On the stability of a scoring rules set under the IAC hypothesis

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Abstract

A society facing a choice problem has also to choose the voting rule itself from a set of different possible voting rules. In such situations, the consequentialism property allows us to induce voters' preferences on voting rules from preferences over alternatives. A voting rule employed to resolve the society's choice problem is self-selective if it chooses itself when it is also used in choosing the voting rule. A voting rules set is said to be stable if it contains at least one self-selective voting rule at each profile of preferences on voting rules. We consider in this paper a society which will make a choice from a set constituted by three alternatives {a, b, c} and a set of the three well-known scoring voting rules {Borda, Plurality, Antiplurality}. Under the Impartial Anonymous Culture assumption (IAC), we will derive a probability for the stability of this triplet of voting rules. We use Ehrhart polynomials in order to solve our problems. This method counts the number of lattice points inside a convex bounded polyhedron (polytope). We discuss briefly recent algorithmic solutions to this method and use it to determine the probability of stabillity of {Borda, Plurality, Antiplurality} set.

Key words: Self-selectivity, Stability, Consequentialism, Ehrhart polynomials.
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1 Introduction

Within the social choice theory framework, one always considers a society which seeks to define the collective choice upon the individuals' preferences. That can be regarded like an "ordinary" choice level.

The aim of this paper is to imagine a situation such that the individuals composing the society also have choices to make on the voting rules themselves that will be used in making the ordinary-level choice. In other words, the way of choosing (the voting rule) is itself an option of the set from which the society must make a choice.

Such a situation was introduced for the first time in the social choice literature by Koray [20] together with the notion of *Self-selectivity*. A voting rule is Self-selective if, when it is used to make a choice from a given voting rules set, it chooses itself against any rule. The self-selective

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rule is equivalent to the undominated rule. Thus, the characteristic of self-selectivity depends on two elements: the other rules in the set, and individuals' preferences.

To illustrate this concept, let us consider two examples. Imagine that the votes are made according to the majority rule: The majority decision is self-selective when there is a majority favorable to the majority voting rule. On the other hand, if a dictator is favorable to the majority voting rule, his dictatorship is not self-selective. This exactly what happened in Spain: After the General Franco's death, the King Carlos asked for a new constitution in order to establish a democracy.

The self-selective notion also makes possible the introduction of a new concept if we focus on the voting rules set. The second principle introduced here is the *stability of voting rules set* which can be defined as follows: A voting rules set is (weakly) stable if it has at least one self-selective element whatever the individuals' preferences. We say that this set satisfies the stability axiom. We will illustrate this definition by simple examples in the following sections.

The researchers in self-selectivity are divided into consequentialists and non-consequentialists. The first group assumes a complete information about preferences of others and that, given a profile and a set of social choice rules, the voters form their preferences on the set of rules estimating the consequences of their application, i.e. comparing the alternatives that these rules would select if implemented. The second group, mainly represented by Diss and Merlin [10] and Houy [17, 18, 19], rejects consequentialism and propose the so-called stability of a voting rules set.

Consequentialists can be also of different kinds, those who allow comparisons of only two rules (the status quo and another rule representing change), like in Barberà and Jackson [2] and Barberà and Bevià [1], and those who allow many rules compared at once particularly in Koray [20] and Koray and Unel [21].

Our paper aims to highlight the notion of the stability of voting rules set by proposing an application of these new axiom to a set composed by three scoring voting rules and especially by considering the consequentialism property. We propose in this paper to deal with the three most well-known rules in this category: the Borda rule (B), the Plurality rule (P) and the Antiplurality rule (A). During this work, we will try to answer the following question: is the set $\{$B, P, A$\}$ stable ? In other words does there always exist a self-selective rule in this set whatever the individuals' preferences are ? We will show initially that this set can be not stable. Next, we will evaluate the probability of the set's stability when the number of voters tends to infinity.

Our society here is assumed to be endowed with a preference profile on three alternatives $\{a, b, c\}$ and to have a set $\{$B, P, A$\}$ to make its choice of a voting rule. Consequentialism property [20, 21] stipulates that if each voter's preferences on $\{a, b, c\}$ are represented by a linear order, then our voters will rank the available voting rules in $\{$B, P, A$\}$ in accordance with what they will choose from $\{a, b, c\}$.

Note that before calculating the probability that the set $\{$B, P, A$\}$ is stable we need to define a probability model for all voters types. There are two standard models to be found in the social choice literature: the so-called Impartial Culture (IC) assumption and the Impartial Anonymous Culture (IAC) assumption. In this paper we take into account the latter one. This model was proposed for the first time in the literature by Gehrlein and Fishburn [14] in 1976. We will give more details on this assumption in the following section.

Under the IAC hypothesis on voter preferences, the involved calculations amount simply to count the number of points with integer coordinates inside a set that is characterized by linear equations and inequalities. The variables are usually the numbers of voters with each of the $m!$ possible preference orders, where m is the number of alternatives. The probability calculations of the $\{$B, P, A$\}$ stability will be done by using the *"Ehrhart polynomials"* method. This new method was recently introduced in the social choice literature by Lepelley, Louichi and Smaoui [22] and Pritchard and Wilson [23].

The rest of the paper will be organized as follows: Section 2 formalizes the first ingredient of our paper dealing with preferences, scoring voting rules and the stability notion. In Section 3, we present Ehrhart's polynomial theory and we use it to determine the probability's stability of {B, P, A} set. Finally, Section 4 concludes.

2 Basic notions

2.1 Voting on alternatives

The voting rules considered in this paper are all scoring rules. In a scoring rule, each voter's choice must be a vector that specifies the number of points that the voter gives to each alternative. We will focus in this paper on three rules: the Borda rule (B), the Plurality rule (P) and the Antiplurality rule (A). For Borda rule, for each voter, a candidate receives 2 points if it is ranked first by the voter, 1 point if it is ranked second and 0 if it is ranked third. The Plurality rule is the scoring rule with the vector $(1,\ 0,\ 0)$ and finally the Antiplurality rule is the scoring rule with the vector $(1,\ 1,\ 0)$. For all these rules, the score of a candidate is the total number of points the candidate receives and the winner is the one who receives the highest number of points.

Then, let a society of n ($n \to \infty$) individuals that is to choose one alternative among the set {a, b, c}. In addition, the individuals' preferences on {a, b, c} should always be complete, strict and transitive. Therefore there are 6 types of preferences numbered from 1 to 6 in Table 1, n_i being the number of voters having the preferences number i. For example, $a \succ b \succ c$ (n_1) means that n_1 individuals prefer a to b, which is preferred to c.

Table 1: All types of preferences among $\{a, b, c\}$

$a \succ b \succ c\ (n_1)$	$a \succ c \succ b\ (n_2)$	$c \succ a \succ b\ (n_3)$
$c \succ b \succ a\ (n_4)$	$b \succ c \succ a\ (n_5)$	$b \succ a \succ c\ (n_6)$

The existence or not, of a self-selective rule will be determined by the study of the situation represented by the positive integer vector $\tilde{n} = (n_1, ..., n_6)$ which points out the number of voters according to the preferences order such that $\sum_{i=1}^{6} n_i = n$. Then, we will have the following scores (Table 2), where $S_R(\tilde{n}, x)$ is the score of alternative x with the rule R for the profile $\tilde{n}$:

Table 2: The scores

	$R = A$	$R = B$	$R = P$
$S_R(\tilde{n}, a)$	$n_1 + n_2 + n_3 + n_6$	$2(n_1 + n_2) + (n_3 + n_6)$	$n_1 + n_2$
$S_R(\tilde{n}, b)$	$n_5 + n_6 + n_1 + n_4$	$2(n_5 + n_6) + (n_1 + n_4)$	$n_5 + n_6$
$S_R(\tilde{n}, c)$	$n_3 + n_4 + n_2 + n_5$	$2(n_3 + n_4) + (n_2 + n_5)$	$n_3 + n_4$

2.2 Voting on Rules

Our society here is assumed to be endowed with a preference profile on {a, b, c} and to have a set of three scoring voting rules {B, P, A} available to make its choices. Consequentialism property illustrates that if each voter's preferences on {a, b, c} are represented by a linear order, the voters in this society are naturally expected to rank the voting rules in accordance with their preferences on {a, b, c}. This induces a preference profile on {B, P, A} which will be complete preorders since an agent will be indifferent between two voting rules choosing the same alternative from {a, b, c}.

Example 1 *Consider an example with three alternatives $\{a, b, c\}$ and 41 voters.*

$$a \succ b \succ c\,(10) \quad a \succ c \succ b\,(10) \quad c \succ a \succ b\,(10) \;\; c \succ b \succ a\,(8) \quad b \succ c \succ a\,(3)$$

The Antiplurality rule selects the outcome b, the Borda count picks out a and finally the Plurality rule selects a. Using the consequentialism property, the complete preorder on $\{$B, P, A$\}$ is represented as follows:

$$A \succ B \sim P\,(21) \qquad B \sim P \succ A\,(20)$$

Where $A \succ B \sim P$ means that individuals prefer A to B, which is equal to P.

For this reason we can find in the social choice literature two different principles to decide between the voting rules in such situations.

2.3 Extensions

For any $-1 \leq s \leq 1$, let us consider the following family of points vector $W_s = (2, 1+s, 0)$; This vector describes all the possible scoring rules for the case of 3 options. In particular, W_s is the Plurality rule for $s = -1$, the Borda rule for $s = 0$ and the Antiplurality rule if $s = 1$.

Principle 1 *When individuals are indifferent between two voting rules, all the scoring rules are similar: they give 1 point for the rule(s) ranked first and 0 point for the rule(s) classed last.*

Principle 2 *When individuals are indifferent between two voting rules, the vector $W_s = (2, 1+s, 0)$ is transformed to $W_s^{'} = (\frac{3+s}{2}, \frac{3+s}{2}, 0)$ if there were a tie in the first position and to $W_s^{''} = (2, \frac{1+s}{2}, \frac{1+s}{2})$ if there were a tie in the last position.*

For the example 1, applying Principle 1, Borda, Plurality and Antiplurality are similar in the sense that their points vector is always $(1, 0)$ (1 point for the option(s) ranked first and 0 for the alternative(s) ordered second). Then, applying these rules for the preference $A \succ B \sim P$, while Antiplurality will receive 1 point, Borda and Plurality get 0 point. However, for the preference $B \sim P \succ A$, Borda and Plurality obtain 1 point and Antiplurality receives 0 point. By this way, for this example, Antiplurality gets 21 points and Borda and Plurality will receive 20 points.

By contrast, applying Principle 2, for the preference $A \succ B \sim P$ the points vector will be $W_0^{''} = (2, \frac{1}{2}, \frac{1}{2})$ for Borda, $W_{-1}^{''} = (2, 0, 0)$ for Plurality and finally $W_1^{''} = (2, 1, 1)$ for Antiplurality. Whereas for the preference $B \sim P \succ A$, the points vector is $W_0^{'} = (\frac{3}{2}, \frac{3}{2}, 0)$ for Borda, $W_{-1}^{'} = (1, 1, 0)$ for Plurality and $W_1^{'} = (2, 2, 0)$ for Antiplurality. According to these vectors, Borda selects Antiplurality, Plurality picks out Antiplurality. Finally Antiplurality chooses Borda and Plurality.

Note that, we don't need to use a principle to decide between voting rules in the case that individuals are indifferent between three options in the sense that for this case we will show that the studied set will always be stable, provided that we break ties in favor of status quo.

2.4 Definition of the stability

We will now present the definition of stability and apply it to the triplet $\{$B, P, A$\}$. Under the hypothesis of a universal domain, if one proposes to a society a set of neutral and different voting rules, would we be sure that the preferences will lead to a self-selective rule chosen by the individuals ? In a global aspect, the notion of stability concerns the choice of one rule among many of them, independent of initial preferences. Moreover, on applying this notion, one rule may be considered as preferred to another if it is more stable within a given initial set of voting rules.

Definition 1 *Consider a voting rules set E and a complete preorder on these voting rules induced by the preferences on the alternatives. A voting rule $R \in E$ is self-selective, at some profile, if it sustains itself against any rule $R^{'} \in E$. That is, when the rule R is used to choose between the rules in E.*

Definition 2 *The set E is (weakly) stable if, at any profile, there exists at least one self-selective voting rule.*

Let us consider three examples to highlight these ideas.

Example 2 *Consider 3 alternatives $\{a, b, c\}$, 11 voters and the preference profile such that:*

$$b \succ c \succ a\ (3) \qquad b \succ a \succ c\ (2) \qquad c \succ a \succ b\ (2) \qquad a \succ c \succ b\ (4)$$

For this example, the application of the Borda rule gives out the outcome a. The Plurality rule selects b and finally the Antiplurality rule chooses c. Then, the complete preorder on $\{$B, P, A$\}$ is represented as follows:

$$P \succ A \succ B\ (3) \qquad P \succ B \succ A\ (2) \qquad A \succ B \succ P\ (2) \qquad B \succ A \succ P\ (4)$$

Then, the three rules composing the set are self-selective in the sense that Borda selects Borda, Plurality opts for Plurality and finally the Antiplurality rule selects Antiplurality. However, example 3 shows that the $\{B, P, A\}$ may be not stable for Principle 2.

Example 3 *Consider the preference profile displayed in Example 1. The complete preorder on $\{$B, P, A$\}$ is represented as follows:*

$$A \succ B \sim P\ (21) \qquad B \sim P \succ A\ (20)$$

The three rules composing the set are not self-selective in the sense that Borda selects Antiplurality, Plurality opts for Antiplurality and finally the Antiplurality rule selects Plurality and Borda with a tie.

Example 4 *Consider 3 alternatives $\{a, b, c\}$, 5 voters and the preference profile represented as follows:*

$$b \succ c \succ a\ (2) \qquad b \succ a \succ c\ (2) \qquad c \succ b \succ a\ (1)$$

For this example, the application of each scoring rule (in $\{$B, P, A$\}$) gives out the outcome b. Then, considering the consequentialism property, the individuals complete preorder on $\{$B, P, A$\}$ corresponds to:

$$A \sim B \sim P\ (5)$$

Society is indifferent on the three voting scoring rules and then, by breaking a tie in favor of the status quo, each time there is a self-selective rule in the set $\{$B, P, A$\}$.

Thus, our objective is to try to calculate the stability's probability of this set of 3 voting rules under the Impartial Anonymous Culture (IAC) hypothesis. To fulfill this programme, we must first enumerate all the possible relationships among the three alternatives.

3 Probability computation

First, we define the conceptual framework of our paper. The well-known Impartial Anonymous Culture (IAC) assumption stipulates that all voting situations $\tilde{n}$ are equiprobable. For n agents and m alternatives, the total number of voting situations is $\mathcal{C}^{n}_{n+m!-1}$. Using a simple calculation, for n agents and 3 alternatives, the number of voting situations is given by the fifth-degree polynomial below: $\psi(n) = \frac{1}{120}n^5 + \frac{1}{8}n^4 + \frac{17}{24}n^3 + \frac{15}{8}n^2 + \frac{137}{60}n + 1$.

3.1 The Different Cases

In this study, we will first assume that the size of the population is large, which will make the probabilities of tied outcomes on {a, b, c} extremely unlikely. Thus, the application of each rule to the {a, b, c} set gives only three possible results (a alone, b alone or c alone). In fact, we will obtain $(3)^3 = 27$ possible cases (Table 3).

Table 3: All possible cases

Case	*B*	*P*	*A*
1	a	b	c
2	a	c	b
3	b	a	c
4	b	c	a
5	c	a	b
6	c	b	a

Table 3.a: Category I

Case	*B*	*P*	*A*
7	a	a	b
8	a	a	c
9	b	b	a
10	b	b	c
11	c	c	a
12	c	c	b

Table 3.b: Category II

Case	*B*	*P*	*A*
13	a	b	a
14	a	c	a
15	b	a	b
16	b	c	b
17	c	a	c
18	c	b	c

Table 3.c: Category III

Case	*B*	*P*	*A*
19	b	a	a
20	c	a	a
21	a	b	b
22	c	b	b
23	a	c	c
24	b	c	c

Table 3.d: Category IV

Case	*B*	*P*	*A*
25	a	a	a
26	b	b	b
27	c	c	c

Table 3.e: Category V

3.1.1 Case 1 in detail

Let us consider, for example, the inequalities that characterize the first case. Since B gives a, P gives b and A gives c, the domains of profile leading to these results is represented by:

$$\left\{\begin{array}{l} n_1 + 2n_2 + n_3 - n_4 - 2n_5 - n_6 > 0 \\ 2n_1 + n_2 - n_3 - 2n_4 - n_5 + n_6 > 0 \\ -n_3 - n_4 + n_5 + n_6 > 0 \\ -n_1 - n_2 + n_5 + n_6 > 0 \\ -n_1 + n_4 + n_5 - n_6 > 0 \\ -n_1 + n_2 + n_3 - n_6 > 0 \end{array}\right. \tag{1}$$

Using the consequentialism property, the complete preorder on {B, P, A} is represented as follows:

$$\begin{array}{lll} B \succ P \succ A\ (n_1) & B \succ A \succ P\ (n_2) & A \succ B \succ P\ (n_3) \\ A \succ P \succ B\ (n_4) & P \succ A \succ B\ (n_5) & P \succ B \succ A\ (n_6) \end{array}$$

Taking into account the inequalities in (1), we can easily find that the three rules are self-selective. This is true using either Principle 1 or Principle 2. Then calculating the probability of the stability of this case is only tantamount to computing the solution of the system (1). We can easily show that this case is similar to the cases from 2 to 6 in the sense that Borda, Plurality and Antiplurality are always self-selective.

3.1.2 Case 7 in detail

This case denotes a voting situation with n voters for which Borda and Plurality choose a and Antiplurality selects b. The conditions for such a situation are written as follows:

$$\begin{cases} n_1 + 2n_2 + n_3 - n_4 - 2n_5 - n_6 > 0 \\ 2n_1 + n_2 - n_3 - 2n_4 - n_5 + n_6 > 0 \\ n_1 + n_2 - n_5 - n_6 > 0 \\ n_1 + n_2 - n_3 - n_4 > 0 \\ n_1 - n_3 - n_4 + n_6 > 0 \\ -n_2 - n_3 + n_4 + n_5 > 0 \end{cases} \tag{2}$$

Applying the consequentialism assumption, the complete preorder on {B, P, A} is represented in the following profile:

$$A \succ B \sim P \ (n_4 + n_5 + n_6) \qquad B \sim P \succ A \ (n_1 + n_2 + n_3)$$

Let $n_1 + n_2 + n_3 = \alpha$ and $n_4 + n_5 + n_6 = n - \alpha$. Let us use the second Principle. $S_P(\tilde{n},\ B) = S_P(\tilde{n},\ P) = \frac{\alpha}{2}$, $S_P(\tilde{n},\ A) = n - \alpha$, thus Plurality is self-selective if $\alpha > \frac{2n}{3}$. Applying Borda we find $S_B(\tilde{n},\ B) = S_B(\tilde{n},\ P) = 0.5n + \alpha$, $S_B(\tilde{n},\ A) = 2n - 2\alpha$ and the Borda rule is self-selective if $\alpha > \frac{n}{2}$. Similarly, we find that Antiplurality is self-selective if $\alpha < \frac{n}{3}$. Taking into account these conditions, we can show that no rule is self-selective if $\frac{n}{3} < \alpha < \frac{n}{2}$. Outside this interval {B, P, A} set is stable in the sense that at least one rule is self-selective. Then calculating the stability of {B, P, A} when Borda and Plurality give a and Antiplurality chooses c is equivalent to computing the solution of the system (2) with two added inequalities $n_1 + n_2 + n_3 > \frac{n}{3}$ and $n_1 + n_2 + n_3 < \frac{n}{2}$. Similarly, for every case of category II, finding the probability of the stability of {B, P, A} set is always equivalent to solve a system of 8 inequalities.

3.1.3 Case 13 in detail

For this case, Borda gives a, Plurality selects b and Antiplurality chooses a. The system of inequalities of such situation is given as follows:

$$\begin{cases} n_1 + 2n_2 + n_3 - n_4 - 2n_5 - n_6 > 0 \\ 2n_1 + n_2 - n_3 - 2n_4 - n_5 + n_6 > 0 \\ -n_1 - n_2 + n_5 + n_6 > 0 \\ -n_3 - n_4 + n_5 + n_6 > 0 \\ n_2 + n_3 - n_4 - n_5 > 0 \\ n_1 - n_4 - n_5 + n_6 > 0 \end{cases} \tag{3}$$

Then, the society will have the following preferences on the voting scoring rules:

$$A \sim B \succ P \ (n_1 + n_2 + n_3) \qquad P \succ A \sim B \ (n_4 + n_5 + n_6)$$

Let $n_1+n_2+n_3 = \alpha$ and $n_4+n_5+n_6 = n-\alpha$. Using simple mathematical tools we find that the {B, P, A} set is stable in the sense that at least one rule is self-selective. This is true using either Principle 1 or Principle 2. Then, to determine the probability of the cases described in category III, we should only solve systems of 6 inequalities in the sense that introducing consequentialism does not affect the probability of these cases.

3.1.4 Other cases

Notice that with the help of elementary mathematical operations, we can easily demonstrate that the cases described in category IV are impossible. Also, for the cases in category V, using either Principle 1 or Principle 2, the $\{$B, P, A$\}$ set is always stable in the sense that employing the consequentialism property, society will be indifferent on the three voting scoring rules and then there is always a self-selective rule in the set.

Using these considerations we can first give the following proposition for Principle 1.

Proposition 1 *Consider elections using scoring rules systems for which there are three candidates and n voters, and for which society will choose the voting rule itself among $\{$Borda, Plurality, Antiplurality$\}$ set. Assume that all voting situations are equally likely (IAC). Under Principle 1, the $\{$Borda, Plurality, Antiplurality$\}$ set is stable for all the profiles.*

For Principle 2 we have to solve all systems illustrated above. The Ehrhart polynomials theory will allow us to count the number of solutions for each system.

3.2 Ehrhart Polynomials

We will not explain Ehrhart theory and attached algorithms in detail but we will try to give a brief description of its main results. For more information we recommend the paper of Lepelley, Louichi and Smaoui [22].

The basic idea is that all sets of voting situations that are of interest in our paper can be characterized by linear equations and inequalities. The set of such (in)equalities defines a convex polytope that can be described by a set of linear contraints, i.e., $P = \{x \in \mathbb{Z}^d | Ax + b \geq 0\}$ or $P = \{x \in \mathbb{Q}^d | Ax + b \geq 0\} \cap \mathbb{Z}^d$. The extremal points of such polytopes are called its vertices. The calculations involved amount simply to find the number of points in P which is equivalent to counting the number of integer lattice points inside the convex polytope.

In the 1960's Eugène Ehrhart [13] showed that for any rational d-polytope P, the number of integer points in the dilatation $nP = \{nx | x \in P\}$ on a rational polytope P can be represented by a pseudo-polynomial (aka, Ehrhart polynomial) $\mathrm{L}(P, n)$ in n. The degree of $\mathrm{L}(P, n)$ is less than or equal to the dimension of P. The coefficients of $\mathrm{L}(P, n)$ are periodic numbers, which means that they depend periodically on the parameter n. Basically, the periodic numbers can be represented by a n-dimensional lookup-table $U_n = U[n \ mod \ s]$, where s is called the period of U_n. The periods of the coefficients are all less or equal to the lcm (least common multiple) of the denominators of the vertices. In particular, if P is an integral polytope, $\mathrm{L}(P, n)$ is a polynomial.

Note that the coefficient of the leading term is always independent of n and is equal to the Euclidean volume of P. Knowledge of this coefficient is very important and sufficient in our paper in the sense that the limiting probability under IAC as $n \to \infty$ is simply the volume of P divided by the total number of all possible voting situations $\psi(n) = \frac{1}{120}n^5 + \frac{1}{8}n^4 + \frac{17}{24}n^3 + \frac{15}{8}n^2 + \frac{137}{60}n + 1$.

To our knowledge, there exist two famous methods for computing Ehrhart polynomials: Clauss's algorithm (1998) and the parameterized Barvinok's algorithm (2004).

Based on the structure of the Ehrhart polynomials, Clauss and Loechner [8] developed an algorithm to compute the Ehrhart quasi-polynomials. They calculate the number of points in a set of instances of P for fixed values of n in a given validity domain, called initial countings, and then calculate the Ehrhart polynomial for this validity domain through interpolation. During this calculation, they directly determine the elements in the lookup-tables representing the periodic numbers. In order to interpolate an Ehrhart polynomials of degree d in n parameters with period s, $\prod_{i=1}^{n}(d+1)s_i$ instances are needed.

This leads to the first disadvantages of the interpolation method which has been explicitly mentioned by Turjan et al.[9]. If a hyperrectangle instance with the required integer size vector does not exist, the validity domain is called "degenerate", and the algorithm fails to compute the Ehrhart quasi-polynomial. The implementation of Clauss's method uses some heuristics to circumvent the problem of degenerate domains. These heuristics are not covering all situations in the sense that in several applications degeneracy is an occurring problem.

The second disadvantage of this method concern the time of the interpolation. Since Clauss's method is based on the lookup-table representation of periodic numbers, if the periods are large, then the number of instances will be very large and the interpolation will take a long time. Also, if any of the instances of P contains a large number of points, the computation time will increase accordingly.

In the 1990's, on the basis of work made by the geometers Brion, Khovanski, Lawrence, and Pukhlikov, Barvinok [3, 4] proposed an algorithm for counting integer points inside polyhedra that runs in polynomial time for fixed dimension and allows to solve the above problems to a large extent. This algorithm was latter refined by De Loeara et al. [16, 15], especially with respect to a practical implementation and began an important area of research of a wide variety of topics in pure and applied mathematics.

The key ideas are using rational functions as efficient data structures and the unimodular signed decomposition of polyhedra. Given a convex polyhedron P, define the multivariate generating function attached to P as:

$$f(P;x) = \sum_{\alpha \in P \cap \mathbb{Z}^d} x^\alpha \quad where \ x^\alpha = x_1^{\alpha_1} ... x_d^{\alpha_d} \ and \ x = (x_1, ..., x_d)$$

This is an infinite formal power series if P is not bounded, but if P is a polytope it is a (Laurent) polynomial with one monomial per lattice point. The number of points in P is then simply the number of monomials in the generating function, which can be calculated by evaluating the generating function at 1, i.e. $f(P;1)$. The generating function is obviously not constructed enumerating all the integer points in P, but rather as a signed sum of short rational functions that can be derived from the description of P.

Brion theorem [6] allows the evaluation of $f(P;x)$ by computing the generating function of the supporting cones of P.

$$f(P;x) = \sum_{v \in V(p)} f(K(P,v);x)$$

where K is the supporting cone, v is a vertex and $V(P)$ is the set of all vertices of P. Notice that to apply this formula we must decompose the supporting cones to unimodular ones (determinant = $\pm$ 1). Thus unimodular cones contains only one point. Simplifying and expending (in term of Laurent series) the sum of all generating functions and evaluating them at $x = (1, ..., 1)$, we finally obtain the result of the number of lattice points in the polytope P. (See algorithms in the appendix).

3.3 Results

3.3.1 Example: case 7

The system of linear (in)equalities that characterized the instability of the {B, P, A} set in this case is given as:

$$
\begin{cases}
n_1 + 2n_2 + n_3 - n_4 - 2n_5 - n_6 > 0 \\
2n_1 + n_2 - n_3 - 2n_4 - n_5 + n_6 > 0 \\
n_1 + n_2 - n_5 - n_6 > 0 \\
n_1 + n_2 - n_3 - n_4 > 0 \\
n_1 - n_3 - n_4 + n_6 > 0 \\
-n_2 - n_3 + n_4 + n_5 > 0 \\
n_1 + n_2 + n_3 > \frac{n}{3} \\
n_1 + n_2 + n_3 < \frac{n}{2} \\
n_1 + n_2 + n_3 + n_4 + n_5 + n_6 = n \\
n_i \geq 0 \text{ for } i\text{=1, 2,...,6.}
\end{cases} \tag{4}
$$

Using Barvinok's algorithm, we obtain (for $n \geq 8$) the following output:

$$
\begin{aligned}
&\frac{79}{3110400}n^5 + [\frac{263}{622080}, \frac{13}{311040}]_n n^4 + [\frac{11}{5760}, \frac{-473}{311040}, \frac{11}{5760}, \frac{-91}{103680}, \frac{197}{155520}, \frac{-91}{103680}]_n n^3 \\
&+ [\frac{-1}{864}, \frac{-449}{155520}, \frac{25}{7776}, \frac{-41}{17280}, \frac{-29}{3888}, \frac{311}{155520}]_n n^2 + [\frac{-1}{120}, \frac{6247}{622080}, \frac{149}{9720}, \frac{341}{23040}, \frac{-163}{4860}, \frac{7607}{622080}]_n n \\
&+ [0, \frac{-8867}{1555200}, \frac{-4301}{64800}, \frac{-57}{6400}, \frac{232}{6075}, \frac{-739}{6912}, \frac{-53}{800}, \frac{101941}{1555200}, \frac{263}{2025}, \frac{-481}{6400}, \frac{365}{7776}, \frac{15437}{172800}, \frac{-3}{25}, \frac{115549}{1555200}, \\
&\frac{883}{64800}, \frac{-33}{256}, \frac{-497}{6075}, \frac{-4651}{172800}, \frac{11}{800}, \frac{-84683}{1555200}, \frac{17}{81}, \frac{31}{6400}, \frac{-14203}{194400}, \frac{29261}{172800}, \frac{-1}{25}, \frac{-2843}{62208}, \frac{-6893}{64800}, \frac{-313}{6400}, \\
&\frac{-11}{6075}, \frac{-25387}{172800}, \frac{3}{32}, \frac{39733}{1555200}, \frac{182}{2025}, \frac{543}{6400}, \frac{1349}{194400}, \frac{341}{6912}, \frac{-4}{25}, \frac{53341}{1555200}, \frac{-1709}{64800}, \frac{-1081}{6400}, \frac{19}{243}, \\
&\frac{-11563}{172800}, \frac{-21}{800}, \frac{164149}{1555200}, \frac{344}{2025}, \frac{-9}{256}, \frac{-21979}{194400}, \frac{22349}{172800}, \frac{-2}{25}, \frac{-133283}{1555200}, \frac{139}{2592}, \frac{-569}{6400}, \frac{-254}{6075}, \frac{2261}{172800}, \\
&\frac{43}{800}, \frac{-899}{62208}, \frac{101}{2025}, \frac{287}{6400}, \frac{-6427}{194400}, \frac{1613}{172800}]_n
\end{aligned}
$$

It means that if $n \rightarrow \infty$ the number of solutions of this systems is $\frac{79}{3110400}n^5$. Notice that without taking into account the (in)stability of the $\{$Borda, Plurality, Antiplurality$\}$ set (system 2) and using the same algorithm, the number of solutions is $\frac{13}{31104}n^5$ when $n \rightarrow \infty$. Then the number of stable solutions is: $(\frac{13}{31104} - \frac{79}{3110400})n^5$. These relations allow us to conclude that, for large electorates, the likelihood of the stability of $\{$B, P, A$\}$ set in the case 7 is: $\frac{(\frac{13}{31104} - \frac{79}{3110400})n^5}{\frac{1}{120}n^5} = 4,7106\%$. $\frac{1}{120}n^5$ is the total number of voting situations $\psi(n)$ when $n \rightarrow \infty$. The likelihood of instability is given as follows: $\frac{\frac{79}{3110400}n^5}{\frac{1}{120}n^5} = 0,3048\%$

This case is similar to cases from 8 to 12. The Ehrhart's polynomials and probabilities of other cases can be obtained in a similar way.

3.3.2 General results

Computed values for each cases are displayed in Table 4. Table 5 represents the probability and the number of self-selective rules (SSR) by categories.

We have defined a stable set as being a set which has at least one self-selective rule at any profile. We can thus give the following proposition:

Proposition 2 *Consider elections using scoring rules systems for which there are three candidates and n voters, and for which society will choose the voting rule itself among $\{$Borda, Plurality, Antiplurality$\}$ set. Assume that all voting situations are equally likely (IAC). Under Principle 2, the $\{$Borda, Plurality, Antiplurality$\}$ set is stable for 98.17% of the profiles.*

Table 4: Probability by cases under Principle 2

Cases	Stability	Instability
From 1 to 6	0,1984%	0%
From 7 to 12	4,7106%	0,3048%
From 13 to 18	2,7337%	0%
From 19 to 24	0%	0%
From 25 to 27	17,4383%	0%

Table 5: Probability by category under Principle 2

Categories	Stability	Number of SSR	Instability	Number of SSR
I	1,1905%	always 3	0%	0
II	28,2636%	at least 1	1,8288%	0
III	16,4021%	at least 1	0%	0
IV	0%	impossible cases	0%	impossible cases
V	52,3148%	at least 1	0%	0

4 Concluding Remarks

In this paper, we have illustrated one of the recently growing literature on the self-selectivity of voting rules with a classical example. We have considered a set composed by {Borda, Plurality, Antiplurality} under the Impartial Anonymous Culture hypothesis. The paper provides some nice examples of profiles; for one of these all three rules are self-selective, and for the other none of the three are self-selective. The latter profile answers the stability question for {A, B, P} in the negative. The rest of the paper calculates the probability for stability of {A, B, P} under IAC. From these representations, it follows that the occurrence of instability of such voting rules set can be considered as rare (or null) in three-candidate elections.

Notice that, while we chose to work with a special case of three alternatives {a, b, c} and a set of three voting scoring rules {Borda, Plurality, Antiplurality}, it is still interesting to ask questions like: what would happen with different sets of scoring rules (Plurality with Run-off, Copeland rule, etc.) ? Among sets of three alternatives, does expanding the number of scoring rules eventually yield a stable set ?, etc.

Appendix: Algorithms

(From [5, 7])

Algorithm 1: Barvinok's algorithm

1. For each vertex v_i of P
   (a) Determine supporting cone, $K(P, v_i)$
   (b) Let $K = K(P, v_i) - v_i$
   (c) Decompose K into unimodular cones K_j such that: $[K] = \sum_j \epsilon_j [K_j]$
   (d) For each K_j, determine $f(K_j; x)$
   (e) $f(K(P, v_i); x) = \sum_j \epsilon_j x^{E(v_i, K_j)} f(K_j; x)$, $\epsilon_j \in \{-1, 1\}$ and $E(v_i, K_j)$ is the unique lattice point belonging to the fundamental half-open parallelepiped corresponding to the translated cone $K_j + v_i$
2. $f(P; x) = \sum_{v_i \in D} f(K(P, v_i); x)$
3. evaluate $f(P; 1)$

Algorithm 2: Parameterized Barvinok

1. For each (parametric) vertex $v_i(\mathbf{p})$ of P
   (a) Determine supporting cone, $K(P, v_i(\mathbf{p}))$
   (b) Let $K = K(P, v_i(\mathbf{p})) - v_i(\mathbf{p})$
   (c) Decompose K into unimodular cones: $[K] = \sum_j \epsilon_j [K_j]$
   (d) For each K_j, determine $f(K_j; x)$
   (e) $f(K(P, v_i(\mathbf{p})); x) = \sum_j \epsilon_j x^{E(v_i(\mathbf{p}), K_j)} f(K_j; x)$
2. For each validity domain D of P
   (a) $f(P; x) = \sum_{v_i(\mathbf{p}) \in D} f(K(P, v_i(\mathbf{p})); x)$
   (b) evaluate $f(P; 1)$

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