d_G(v, u)$ .  $\Gamma(S)$  denotes the set of neighbors of  $S$  in  $G$ . In a connected graph  $G$ , a separating set is a set of vertices such that deleting the vertices in  $S$  (and the edges connected to them) produces more than one connected component. For  $G = (V, E)$  and two disjoint subsets  $U, W$  of  $V$  we use the following notion for an *induced bipartite subgraph*  $B_G[U, W]$  of  $G$  on vertex set  $U \cup W$  with edge set  $\{\{u, w\} \in E \mid u \in U, w \in W\}$ . Let  $G[U]$  be the *induced subgraph* of  $G$  on vertex set  $V \setminus U$ .

A *tree decomposition* of a graph  $G = (V, E)$  is a pair  $(\{X_i \mid i \in I\}, T = (I, F))$ , where  $\{X_i \mid i \in I\}$  is a collection of subsets of  $V$  called bags, and  $T$  is a tree with node set  $I$  and edge set  $F$ , satisfying the following properties:

- i)  $\bigcup_{i \in I} X_i = V$
- ii) for each  $\{u, v\} \in E$ , there is an  $i \in I$  with  $u, v \in X_i$  and
- iii) for each  $v \in V$ , the set of nodes  $\{i \mid v \in X_i\}$  forms a subtree of  $T$ .

The *width* of a tree decomposition of  $G$ , is defined as  $\max\{|X_i| \mid i \in I\} - 1$ . The *treewidth* of  $G$  is the minimum width over all tree decompositions of  $G$ .

A *tree distance decomposition* of a graph  $G = (V, E)$  is a triple  $(\{X_i \mid i \in I\}, T = (I, F), r)$ , where  $\{X_i \mid i \in I\}$  is a collection of subsets of  $V$  called bags,  $X_r = S$  a set of vertices and  $T$  is a tree with node set  $I$ , edge set  $F$  and root  $r$ , satisfying:

- i)  $\bigcup_{i \in I} X_i = V$  and for all  $i \neq j, X_i \cap X_j = \emptyset$
- ii) for each  $v \in V$ , if  $v \in X_i$  then  $d_G(X_r, v) = d_T(r, i)$  and
- iii) for each  $\{u, v\} \in E(G)$ , there are  $i, j \in I$  with  $u \in X_i, v \in X_j$  and  $i = j$  or  $\{i, j\} \in F$  (for every edge in  $G$  its two endpoints belong to the same or to adjacent bags in  $T$ ).

Let  $D = (\{X_i \mid i \in I\}, T = (I, F), r)$  be a tree distance decomposition of  $G$ .  $X_r$  is the *root bag* of  $D$ . The *width* of  $D$  is the maximum number of elements of a bag  $X_i$ . The *tree distance width* of  $G$  is the minimum width over all tree distance decompositions of  $G$ .

The tree distance decomposition  $D$  is called *minimal* if for each  $i \in I$ , the set of vertices in the bags with labels in the subtree rooted at  $i$  in  $T$  induce a connected subgraph in  $G$ . In [19] it is shown that for every root set  $S \subseteq V$  there is a unique minimal tree distance decomposition of  $G$  with root set  $S$ . The width of such a decomposition is minimal among the tree distance decompositions of  $G$  with root set  $S$ .

An isomorphism from  $G$  onto  $H$  *respects* their tree (distance) decompositions  $D, D'$  if vertices in a bag of  $D$  in  $G$  are mapped blockwise onto vertices in a bag of  $D'$  in  $H$ . Not every isomorphism has this property.

$Sym(V)$  is the *symmetric group* on a set  $V$ .

### 3. Graphs of bounded tree distance width

#### 3.1. Tree distance decomposition in L

We describe an algorithm that on input a graph  $G$  and a subset  $S \subset V$  produces the minimal tree distance decomposition  $D = (\{X_i \mid i \in I\}, T = (I, F), r)$  of  $G$  with root set  $X_r = S$ . The algorithm works within space  $c \cdot k \log n$  for some constant  $c$ , where  $k$  is the width of the minimal tree distance decomposition of  $G$  with root set  $S$ . The output of the algorithm is a sequence of strings of the form ( bag label, bag depth,  $v_{i_1}, v_{i_2}, \dots, v_{i_l}$ ), indicating the number of the bag, the distance of its elements to  $S$  and the list of the elements in the bag.

The algorithm basically performs a depth first traversal of the tree  $T$  in the decomposition while constructing it. Starting at  $S$  the algorithm uses three functions for traversing  $T$ . These functions perform queries to a logspace subroutine computing reachability [16].

**Parent**( $X_i$ ): On input the elements of a bag  $X_i$  the function returns the elements of the parent bag in  $T$ . These are the vertices  $v \in V$  with the following two properties:  $v \in \Gamma(X_i) \setminus X_i$  and  $v$  is reachable from  $S$  in  $G \setminus X_i$ . For a vertex  $v$  these two properties can be tested in space  $O(\log n)$  by an algorithm with input  $G, S$  and  $X_i$ . In order to find all the vertices in the parent set, the algorithm searches through all the vertices in  $V$ .

**First Child**( $X_i$ ): This function returns the elements of the first child of  $i$  in  $T$ . This is the child with the vertex  $v_j \in V$  with the smallest index  $j$ .  $v_j$  satisfies that  $v_j \in \Gamma(X_i) \setminus X_i$  and that  $v_j$  is not reachable from  $S$  in  $G \setminus X_i$ . It can be found cycling in order through the vertices of  $G$  until the first one satisfying the properties is found. The other elements  $w \in X_i$  must satisfy the same two properties as  $v_j$  and additionally, they must be in the same connected component in  $G \setminus X_i$  where  $v_j$  is contained. In case  $X_i$  does not have any children, the function outputs some special symbol.

**Next Sibling**( $X_i$ ): This function first computes  $X_p := \text{Parent}(X_i)$  and then searches for the child of  $p$  in  $T$  next to  $X_i$ . Let  $v_i$  be the vertex with the smallest label in  $X_i$ . This is done similarly as the computation of First Child. The next sibling is the bag containing the unique vertex  $v_j$  with the following properties:  $v_j$  is the vertex with the smallest label in this bag,  $\text{label}(v_j) > \text{label}(v_i)$  and there is no other bag which has a vertex with a label  $> v_i$  and  $< v_j$ . The vertex  $v_j$  is not reachable from  $S$  in  $G \setminus X_p$ . The other elements in the bag are the vertices satisfying these properties and which are in the same connected component of  $G \setminus X_p$  where  $v_j$  is contained.

With these three functions the algorithm performs a depth-first traversal of  $T$ . It only needs to remember the initial bag  $X_0 = S$  which is part of the input, and the elements of the current bag. On a bag  $X_i$  it searches for its first child. If it does not exist then it searches for the next sibling. When there are no further siblings the next move goes up in the tree  $T$ . The algorithm finishes when it returns to  $S$ . It also keeps two counters in order to be able to output the number and depth of the bags. The three mentioned functions only need to keep at most two bags ( $X_i$  and its father) in memory, and work in logarithmic space. On input a graph  $G$  with  $n$  vertices, and a root set  $S$ , the space used by the algorithm is therefore bounded by  $c \cdot k \log n$ , for a constant  $c$ , and  $k$  being the minimum width of a tree distance decomposition of  $G$  with root set  $S$ . When considering how the three functions are

defined it is clear that the algorithm constructs a tree distance decomposition with root set  $S$ . Also they make sure that for each  $i$  the subgraph induced by the vertices of the bags in the subtree rooted at  $i$  is connected thus producing a minimal decomposition. As observed in [19], this is the unique minimal tree distance decomposition of  $G$  with root set  $S$ .

### 3.2. Isomorphism Algorithm for Bounded Tree Distance Width Graphs

For our isomorphism algorithm we use a tree called the *augmented tree* which is based on the underlying tree of a minimal tree distance decomposition. This augmented tree, apart from the bags, contains information about the separating sets which separate bags.

**Definition 3.1.** Let  $G$  be a bounded tree distance width graph with a minimal tree distance decomposition  $D = (\{X_i \mid i \in I\}, T = (I, F), r)$ . The *augmented tree*  $\mathcal{T}_{(G,D)} = (I_{(G,D)}, F_{(G,D)}, r)$  corresponding to  $G$  and  $D$  is a tree defined as follows:

- The set of nodes of  $\mathcal{T}_{(G,D)}$  is  $I_{(G,D)}$  which contains two kinds of nodes, namely  $I_{(G,D)} = I \cup J$ . Those in  $I$  form the set of *bag nodes* in  $D$ , and those in  $J$  the *separating set nodes*. For each bag node  $a \in I$  and each child  $b$  of  $a$  in  $T$  we consider the set  $X_a \cap \Gamma(X_b)$ , i.e. the *minimum separating set* in  $X_a$  which separates  $X_b$  from the root bag  $X_r$  in  $G$ . Let  $M_{s_1^a}, \dots, M_{s_{l(a)}^a}$  be the set of all minimum separating sets in  $X_a$ , free of duplicates. There are nodes for these sets  $s_1^a, \dots, s_{l(a)}^a$ , the separating set nodes. We define  $J = \bigcup_{a \in I} \{s_1^a, \dots, s_{l(a)}^a\}$ . The node  $r \in I$  is the root in  $\mathcal{T}_{(G,D)}$ .
- In  $F_{(G,D)}$  there are edges between bag nodes  $a \in I$  and the separating set nodes  $s_1^a, \dots, s_{l(a)}^a \in J$  (edges between bag nodes and their children in the augmented tree). There are also edges between nodes  $b \in I$  and  $s_j^a$  if  $M_{s_j^a}$  is the minimum separating set in  $X_a$  which separates  $X_b$  from  $X_r$  (edges between bag nodes and their parents).

To simplify notation, we later say for example that  $s_1, \dots, s_l$  are the children of a bag node  $a$  if the context is clear. The odd levels of the augmented tree  $T'$  correspond to bag nodes and the even levels correspond to separating set nodes.

Observe that for each node in the augmented tree, we associate a bag to a bag node and a minimum separating set to a separating set node. Hence, every vertex  $v$  in the original graph occurs in at least one associated component and it might occur in more than one, e.g. if  $v$  is contained in a bag and in a minimum separating set.

Let  $T_{(G,D)}$  be an augmented tree of some minimal tree distance decomposition  $D$  of a graph  $G$ . Let  $a$  be a node of  $T_{(G,D)}$ . The subtree of  $T_{(G,D)}$  rooted at  $a$  is denoted by  $T_a$ . Note that  $T_{(G,D)} = T_r$  where  $X_r$  is the bag corresponding to the root of the tree distance decomposition  $D$ . We define  $\text{graph}(T_a)$  as the subgraph of  $G$  induced by all the vertices associated to at least one of the nodes of  $T_a$ . The *size* of  $T_a$ , denoted  $|T_a|$  is the number of vertices which occur in at least one component which is associated to a node in  $T_a$ . Note,  $|T_a|$  is polynomially related to  $|\text{graph}(T_a)|$ , i.e. the number of vertices in the corresponding subgraph of  $G$ .

When given a tree distance decomposition the augmented tree can be computed in logspace. Using the result in Section 3.1 we immediately get:

**Lemma 3.2.** *Let  $G$  be a graph of bounded tree distance width. The augmented tree for  $G$  can be computed in logspace.*

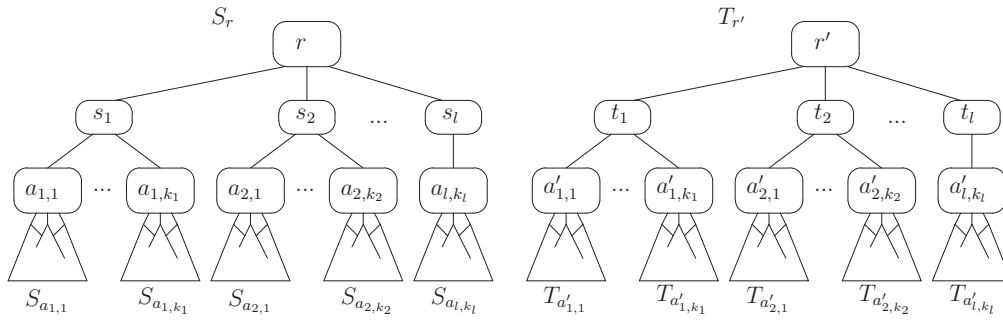


Figure 1: The augmented trees  $S_r$  and  $T_{r'}$  rooted at bag nodes  $r$  and  $r'$ . Node  $r$  has separating set nodes  $s_1, \dots, s_l$  as children. The children of  $s_1$  are again bag nodes  $a_{1,1}, \dots, a_{1,k_1}$ .  $S_{a_{i,j}}$  is the subtree rooted at  $a_{i,j}$ . Bag nodes and separating set nodes alternate in the tree.

**Isomorphism Order of Augmented Trees.** We describe an isomorphism order procedure for comparing two augmented trees  $S_{(G,D)}$  and  $T_{(H,D')}$  corresponding to the graphs  $G$  and  $H$  and their tree distance decompositions  $D$  and  $D'$ , respectively. This isomorphism order algorithm is an extension of the one for trees given by Lindell [12] and it is different from that for planar graphs given by Datta et al. [5]. The trees  $S_{(G,D)}$  and  $T_{(H,D')}$  are rooted at bag nodes  $r$  and  $r'$ . The rooted trees are denoted then  $S_r$  and  $T_{r'}$  as shown in Figure 1.

We will show that two graphs of bounded tree distance width are isomorphic if and only if for some root nodes  $r$  and  $r'$  the augmented trees corresponding to the minimal tree distance decompositions have the same isomorphism order.

The isomorphism order depends on the order of the vertices in the bags  $r$  and  $r'$ . Let  $X_r$  and  $X_{r'}$  be the corresponding bags in  $D$  and  $D'$ . We define the sets of mappings  $\Theta_{(r,r')} = \text{Sym}(X_r) \times \text{Sym}(X_{r'})$ . Let  $(\sigma, \sigma')$  be such a mapping, then the tuples  $(G[X_r], \sigma)$  and  $(G[X_{r'}], \sigma')$  describe a fixed ordering on the vertices of the induced subgraphs. If  $r$  is not the top-level root of the augmented tree then  $\Theta_{(r,r')}$  may become restricted to a subset, when going into recursion. The isomorphism order is defined to be  $S_r <_{\mathbb{T}} T_{r'}$  if there exist mappings  $(\sigma, \sigma') \in \Theta_{(r,r')}$  such that one of the following holds:

- 1)  $(G[X_r], \sigma) < (H[X_{r'}], \sigma')$  via lexicographical comparison of both ordered subgraphs
- 2)  $(G[X_r], \sigma) = (H[X_{r'}], \sigma')$  but  $|S_r| < |T_{r'}|$
- 3)  $(G[X_r], \sigma) = (H[X_{r'}], \sigma')$  and  $|S_r| = |T_{r'}|$  but  $\#r < \#r'$  where  $\#r$  and  $\#r'$  is the number of children of  $r$  and  $r'$
- 4)  $(G[X_r], \sigma) = (H[X_{r'}], \sigma')$  and  $|S_r| = |T_{r'}|$  and  $\#r = \#r' = l$  but  $(S_{s_1}, \dots, S_{s_l}) <_{\mathbb{T}} (T_{t_1}, \dots, T_{t_l})$  where we assume that  $S_{s_1} \leq_{\mathbb{T}} \dots \leq_{\mathbb{T}} S_{s_l}$  and  $T_{t_1} \leq_{\mathbb{T}} \dots \leq_{\mathbb{T}} T_{t_l}$  are ordered subtrees of  $S_r$  and  $T_{r'}$ , respectively. To compute the order between the subtrees  $S_{s_i} \leq_{\mathbb{T}} T_{t_j}$  we consider

- i*: the lexicographical order of the minimal separating sets ( $s_i$  and  $t_j$ ) in  $X_r$  and  $X_{r'}$  according to  $\sigma$  and  $\sigma'$ , as the primary criterion (observe that the separating sets are subsets of  $X_r$  (resp.  $X_{r'}$ ) and are therefore ordered by  $\sigma$  and  $\sigma'$ ) and
- ii*: pairwise the children  $a_{i,i'}$  of  $s_i$  and  $a'_{j,j'}$  of  $t_j$  (for all  $i'$  and  $j'$ ) via cross-comparisons such that the induced bipartite graphs  $B_G[s_i, a_{i,i'}]$  and

$B_H[t_j, a'_{j,j'}]$  can be *matched* according to  $\sigma$  and  $\sigma'$  (i.e.  $\sigma\sigma'^{-1}$  is an isomorphism) and

*iii: recursively* the subtrees rooted at the children of  $s_i$  and  $t_j$ . Note, that these children are again bag nodes. For the cross comparison of bag nodes  $a_{i,i'}$  and  $a'_{j,j'}$  we restrict the set  $\Theta_{(a_{i,i'}, a'_{j,j'})}$  to a subset of  $Sym(X_{a_{i,i'}}) \times Sym(X'_{a'_{j,j'}})$ . Namely,  $\Theta_{(a_{i,i'}, a'_{j,j'})}$  contains the pair  $(\phi, \phi') \in Sym(X_{a_{i,i'}}) \times Sym(X'_{a'_{j,j'}})$  if  $\phi\phi'^{-1}$  extends the partial isomorphism  $\sigma\sigma'^{-1}$  from child  $a_{i,i'}$  onto  $a'_{j,j'}$  blockwise and which induces an isomorphism from  $B_G[s_i, a_{i,i'}]$  onto  $B_H[t_j, a'_{j,j'}]$ .

We say that two augmented trees  $S_r$  and  $T_{r'}$  are *equal according to the isomorphism order*, denoted  $S_r =_{\mathcal{T}} T_{r'}$ , if neither  $S_r <_{\mathcal{T}} T_{r'}$  nor  $T_{r'} <_{\mathcal{T}} S_r$  holds.

**Isomorphism of two subtrees rooted at bag nodes  $r$  and  $r'$ .** We have constant size components associated to the bag nodes. A logspace machine can easily run through all the mappings of  $X_r$  and  $X_{r'}$  and record the mappings which gives the minimum isomorphism order. This can be done with cross-comparison of trees  $(S_r, \sigma)$  and  $(T_{r'}, \sigma')$  with all possible mappings  $\sigma, \sigma'$ . Later we will see, that in recursion not all possible mappings for  $\sigma$  and  $\sigma'$  are considered. Observe that  $|Sym(X_r)| \in O(1)$ .

The comparison of  $(S_r, \sigma)$  and  $(T_{r'}, \sigma')$  itself can be done simply by renaming the vertices of  $X_r$  and  $X_{r'}$  according to the mappings  $\sigma$  and  $\sigma'$  and then comparing the ordered sequence of edges lexicographically. When equality is found then we recursively compute the isomorphism order of the subtrees rooted at the children of  $r$  and  $r'$ .

**Isomorphism of two subtrees rooted at separating set nodes  $s_i$  and  $t_j$ .** Datta et.al. [5] decompose biconnected planar graphs into triconnected components and obtain a tree on these components and separating pairs, i.e. separating sets of size two. We have separating sets of arbitrary constant size.

Since  $s_i$  and  $t_j$  correspond to subgraphs of  $X_r$  and  $X_{r'}$ , we have an order for them given by the fixed mappings  $\sigma$  and  $\sigma'$ . Therefore, we can order the children  $s_1, \dots, s_l$  and  $t_1, \dots, t_l$  according to their occurrence in  $X_r$  and  $X_{r'}$  (e.g. assume  $s_i = (1, 2, 3, 7)$  according to the mapping  $\sigma$  and also  $s_j = (1, 2, 4, 7)$ , then we get  $(s_i, \sigma) <_{\mathcal{T}} (s_j, \sigma)$ ). Hence, when comparing  $s_i$  with  $t_j$  we have to check whether both come on the same position in that order of  $s_1, \dots, s_l$  and  $t_1, \dots, t_l$ . If so, then we go to the next level in the tree, to the children of  $s_i$  and  $t_j$ .

Now we have a cross comparison among the children of  $s_i$  and the children of  $t_j$ . In Steps 4*i*, 4*ii* and 4*iii* we partition the children  $a_{i,1}, \dots, a_{i,l_i}$  of  $s_i$  and  $a'_{j,1}, \dots, a'_{j,l_j}$  of  $t_j$ , respectively, into isomorphism classes, step by step.

The membership of a child to a class according to Step 4*i* and 4*ii* can be recomputed. It suffices to keep counters on the work-tape to notice the current class and traversing the siblings from left to right. After these two steps,  $a_{i,i'}$  and  $a'_{j,j'}$  are in the same class if and only if vertices of  $s_i$  and  $t_j$  appear lexicographically at the same positions in  $\sigma$  and  $\sigma'$  and the bipartite graphs  $B[s_i, a_{i,i'}]$  and  $B[t_j, a'_{j,j'}]$  are isomorphic where  $s_i$  is mapped onto  $t_j$  blockwise corresponding to  $\sigma\sigma'^{-1}$  in an isomorphism. In Step 4*iii* we go into recursion and compare members of one class which are rooted at subtrees of the same size. When going into recursion at  $a_{i,i'}$  and  $a'_{j,j'}$  we consider only those mappings from  $(\phi, \phi') \in \Theta_{(a_{i,i'}, a'_{j,j'})}$  which induce an isomorphism  $\phi\phi'^{-1}$  from  $B[s_i, a_{i,i'}]$  onto  $B[t_j, a'_{j,j'}]$ .

**Correctness of the isomorphism order.** Both, the bag nodes and the separating set nodes correspond to subgraphs which are basically separating sets. A bag separates all its subtrees from the root and the separating set nodes refine the bag to separating sets of minimum size. Hence, a partial isomorphism is constructed and extended from each node to its child nodes, traversing the augmented tree (the whole graph, accordingly) in depth first manner. In the recursion, the isomorphism between the roots of the current subtrees, say  $S_r$  and  $T_{r'}$ , is partially fixed by the partial isomorphism between their parents. With an exhaustive search we check every possible remaining isomorphism from  $X_r$  onto  $X_{r'}$ , and go into recursion again partially fixing the isomorphism for the subtrees rooted at children of  $r$  and  $r'$ . By an inductive argument, the partial isomorphism described for the augmented tree can be followed simultaneously in the original graph and we get:

**Theorem 3.3.** *The graphs  $G$  and  $H$  of bounded tree distance width are isomorphic if and only if there is a choice of a root bag  $r$  and  $r'$  producing augmented trees  $S_r$  and  $T_{r'}$  such that  $S_r \cong T_{r'}$ . The isomorphism order between two augmented trees of  $G$  and  $H$  can be computed in logspace.*

The proof is based on a careful space analysis at each computational step building on concepts of the isomorphism order algorithm of Lindell [12]. The isomorphism order is the basis for a canonization procedure. This is shown in a full version of this paper.

**Theorem 3.4.** *A graph of bounded tree distance width can be canonized in logspace.*

#### 4. Graphs of bounded treewidth

In this section we consider several isomorphism problems for graphs of bounded treewidth. We are interested in isomorphisms *respecting* the decompositions (i.e. vertices are mapped blockwise from a bag to another bag). We show first that if the tree decomposition of both input graphs is part of the input then the isomorphism problem can be decided in L. We also show that if a tree decomposition of only one of the two given graphs is part of the input, then the isomorphism problem is in LogCFL. It follows that the isomorphism problem for graphs of bounded treewidth is also in LogCFL.

Assume the decompositions of both input graphs are given. Let  $(G, D), (H, D')$  be two bounded treewidth graphs together with tree decompositions  $D$  and  $D'$ , respectively. We look for an isomorphism between  $G$  and  $H$  satisfying the condition that the images of the vertices in one bag in  $D$  belong to the same bag in  $D'$ .

We prove that this problem is in L. For this we show that given tree decompositions together with designated bags as roots for  $G$  and  $H$  the question of whether there is an isomorphism between the graphs mapping root to root and respecting the decompositions (i.e. mapping bags in  $G$  blockwise onto bags in  $H$ ) can be reduced to the isomorphism problem for graphs of bounded tree distance decomposition. We argued in the previous section that this problem belongs to L.

**Theorem 4.1.** *The isomorphism problem for bounded treewidth graphs with given tree decompositions reduces to isomorphism for bounded tree distance width graphs under  $AC^0$  many-one reductions.*

Since bounded tree distance width GI is in L, this almost proves the desired result. To obtain it, we have to find roots for the tree decompositions. We fix an arbitrary bag in the one graph and try all bags from the decomposition of the other graph as roots. We get:



**Corollary 4.2.** *For every  $k \geq 1$  there is a logarithmic space algorithm that, on input a pair of graphs together with a tree decompositions of width  $k$  for each of them, decides whether there is an isomorphism between the graphs, respecting the decompositions.*

#### 4.1. A LogCFL algorithm for isomorphism

We consider now the more difficult situation in which only one of the input graphs is given together with a tree decomposition.

**Theorem 4.3.** *Isomorphism testing for two graphs of bounded treewidth, when a tree decomposition for one of them is given, can be done in LogCFL.*

*Proof.* We describe an algorithm which runs on a non-deterministic auxiliary pushdown automaton (NAuxPDA). Besides a read-only input tape and a finite control, this machine has access to a stack of polynomial size and a  $O(\log n)$  space bounded work-tape. On the input tape we have two graphs  $G, H$  of treewidth  $k$  and a tree decomposition  $D = (\{X_i \mid i \in I\}, T = (I, F), r)$  for  $G$ . For  $j \in I$  we define  $G_j$  to be the subgraph of  $G$  induced on the vertex set  $\{v \mid v \in X_i, i \in I \text{ and } i = j \text{ or } i \text{ a descendant of } j \text{ in } T\}$ . That is,  $G_j$  contains the vertices which are separated by the bag  $X_j$  from  $X_r$  and those in  $X_j$ . We define  $D_j = (\{X_i, \mid, i \in I_j\}, T_j = (I_j, F_j), j)$  as the tree decomposition of  $G_j$  corresponding to  $T_j$ , the subtree of  $T$  rooted at  $j$ . We also consider a way to order the children of a node in the tree decomposition:

**Definition 4.4.** Let  $1, \dots, l$  be the children of  $r$  in the tree  $T$ . We define the *lexicographical subtree order*, as the order among the subtrees  $(G_1, D_1), \dots, (G_l, D_l)$  which is given by:  $(G_i, D_i) < (G_j, D_j)$  iff there is a vertex  $w \in V(G_i) \setminus X_r$  which has a smaller label than every vertex in  $V(G_j) \setminus X_r$ .

The algorithm non-deterministically guesses two main structures. First, we guess a tree decomposition of width  $k$  for  $H$ . This is done in a similar way as in the LogCFL algorithm from Wanke [18] for testing that a graph has bounded treewidth. Second, we guess an isomorphism  $\phi$  from  $G$  to  $H$  by extending partial mappings from bag to bag.

Very simplified, Wanke’s algorithm on input a graph  $H$  starts guessing a root bag and it guesses then non-deterministically further bags in the decomposition using the pushdown to test that these bags fulfill the properties of a tree decomposition and that every edge in  $G$  is included in some bag. Our algorithm simulates Wanke’s algorithm as a subroutine. In the description of the new algorithm we concentrate on the isomorphism testing part and hide the details of how to choose the bags. For simplicity the sentence “guess a bag  $X_j$  in  $H$  according to Wanke’s algorithm” means that we simulate the guessing steps from Wanke, checking at the same time that the constructed structure is in fact a tree decomposition. Note, if the bags were not chosen appropriately, then the algorithm would halt and reject.

We start guessing a root bag  $X'_r$  of size  $\leq k + 1$  for a decomposition of  $H$ . With  $X'_r$  as root bag we guess the tree decomposition  $D'$  of  $H$  which corresponds to  $D$  and its root  $r$ . We also construct a mapping  $\phi$  describing a partial isomorphism from the vertices of  $G$  onto the vertices of  $H$ . At the beginning,  $\phi$  is the empty mapping and we guess an extension of  $\phi$  from  $X_r$  onto  $X'_{r'}$ . The algorithm starts with  $a = r$  (and  $a' = r'$ ). Then we describe isomorphism classes for  $1, \dots, l$ , the children of  $a$ . First, the children of  $a$  can be distinguished because  $X_1, \dots, X_l$  may intersect with  $X_a$  differently. Second, we further partition the children within one class according to the number of

isomorphic siblings in that class. This can be done in logspace with cross comparisons of pairs among  $(G_1, D_1), \dots, (G_l, D_l)$ , see Corollary 4.2. It suffices to order the isomorphism classes according to the lexicographical subtree order of the members in the classes. We compare then the children of  $a$  with guessed children of  $a'$  keeping the following information: For each isomorphism class we check whether there is the same number of isomorphic subtrees of  $a'$  in  $H$  and whether those intersect with  $X'_{a'}$ , accordingly. For this we use the lexicographical subtree order to go through the isomorphic siblings from left to right, just keeping a pointer to the current child on the work tape. For two such children, say  $s_1$  of  $a$  and  $t_1$  of  $a'$ , we check then recursively whether  $(G_1, D_1)$  is isomorphic to the corresponding subgraph of  $t_1$  in  $H$ , by an extension of  $\phi$ .

When we go into recursion, we push on the stack  $O(\log n)$  bits for a description of  $X_a$  and  $X'_{a'}$  as well as a description of the partial mapping  $\phi$  from  $X_a$  onto  $X'_{a'}$ .

In general, we do not keep all the information of  $\phi$  on the stack. We only have the partial isomorphism  $\phi : \{v \mid v \in X_r \cup \dots \cup X_a\} \rightarrow \{v \mid v \in X'_{r'} \cup \dots \cup X'_{a'}\}$ , where  $r, \dots, a$  ( $r', \dots, a'$ , respectively) is a simple path in  $T$  from the root to the node at the current level of recursion. After we ran through all children of some node we go one level up in recursion and recompute all the other information which is given implicitly by the subtrees from which we returned. Suppose now, we returned to the bag  $X_a$ , we have to do the following:

- Pop from the stack the partial isomorphism  $\phi$  of the bags  $X_a$  onto  $X'_{a'}$ .
- Compute the lexicographical next isomorphic sibling. For this we consider the partition into isomorphism classes according to  $\phi$  and the lexicographical subtree order of Definition 4.4. Recall, isomorphism testing of two subtrees of  $X_a$  can be done in logspace.
- If there is no such sibling then we compute the lexicographical first child of  $X_a$  inside the same isomorphism class. From this child of  $X_a$  we compute the sibling which is not in the same isomorphism class and which comes next to the right in the lexicographical subtree order.
- If there is neither a further sibling in the same isomorphism class nor a non-isomorphic sibling of higher lexicographical order then we ran through all children of  $X_a$  and we are ready to further return one level up in recursion.

Also for  $X'_{a'}$  we guess all children in an isomorphism class from left to right in lexicographical subtree order. If there is no further level to go up in recursion then the stack is empty and we halt in an accepting state. Algorithm 1 summarizes the above considerations.

In Line 1, we guess an extension of  $\phi$  to include a mapping from  $X_a$  onto  $X'_{a'}$ . We know the partial isomorphism of their parent bags since this information can be found on the top of the stack. In Line 3, we have e.g. the partition  $E_1 = \{T_1, \dots, T_{l_1}\}$ ,  $E_2 = \{T_{l_1+1}, \dots, T_{l_2}\}$  and so on. It can be obtained in logspace by testing isomorphism of the tree structures  $(G_1, D_1), \dots, (G_l, D_l)$ . Two subtrees rooted at  $X_i$  and  $X_j$  are in the same isomorphism class iff there is an automorphism in  $G$  which maps  $X_i$  onto  $X_j$  and fixes their parent  $X_a$  setwise. In Lines 6 to 9, we guess  $X'_{i'}$  in  $H$  which corresponds to  $X_i$ , we test recursively whether the corresponding subgraphs  $G_i$  and  $H_{i'}$  are isomorphic with an extension of  $\phi$ . In Line 7, we check whether  $X'_{i'}$  fulfills the properties of a correct tree-decomposition as in Wanke's algorithm (i.e.  $X'_{i'}$  must be a separating set which separates its split components from the vertices in  $X'_{a'} \setminus X'_{i'}$ ).

To see that the algorithm correctly computes an isomorphism, we make the following observation. A bag  $X_a$  is a separating set which defines the connected subgraphs  $G_1, \dots, G_l$ .

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**Algorithm 1** Treewidth Isomorphism with one tree decomposition

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**Input:** Graphs  $G, H$ , tree decomposition  $D$  for  $G$ , bags  $X_a$  in  $G$  and  $X'_{a'}$  in  $H$ .

**Top of Stack:** Partial isomorphism  $\phi$  mapping the vertices in the parent bag of  $X_a$  onto the vertices in the parent bag of  $X'_{a'}$ .

**Output:** Accept, if  $G$  is isomorphic to  $H$  by an extension of  $\phi$ .

- 1: Guess an extension of  $\phi$  to a partial isomorphism from  $X_a$  onto  $X'_{a'}$
  - 2: **if**  $\phi$  cannot be extended to a partial isomorphism which maps  $X_a$  onto  $X'_{a'}$  **then** reject
  - 3: Let  $1, \dots, l$  be the children of  $a$  in  $T$ . Partition the subtrees of  $T$  rooted at  $1, \dots, l$  into  $p$  isomorphism classes  $E_1, \dots, E_p$
  - 4: **for** each class  $E_j$  from  $j = 1$  to  $p$
  - 5:   **for** each subtree  $T_i \in E_j$  (in lexicographical subtree order)
  - 6:     guess a bag  $X'_{i'}$  in  $H$  (in increasing lexicographical subtree order). Let  $H_{i'}$  be the subgraph of  $H$  induced by the vertices in  $X'_{i'}$  and by those which are separated from  $X'_{r'}$  in  $H \setminus X'_{i'}$
  - 7:     **if**  $X'_{i'}$  is not a correct child bag of  $X'_{a'}$  (see Wanke's algorithm) **then** reject.
  - 8:     Invoke this algorithm with input  $(G_i, H_{i'}, D_i, X_i, X'_{i'})$  recursively and push  $X_a, X'_{a'}$  and the partial isomorphism  $\phi$  on the stack
  - 9:     After recursion pop these informations from the stack
  - 10: **if** the stack is not empty **then** go one level up in recursion
  - 11: accept and halt
- 

These subgraphs do not contain the root  $X_r$  and  $V(G_i) \cap V(G_j) \subseteq X_a$  since we have a tree decomposition  $D$  ( $V(G_i)$  are the vertices of  $G_i$ ). We guess and keep from the partial isomorphism  $\phi$  exactly those parts which correspond to the path from the roots  $X_r$  and  $X'_{r'}$  to the current bags  $X_a$  and  $X'_{a'}$ . Once we verified a partial isomorphism from one child component (e.g.  $G_i$ ) of  $X_a$  onto a child component (e.g.  $H_{i'}$ ) of  $X'_{a'}$ , for the other child components it suffices to know the partial mapping of  $\phi$  from  $X_a$  onto  $X'_{a'}$ .

Observe that for each  $v$  in  $G$  in a computation path from the algorithm there can only be a value for  $\phi(v)$ . Clearly, if  $G$  and  $H$  are isomorphic then the algorithm can guess the decomposition of  $H$  which fits to  $D$ , and the extensions of  $\phi$  correctly. In this case the N AuxPDA has some accepting computation. On the other hand, if the input graphs are non-isomorphic then in every non-deterministic computation either the guessed tree decomposition of  $H$  does not fulfill the conditions of a tree decomposition (and would be detected) or the partial isomorphism  $\phi$  cannot be extended at some point. ■

Wanke's algorithm decides in LogCFL whether the treewidth of a graph is at most  $k$  by guessing all possible tree decompositions. Using a result from [7] it follows that there is also a (functional) LogCFL algorithm that on input a bounded treewidth graph computes a particular tree decomposition for it. Since LogCFL is closed under composition, from this result and Theorem 4.3 we get:

**Corollary 4.5.** *The isomorphism problem for bounded treewidth graphs is in LogCFL.*

**Conclusions and open problems.** We have shown that the isomorphism problem for graphs of bounded treewidth is in the class LogCFL and that isomorphism testing and canonization of bounded tree distance width graphs is complete for L. By using standard

techniques in the area it can be shown that the same upper bounds apply for other problems related to isomorphism on these graph classes. For example the automorphism problem or the functional versions of automorphism and isomorphism can be done within the same complexity classes. The main question remaining is whether the LogCFL upper bound for isomorphism of bounded treewidth graphs can be improved. On the one hand, no LogCFL-hardness result for the isomorphism problem is known, so maybe the result can be improved. We believe that proving a logspace upper bound for the isomorphism problem of bounded treewidth graphs would require to compute tree decompositions within logarithmic space, which is a long standing open question. Another interesting open question is whether bounded treewidth graphs can be canonized in LogCFL.

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