Proportional Response as Iterated Cobb-Douglas

Michael J. Todd *

July 10, 2010

Abstract

We show that the proportional response algorithm for computing an economic equilibrium in a Fisher market model can be interpreted as iteratively approximating the economy by one with Cobb-Douglas utilities, for which a closed-form equilibrium can be obtained. We also extend the method to allow elasticities of substitution at most one.

1 Introduction

In [3] Zhang proposed the proportional response distributed algorithm to compute an economic equilibrium for a Fisher model where each consumer has a constant elasticity of substitution (CES) utility function, viewing it as a method in which agents iteratively refine their bids for the goods. Birnbaum, Devanur, and Xiao [1] consider the algorithm in the case of linear utilities, extend it to allow spending constraint utilities, and interpret it as a generalized gradient-descent or generalized proximal-point method for an associated optimization problem. Both papers also analyze the convergence of the method.

In this note, we show that the algorithm can be viewed as iteratively approximating the economy by economies with Cobb-Douglas utilities, for which the equilibrium can be obtained in closed form, and extend it to allow elasticity coefficients of one or less. From this point of view, the method appears as a variant of Newton's method, in which usually successive linearizations are made.

2 CES utility functions

Suppose there are n goods. A CES utility function assigns to a bundle of goods $x \in \Re^n_+$ the utility

$$u(x) = \left(\sum_j a_j^{1/\sigma} x_j^{1-1/\sigma} \right)^{\sigma/(\sigma-1)}. \qquad (1)$$

*School of Operations Research and Information Engineering, Cornell University, Ithaca, NY 14853, USA. Supported in part by ONR through grant N00014-08-1-0036. This research was inspired by Nikhil Devanur's presentation at the Dagstuhl seminar. E-mail `mjt7@cornell.edu`.

Dagstuhl Seminar Proceedings 10171
Equilibrium Computation
http://drops.dagstuhl.de/opus/volltexte/2010/2671

Here a is a nonzero vector in $\Re^n_+$ representing the relative desirability of the goods, and $\sigma > 0$, $\sigma \neq 1$, is the elasticity coefficient. (For the analysis below, we assume that all components a_j are positive, but the general case can be obtained by summing over just those j for which a_j is positive.) We say the utility function represents an agent's preferences if in making economic decisions she acts as if she were maximizing this function. Hence any strictly monotonic transformation of a utility function also represents the agent's preferences. We can therefore assume without loss of generality that

$$\sum_j a_j = 1.$$

We can also write ρ for $1 - 1/\sigma$ to get

$$u(x) = \left(\sum_j a_j^{1-\rho} x_j^{\rho} \right)^{1/\rho}, \tag{2}$$

where $-\infty < \rho < 1$, $\rho \neq 0$. For ρ positive, an equivalent utility function is

$$u(x) = \sum_j a_j^{1-\rho} x_j^{\rho}, \tag{3}$$

and if we write w_j for $a_j^{1/(\sigma-1)}$ we obtain

$$\bar{u}(x) = \sum (w_j x_j)^{\rho}. \tag{4}$$

This is the form that Zhang [3] assumes. If $\rho = 1$, corresponding to $\sigma \to \infty$, we get the linear utility function $\sum_j w_j x_j$.

Let us take the (natural) logarithm of $u(x)$ in (2) above:

$$\ln u(x) = \frac{\ln \left(\sum_j a_j^{1-\rho} x_j^{\rho} \right)}{\rho}.$$

We can now use L'Hôpital's rule to take limits as $\rho \to 0$, obtaining

$$\ln u^0(x) = \frac{\sum_j a_j \ln(x_j/a_j)}{\sum_j a_j} = \sum_j a_j \ln x_j - \sum_j a_j \ln a_j.$$

The constant part can be eliminated, and we obtain the limiting utility function

$$\ln u^0(x) = \sum_j a_j \ln x_j, \text{ or } u^0(x) = \prod_j x_j^{a_j}, \tag{5}$$

which is a CES utility function with elasticity coefficient 1, and is called a Cobb-Douglas utility function.

3 Economies and equilibria

Suppose now we have m agents, each with a CES utility function u_i given as above by a vector $a_i = (a_{ij}) \in \Re^n_+$ and ρ_i, and a budget $b_i > 0$. We assume without loss of generality that there is one unit of each good available, that $\sum_i b_i = 1$, and that $\sum_i a_i > 0$, so that each good is desired by some agent.

We say a price vector $\bar{p} \in \Re^n_+$ and allocations $\bar{x}_i = (\bar{x}_{ij}) \in \Re^n_+$, $i = 1, \ldots, m$, form an *equilibrium* if

(a) the market clears: $\sum_i \bar{x}_i = e := (1; 1; \ldots; 1)$; and

(b) each agent maximizes her utility subject to her budget constraint: $\bar{x}_i$ solves

$$\max_{x_i} \{u_i(x_i) : p^T x_i \leq b_i, x_i \in \Re^n_+\}.$$

Since each a_i has a positive component, $p^T \bar{x}_i = b_i$, and so we have the normalization $p^T e = p^T \sum_i \bar{x}_i = \sum_i b_i = 1$.

Observe that the optimality conditions for agent i's utility maximization problem involve her utility function only through $\nabla u_i(\bar{x}_i)$, and so this is the critical part of her preferences as far as equilibrium is concerned.

Now suppose that each agent has a Cobb-Douglas utility function, which we write for convenience in its logarithmic form:

$$u_i(x_i) = \sum_j a_{ij} \ln x_{ij}. \tag{6}$$

This is a concave function of x_i. Suppose for a moment that all a_{ij}'s are positive. Then from the (necessary and sufficient) optimality conditions, agent i's utility maximization problem at prices p is solved by $\bar{x}_i$ with $a_{ij}/\bar{x}_{ij} = \lambda_i p_j$ for all j, where $\lambda_i \geq 0$ is the Lagrange multiplier associated with her budget constraint. Hence λ_i is positive (and all components of p must be positive) and $\bar{x}_{ij} = a_{ij}/(\lambda_i p_j)$, and then from the budget constraint, $\lambda_i = 1/b_i$ and so $\bar{x}_{ij} = b_i a_{ij}/p_j$. Thus agent i spends a constant fraction a_{ij} of her budget on good j whatever its price is. It is easy to check that this remains true if some a_{ij}'s are zero (and then the corresponding prices p_j can also be zero). To obtain an equilibrium, we require $\sum_i \bar{x}_{ij} = 1$ for each j, and this gives $p_j = \sum_i b_i a_{ij}$. Hence we find that $\bar{p} = (\bar{p}_j := \sum_i b_i a_{ij})$, $\bar{x}_i = (\bar{x}_{ij} := b_i a_{ij}/\bar{p}_j)$, $i = 1, \ldots, m$, give an equilibrium.

4 The proportional response algorithm

This method proceeds as follows (assuming for now that all ρ_i's are positive, as does Zhang). At each stage, the agents make bids b_{ij} for each good, where b_{ij} is positive if a_{ij} is and $\sum_j b_{ij} = b_i$. These bids then determine the prices p via $p_j = \sum b_{ij}$, and the corresponding allocations are $x_i = (x_{ij})$, $i = 1, \ldots, m$, where $x_{ij} = b_{ij}/p_j$. Agent i obtains the utility (in the form (3))

$$u_i(x_i) = \sum_j a_{ij}^{1-\rho_i} x_{ij}^{\rho_i} =: \sum_j u_{ij}(x_{ij}).$$

(An analogous analysis, with the same result, goes through if the utilities are written in the form (4).) At the next stage, the agents make bids proportional to the fraction of utility they obtained from each good in the previous round:

$$b_{ij}^{+} := \frac{u_{ij}(x_{ij})}{u_i(x_i)} b_i.$$

We now show how this algorithm can be interpreted as successive approximation by Cobb-Douglas economies. Indeed, suppose at some stage the agents are considering allocations $x_i = (x_{ij})$, $i = 1, \ldots, m$. The components of the gradient of u_i at x_i are then

$$\frac{\partial u_i(x_i)}{\partial x_{ij}} = \rho_i a_{ij}^{1-\rho_i} x_{ij}^{\rho_i - 1}, \tag{7}$$

while those of the Cobb-Douglas utility function $\hat{u}_i(x_i) := \sum_j \hat{a}_{ij} \ln x_{ij}$ are

$$\frac{\partial \hat{u}_i(x_i)}{\partial x_{ij}} = \hat{a}_{ij}/x_{ij}.$$

These two gradients will be proportional to each other if we set

$$\hat{a}_{ij} := \frac{a_{ij}^{1-\rho_i} x_{ij}^{\rho_i}}{\sum_k a_{ik}^{1-\rho_i} x_{ik}^{\rho_i}}$$

(the normalization is added so that $\sum_j \hat{a}_{ij} = 1$). We now consider the economy where each agent's utility function u_i is replaced by the approximating utility function $\hat{u}_i$ with the $\hat{a}_{ij}$'s as above. By the results of the previous section, the equilibrium in this approximating economy is easy to write down: agent i spends an amount

$$\hat{b}_{ij} := b_i \hat{a}_{ij}$$

on good j and the resulting equilibrium price vector is $\hat{p} = (\hat{p}_j = \sum_i \hat{b}_{ij})$. But by substituting the expression for $\hat{a}_{ij}$ in that for $\hat{b}_{ij}$, we find that the latter agrees with b_{ij}^{+} in the proportional response algorithm. And then so does the next price vector and the corresponding allocations.

We can extend the algorithm to the case that some agents' elasticity coefficients are one or less, or some ρ_i's are zero or less. First suppose that some ρ_i is negative. Then u_i in the form (3) is not a monotonic transformation of u_i in the form (2). Instead, we can use the monotonic transformation

$$u_i(x_i) = \sum_j (-a_{ij}^{1-\rho_i} x_{ij}^{\rho_i}) =: \sum_j u_{ij}(x_{ij}). \tag{8}$$

We can then, exactly as above, define the new bids by

$$b_{ij}^{+} := \frac{u_{ij}(x_{ij})}{u_i(x_i)} b_i = \frac{-u_{ij}(x_{ij})}{-u_i(x_i)} b_i.$$

We have included the second expression because it may be more natural to think of the agent's bid for good j as the fraction of the total (positive) disutility attributable to the jth good times her budget b_i, rather than as a ratio of negative utilities times her budget. If we now take the gradient of the utility function above, we find that its jth component is $|\rho_i| a_{ij}^{1-\rho_i} x_{ij}^{\rho_i - 1}$ in contrast to (7). Proceeding exactly as above, we find

$\hat{a}_{ij}$ is as before, and so the amounts spent on each good at equilibrium in the approximating economy are again $\hat{b}_{ij}$, and these still agree with the new bids b_{ij}^+.

Finally, suppose some agent i has a Cobb-Douglas utility function. Of course, this needs no approximating by a Cobb-Douglas utility function, and so we would have $\hat{b}_{ij} = a_{ij}b_i$. To make this agree with the proportional response algorithm, we need a_{ij} to be the proportion of total utility attributed to the jth good. This is true for neither the logarithmic nor the product form of the utility function (see (5)). It is true for the "function" $\sum_j a_{ij}$, which is the limit as $\rho \to 0$ of the form (3), but since this is a constant, it does *not* represent the agent's preferences. The only way to salvage the algorithm is to just define

$$b_{ij}^+ = a_{ij}b_i$$

for agents i with Cobb-Douglas utility functions.

With these modifications, we have proved

Theorem 1 *The proportional response algorithm for agents with CES utility functions generates exactly the same bids, price vectors, and allocations as the algorithm that at each iteration approximates the agents' utility functions as above by Cobb-Douglas utility functions and moves to the equilibrium of the resulting approximating equilibrium.*

The discussion preceding the theorem highlights a difficulty with the proportional response algorithm: it depends crucially on the particular form taken for the agents' utility functions, whereas several other forms are also perfectly adequate representations of their preferences.

As a final observation, we note that approximating a nonlinear problem for which an analytical solution cannot be obtained by a simpler problem for which it can is exactly the principle behind Newton's method. In Newton's method for finding a zero of a function, a linear approximation is made; in Newton's method for minimizing a function, a quadratic approximation is used. In the pure exchange economy, approximating utility functions by Cobb-Douglas utility functions seems natural because the resulting simpler problems can be solved explicitly. This general philosophy of Newton-like methods and its application to computing economic equilibria were described in [2]. Since we are approximating the objective function of an optimization problem (the utility maximization problem) by a function whose gradient agrees with the nominal function but whose Hessian matrix may be much different, we cannot expect faster than linear or even sublinear convergence, and this is what Zhang [3] and Birnbaum, Devanur, and Xiao [1] obtain.

References

[1] B. Birnbaum, N. Devanur, and L. Xiao, New convex programs and distributed algorithms for Fisher markets with linear and spending constraint utilities, manuscript, available at `http://research.microsoft.com/en-us/um/people/nikdev/`, 2010.

[2] M. Todd, Reformulation of economic equilibrium problems for solution by quasi-Newton and simplicial algorithms, in: The Computation and Modelling of Economic Equilibria, pp. 19–37, A. Talman and G. van der Laan, eds; North-Holland, Amsterdam, 1987.

[3] L. Zhang, Proportional response dynamics in the Fisher market, in ICALP '09, Proceedings of the 36th International Colloquium on Automata, Languages, and Programming, pp. 583–594, Springer, Berlin, 2009.