Diameter of Polyhedra: Limits of Abstraction

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Abstract

We investigate the diameter of a natural abstraction of the 1-skeleton of polyhedra. Even if this abstraction is more general than other abstractions previously studied in the literature, known upper bounds on the diameter of polyhedra continue to hold here. On the other hand, we show that this abstraction has its limits by providing an almost quadratic lower bound.

One of the most prominent mysteries in convex geometry is the question whether the diameter of polyhedra is polynomial in the number of its facets or not. If the largest diameter of a d-dimensional polyhedron with n facets is denoted by $\triangle_u(d, n)$, then the best known lower and upper bounds are $n - d + \lfloor d/5 \rfloor \leq \triangle_u(d, n) \leq n^{1+\log d}$, shown by Klee and Walkup [15] and Kalai and Kleitman [13] respectively. The gap which is left open here is huge, even after decades of intensive research on this problem.

Interestingly, the above upper bound holds also for simple *combinatorial abstractions of polyhedra* by which term we (loosely) mean a rigorously defined set of *purely combinatorial* properties of the polyhedra in question that are strong enough to allow non-trivial conclusions about its geometry.

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In the quest of bounding $\triangle_u$ one can restrict attention to non-degenerate polyhedra (we call a polyhedron *non-degenerate* if each vertex is contained in exactly d facets) since, by perturbation, any polyhedron can be turned into a non-degenerate polyhedron whose diameter is at least as large as the one of the original polyhedron. For this reason we also allow ourselves this simplifying assumption of non-degeneracy (all our results, though, perfectly hold without it).

Combinatorial abstractions have been studied in the literature for a long time [11, 2, 1]. The subject of this paper is a simple *base abstraction* which is defined by *one single feature*, common to all previously studied abstractions from which lower and upper bounds have been previously derived. As an extra evidence (besides simplicity) that our framework is quite natural, we give for it three different descriptions that all turn out to be pairwise equivalent.

Even if our abstraction is more general than previously considered ones, we nonetheless show that all known upper bounds do hold here with natural and simple proofs. On the other hand, we prove an almost quadratic lower bound on the diameter in this abstraction, and this constitutes the main concrete result of this paper.

While only one feature of the previously studied abstractions suffices to derive the best known upper bounds, our lower bound also shows the limits of this natural base abstraction for the purpose of proving *linear* upper bounds on the diameter. To prove such a bound, more features of the geometry of polyhedra will have to be understood and used than the single one that we identify here. Let us, however, note that a polynomial (or even quadratic!) upper bound in this framework still remains a possibility.

In the first description, our base abstraction is given by a connected graph $G = (V, E)$. Here[1] $V \subseteq [n]^d$ and the edges E of G are such that the following connectivity condition holds:

i) For each $u, v \in V$ there exists a path connecting u and v whose intermediate vertices all contain $u \cap v$.

Let $\mathcal{B}_{d,n}$ be the set of all graphs G with the above property; the largest diameter of a graph in $\mathcal{B}_{d,n}$ will be denoted by $D(d, n)$. We call d the *dimension* and n the *number of facets* of the abstraction.

Before we proceed, let us understand why this class contains the 1-skeletons of non-degenerate polyhedra in dimension d having n facets. In

[1] $[n]^d$ is the family of all d-element subsets of $[n] = \{1, \ldots, n\}$

this setting, each vertex is uniquely determined by the d facets in which it is contained. If the facets are named $\{1, \ldots, n\}$, then a vertex is uniquely determined by a d-element subset of $\{1, \ldots, n\}$. Furthermore, for every pair of vertices u, v there exists a path which does not leave the minimal face in which both u and v are contained. This is reflected in condition i). Thus if $\triangle_u(d, n)$ is the maximum diameter of a non-degenerate polyhedron with n facets in dimension d, then $\triangle_u(d, n) \leq D(d, n)$ holds.

Our main result is a super-linear lower bound on $D(d, n)$, namely $D(n/4, n) = \Omega(n^2/\log n)$. The non-trivial construction relies on the notion of *disjoint covering designs* and in order to prove the existence of such designs with desired parameters we use Lovász Local Lemma.

At the same time the bound of Kalai and Kleitman [13], $\Delta_u(d, n) \leq n^{1+\log d}$, as well as the upper bound of Larman [16], $\Delta_u(d, n) \leq 2^{d-1} \cdot n$, which is linear when the dimension is fixed, continue to hold in the base abstraction. While the first bound is merely an adaption of the proof in [13], our proof of the second bound is much simpler than the one which was proved for polyhedra in [16].

We strongly believe that the study of abstractions, asymptotic lower and upper bounds for those and the development of algorithms to compute bounds for fixed parameters d and n should receive more attention since they can help to understand the important features of the geometry of polyhedra that may help to improve the state-of-the art of the diameter question.

References

[1] Ilan Adler. Lower bounds for maximum diameters of polytopes. *Math. Programming Stud.*, pages 11–19, 1974. Pivoting and extensions.

[2] Ilan Adler, George Dantzig, and Katta Murty. Existence of A-avoiding paths in abstract polytopes. *Math. Programming Stud.*, (1):41–42, 1974. Pivoting and extensions.

[3] Ilan Adler and George B. Dantzig. Maximum diameter of abstract polytopes. *Mathematical Programming Study*, (1):20–40, 1974. Pivoting and extensions.

[4] Noga Alon and Joel Spencer. *The probabilistic method.* Wiley-Interscience, 3 edition, 2008.

[5] David Bremner and Lars Schewe,. Edge-Graph Diameter Bounds for Convex Polytopes with Few Facets. *arXiv:0809.0915v3 [math.CO]*, 2009.

[6] Reinhard Diestel. *Graph theory*. Springer-Verlag, New York, 2 edition, 2000.

[7] Paul Erdős and Haim Hanani. On a limit theorem in combinatorial analysis. *Publ. Math. Debrecen*, 10:10–13, 1963.

[8] Bernd Gärtner. A subexponential algorithm for abstract optimization problems. *SIAM Journal on Computing*, 24(5):1018–1035, 1995.

[9] Bernd Gärtner. The random-facet simplex algorithm on combinatorial cubes. *Random Structures & Algorithms*, 20(3):353–381, 2002. Probabilistic methods in combinatorial optimization.

[10] Gil Kalai. A subexponential randomized simplex algorithm (extended abstract). In *STOC*, pages 475–482. ACM, 1992.

[11] Gil Kalai. Upper bounds for the diameter and height of graphs of convex polyhedra. *Discrete & Computational Geometry*, 8:363–372, 1992.

[12] Gil Kalai. Linear programming, the simplex algorithm and simple polytopes. *Mathematical Programming*, 79(1-3, Ser. B):217–233, 1997. Lectures on mathematical programming (ismp97) (Lausanne, 1997).

[13] Gil Kalai and Daniel J. Kleitman. A quasi-polynomial bound for the diameter of graphs of polyhedra. *Bulletin of the American Mathematical Society*, 26:315–316, 1992.

[14] Victor Klee and Peter Kleinschmidt. The d-step conjecture and its relatives. *Math. Oper. Res.*, 12(4):718–755, 1987.

[15] Victor Klee and David W. Walkup. The d-step conjecture for polyhedra of dimension $d < 6$. *Acta Math. 133*, pages 53–78, 1967.

[16] David G. Larman. Paths on polytopes. *Proceedings London Math. Soc.*, pages 161–178, 1970.

[17] Peter Mani and David W. Walkup. A 3-sphere counterexample to the W_v-path conjecture. *Math. Oper. Res.*, 5(4):595–598, 1980.

[18] Jiří Matoušek, Micha Sharir, and Emo Welzl. A subexponential bound for linear programming. *Algorithmica*, 16(4/5):498–516, October/November 1996.

[19] Vojtěch Rödl. On a packing and covering problem. *European Journal of Combinatorics*, (5):69–78, 1985.

[20] Ingo Schurr and Tibor Szabó. Finding the sink takes some time: an almost quadratic lower bound for finding the sink of unique sink oriented cubes. *Discrete & Computational Geometry. An International Journal of Mathematics and Computer Science*, 31(4):627–642, 2004.