New Exact and Approximation Algorithms for the Star Packing Problem in Undirected Graphs

Maxim Babenko[1] **and Alexey Gusakov**[2]

1 **Moscow State University, Yandex**
`maxim.babenko@gmail.com`
2 **Moscow State University, Google**
`agusakov@gmail.com`

Abstract

By a *T-star* we mean a complete bipartite graph $K_{1,t}$ for some $t \le T$. For an undirected graph G, a *T-star packing* is a collection of node-disjoint T-stars in G. For example, we get ordinary matchings for $T = 1$ and packings of paths of length 1 and 2 for $T = 2$. Hereinafter we assume that $T \ge 2$.

Hell and Kirkpatrick devised an ad-hoc augmenting algorithm that finds a T-star packing covering the maximum number of nodes. The latter algorithm also yields a min-max formula.

We show that T-star packings are reducible to network flows, hence the above problem is solvable in $O(m\sqrt{n})$ time (hereinafter n denotes the number of nodes in G, and m — the number of edges).

For the edge-weighted case (in which weights may be assumed positive) finding a maximum T-packing is NP-hard. A novel $\frac{9}{4}\frac{T}{T+1}$-factor approximation algorithm is presented.

For non-negative node weights the problem reduces to a special case of a max-cost flow. We develop a divide-and-conquer approach that solves it in $O(m\sqrt{n}\log n)$ time. The node-weighted problem with arbitrary weights is more difficult. We prove that it is NP-hard for $T \ge 3$ and is solvable in strongly-polynomial time for $T = 2$.

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1 Introduction

1.1 Preliminaries

Recall the classical *maximum matching problem*: given an undirected graph G the goal is to find a collection M (called a *matching*) of node-disjoint edges covering as many nodes as possible. Motivated by this definition, one may consider an arbitrary (possibly infinite) collection of undirected graphs $\mathcal{G}$, called *allowed*, and ask for a collection $\mathcal{M}$ of node-disjoint subgraphs of G (not necessarily spanning) such that every member of $\mathcal{M}$ is isomorphic to some graph in $\mathcal{G}$. Let the *size* of $\mathcal{M}$ be the total number of nodes covered by the elements of $\mathcal{M}$. The generalized matching problem [8] asks for a $\mathcal{G}$-matching of maximum size.

Clearly, the tractability of the generalized problem depends solely on the choice of $\mathcal{G}$. The case when all graphs in $\mathcal{G}$ are bipartite was investigated by Hell and Kirkpatrick [8]. Roughly speaking, in this case the maximum $\mathcal{G}$-matching problem is NP-hard unless $\mathcal{G} =$

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$\{K_{1,1}, \ldots, K_{1,T}\}$ for some $T \geq 1$. (For a precise statement, see [8, Sec. 4].) This is exactly the case we study throughout the paper.

▶ **Definition 1.** A *T-star* is a graph $K_{1,t}$ for some $1 \leq t \leq T$. For an undirected graph G, a *T-star packing* in G is a collection of node-disjoint subgraphs in G (not necessary spanning) that are isomorphic to some T-stars.

Since 1-star packings are just ordinary matchings and are already extensively studied (see, e.g., [14]), we restrict our attention to the case $T \geq 2$.

The max-size T-star packing problem was addressed in [13, 1, 8] and others. An $O(mn)$-time ad-hoc augmenting path algorithm (hereinafter $n := |VG|$, $m := |EG|$) and a min-max formula are known. In [8] it is noted that a faster $O(m\sqrt{n})$-time algorithm can be derived using the blocking augmentation strategy (see [2, 9]), but we are not aware of any publicly available exposition. A more restrictive variant of the problem, where the stars are required to be node-induced subgraphs, is presented in [12]. An extension to node capacities is given in [15].

1.2 Our Contribution

This paper presents an alternative treatment of T-star packings that is based on network flows. In Section 2 we show how the max-size T-star packing problem reduces to finding a max-value flow in a digraph with $O(n)$ nodes and $O(m)$ arcs. This immediately implies an $O(m\sqrt{n})$-time algorithm for the max-size T-star packing problem.

The above reduction serves two purposes. Firstly, it mitigates the need for ad-hoc tricks and fits star packings into a widely studied field of network flows. Secondly, this reduction provides interesting opportunities for attacking other optimization problems that are related to T-star packings.

Let G be an edge-weighted graph and the goal is to find a T-star packing such that the sum of weights of edges belonging to stars is maximum. This problem is NP-hard and in Section 3 we present a $\frac{9}{4}\frac{T}{T+1}$-factor approximation algorithm, which is based on max-cost flows.

Finally let G be a node-weighted graph and the objective function is the sum of weights of nodes covered by stars. This case is studied in Section 4. For non-negative weights, a divide-and-conquer approach yields a nice $O(m\sqrt{n}\log n)$-time algorithm. For general weights, the complexity of the resulting problem depends on T. For $T = 2$, we give a strongly-polynomial algorithm that employs bidirected network flows. If $T \geq 3$, the problem is NP-hard.

2 Reduction to Network Flows

2.1 Auxiliary Digraphs

In this section we explain the core of our approach that relates star packings to network flows. We employ some standard graph-theoretic notation throughout the paper. For an undirected graph G we denote its sets of nodes and edges by VG and EG, respectively. For a directed graph we speak of arcs rather than edges and denote the arc set of G by AG. A similar notation is used for paths, trees, and etc.

For $U \subseteq VG$, the set of arcs entering (respectively leaving) U is denoted by $\delta^{\text{in}}_G(U)$ and $\delta^{\text{out}}_G(U)$. Also, $\gamma_G(U)$ denotes the set of arcs (or edges) with both endpoints in U and $G[U]$ denotes the subgraph of G induced by U, i.e. $G[U] = (U, \gamma_G(U))$. When the (di-)graph is clear from the context, it is omitted from notation. Also for a function $\varphi\colon U \to \mathbb{R}$ and a subset $U' \subseteq U$, let $\varphi(U')$ denote $\sum_{u \in U'} \varphi(u)$.

Let, as earlier, G be an undirected graph and $T \geq 2$ be an integer. Replace each edge in G by a pair of oppositely directed arcs and denote the resulting digraph by $\overrightarrow{G}$. The following definition is crucial:

▶ **Definition 2.** A subset of arcs $F \subseteq A\overrightarrow{G}$ is called *T-feasible* if for each node $v \in VG$ at most T arcs in F leave v and at most one arc in F enters v.

The above T-feasible arc sets are equivalent to T-star packings in the following sense:

▶ **Theorem 3.** *The maximum size of a T-feasible arc set in G is equal to the maximum size of a T-star packing. Moreover, given a T-feasible arc set F one can turn it in linear time into a T-star packing of size at least $|F|$.*

Before presenting the proof of Theorem 3, let us explain how a max-size T-feasible arc set size can be found. To this aim, split each node $v \in V\overrightarrow{G}$ into two copies, say v^1 and v^2. Each arc $(u, v) \in A\overrightarrow{G}$ is transformed into an arc (u^1, v^2). Two auxiliary nodes are added: a *source* s that is connected to every node v^1, $v \in V\overrightarrow{G}$, by arcs (s, v^1), and a *sink* t that is connected to every node v^2, $v \in V\overrightarrow{G}$, by arcs (v^2, t). We endow each arc (s, v^1) with capacity equal to T, each arc (v^2, t) with unit capacity, and the remaining arcs with infinite capacities. The resulting digraph is denoted by H.

We briefly remind the basic terminology and notation on network flows (see, e.g., [5, 18] and [16, Ch. 10]). Let Γ be a digraph with a distinguished source node s and a sink node t. The nodes in $V\Gamma - \{s, t\}$ are called *inner*. Let $u\colon A\Gamma \to \mathbb{Z}_+$ be integer arc capacities.

▶ **Definition 4.** An integer *u-feasible flow* (or just *feasible flow* if capacities are clear from the context) is a function $f\colon A\Gamma \to \mathbb{Z}_+$ such that: (i) $f(a) \leq u(a)$ for each $a \in A\Gamma$; and (ii) $\operatorname{div}_f(v) = 0$ for each inner node v.

Here $\operatorname{div}_f(v) := f(\delta^{\text{out}}(v)) - f(\delta^{\text{in}}(v))$ denotes the *divergence* of f at v. The *value* of f is $\operatorname{val}(f) := \operatorname{div}_f(s)$. A max-value feasible integer flow can be found in strongly polynomial time (see [18] and [16, Ch. 10]).

Let f is a feasible integer flow in H (regarded as a network with a source s, a sink t, and capacities u). Then $f(u^1, v^2) \in \{0, 1\}$ for each $(u, v) \in A\overrightarrow{G}$, since at most one unit of flow may leave v^2. (Hereinafter we abbreviate $f((u, v))$ to $f(u, v)$.) Define

$$F := \left\{(u, v) \in A\overrightarrow{G} \mid f(u^1, v^2) = 1\right\}.$$

Then the u-feasibility of f implies the T-feasibility of F. Moreover, this correspondence between u-feasible integer flows f and T-feasible arc sets F is one-to-one.

The augmenting path algorithm of Ford and Fulkerson [5] computes a max-value flow in H in $O(mn)$ time. Applying blocking augmentations [9, 2], the latter bound can be improved to $O(m\sqrt{n})$. (In fact for networks of the above “bipartite” type, one can prove the bound of $O(m\sqrt{\Delta})$. Here $\Delta := \min(\Delta_s, \Delta_t)$, Δ_s is the sum of capacities of arcs leaving s, and Δ_t is the sum of capacities of arcs entering t.)

Therefore by Theorem 3, a maximum T-star packing can be found in $O(m\sqrt{n})$ time. (The *clique compression technique* [4] implies a somewhat better time bound; however, the speedup is only sublogarithmic.)

2.2 Proof of Theorem 3

The proof consists of two parts. For the easy one, let $\mathcal{P}$ be a T-star packing in G. To construct a T-feasible arc set F, take every star $S \in \mathcal{P}$. Let v be its *central* node (i.e. a node of maximum degree) and $u_1, \ldots, u_t$ be its *leafs* (i.e. the remaining nodes). For $S = K_{1,1}$ the notion of a central node is ambiguous but any choice will do. Add arcs $(v, u_1), \ldots, (v, u_t)$ and also (u_1, v) to F. Clearly F is T-feasible and its size coincides with the number of nodes covered by $\mathcal{P}$.

The reverse reduction is more involved. Consider a T-feasible arc set F. Then F decomposes into a collection of node-disjoint weakly connected components. We deal with each of these components separately and construct a T-star packing $\mathcal{P}$ of size at least $|F|$. Let Q be one of the above components. One can easily see that two cases are possible:

Case I: Q forms a directed out-tree $\mathcal{T}$ where each node has at most T children and the arcs are directed towards leafs. The following pruning is applied iteratively to $\mathcal{T}$. Pick an arbitrary leaf u_1 in $\mathcal{T}$ of maximum depth, let v be the parent of u_1 and $u_2, \ldots, u_t$ be the siblings of u_1. Clearly $t \leq T$. Remove nodes $v, u_1, \ldots, u_t$ together with incident arcs from $\mathcal{T}$ and add to $\mathcal{P}$ a copy of $K_{1,t}$, where v is its center and $u_1, \ldots, u_t$ are the leafs. Repeat the process until $\mathcal{T}$ is empty or consists of a single node (the root r). Each time a star covering $t+1$ nodes is added to $\mathcal{P}$, either $t+1$ (if $u \neq r$) or t (if $u = r$) arcs are removed from $\mathcal{T}$. At the end one gets a T-star packing of size at least $|AQ|$ nodes, as required.

Case II: Q consists of a directed cycle Ω and a number (possibly zero) of directed out-trees attached to it (see Fig. 1(a) for an example). Let $g_0, \ldots, g_{l-1}$ be the nodes of Ω (in the order of their appearance on the cycle). For $i = 0, \ldots, l-1$, let $\mathcal{T}_i$ be the directed out-tree rooted at g_i in Q. (If no tree is attached to g_i, then we regard $\mathcal{T}_i$ as consisting solely of its root node g_i.) Each node in the latter trees has at most T children, and the roots of these trees have at most $T-1$ children. We process the trees $\mathcal{T}_0, \ldots, \mathcal{T}_{l-1}$ like in Case I and obtain a partial packing $\mathcal{P}$. Our final task is to modify $\mathcal{P}$ to satisfy the following condition: each node $v \in VQ$ that has an incoming arc in F is covered by a star in $\mathcal{P}$. So far, the above condition is only violated for nodes in Ω that are not covered by $\mathcal{P}$.

Two subcases are possible. First, suppose that all nodes of Ω are not covered. Then one can cover Ω by a collection of node-disjoint (and also disjoint from $\mathcal{P}$) paths of lengths 1 and 2. Adding these paths to $\mathcal{P}$ finishes the job. (Note that this is exactly where we use the condition $T \geq 2$.)

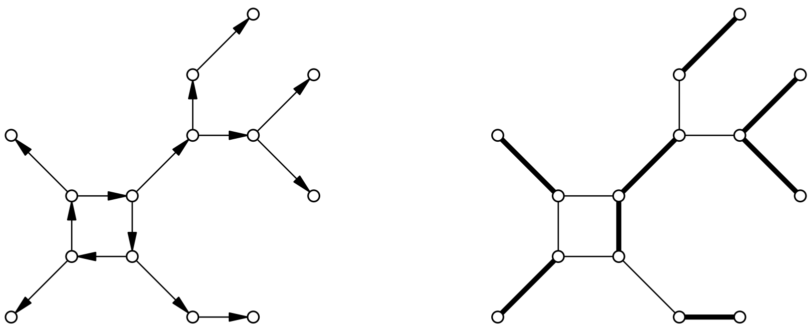
Second, suppose that Ω contains both covered and not covered nodes. Let $g_i, \ldots, g_j$ be a maximal consecutive segment of uncovered nodes, i.e. g_{i-1} and g_{j+1} are covered (indices are taken modulo l). If $j-i$ is odd, then adding $(j-i+1)/2$ disjoint copies of $K_{1,1}$ covering $g_i, \ldots, g_j$ completes the proof. Otherwise let $j-i$ be even. Recall that g_{i-1} is covered by some star $S \in \mathcal{P}$ and g_{i-1} is its central node. Since the degree of g_{i-1} in S is at most $T-1$, one can augment S by adding a new leaf g_i. This way g_i gets covered and the case reduces to the previous one. An example is depicted in Fig. 1(b).

Clearly F can be converted into $\mathcal{P}$ in linear time. ◀

3 Edge-Weighted Packings

3.1 Hardness

Consider arbitrary *edge weights* $w \colon EG \to \mathbb{Q}$ and let the *edge weight* $w(S)$ of a star S be the sum of weights of its edges. In this section we focus on finding a T-star packing $\mathcal{P}$ that maximizes $w(\mathcal{P}) := \sum_{S \in \mathcal{P}} w(S)$. Allowing negative edge weights is redundant since such



(a) Set F. **(b)** Packing $\mathcal{P}$.

Figure 1 Transforming F into $\mathcal{P}$ ($T = 2$).

edges may be removed from G without changing the optimum. Therefore we assume that edge weights are non-negative.

▶ **Theorem 5.** *The problem of deciding, for given G, T, w, and $\lambda \in \mathbb{Q}_+$, if G contains a T-star packing of edge weight at least λ, is NP-hard even in the all-unit weight case.*

Proof. It is known (see, e.g. [8]) that deciding if G admits a perfect (i.e. covering all the nodes) $\mathcal{G}$-matching is NP-hard for $\mathcal{G} = \{K_{1,T}\}$. We reduce the latter to the edge-weighted T-star packing problem as follows. If $|VG|$ is not divisible by $|T| + 1$, then the answer is negative. Otherwise set $w(e) := 1$ for all $e \in EG$. A T-star packing $\mathcal{P}$ obeys $w(\mathcal{P}) = \frac{nT}{T+1}$ if and only if all stars in $\mathcal{P}$ are isomorphic to $K_{1,T}$. Hence solving the edge-weighted T-star packing problem enables to check if G has a perfect $\mathcal{G}$-matching. ◀

3.2 Approximation

We show how to compute, in strongly-polynomial time, a T-star packing $\mathcal{P}$ such that $w(\mathcal{P}) \geq \mathrm{OPT} \cdot \frac{4}{9}\frac{T+1}{T}$, where OPT denotes the maximum weight of a T-star packing in G. Let us extend the weights from G to $\overrightarrow{G}$, i.e. define $w(u, v) := w(v, u) := w(e)$ for $e = \{u, v\} \in EG$. Let OPT' be the maximum weight of a T-feasible arc set in $\overrightarrow{G}$.

▶ **Lemma 6.** $\mathrm{OPT}' \geq \mathrm{OPT} \cdot \frac{T+1}{T}$.

Proof. Fix a max-weight packing of T-stars $\mathcal{P}_{\mathrm{OPT}}$. Consider a star $S \in \mathcal{P}_{\mathrm{OPT}}$, and let $e_1 = \{u, v_1\}, \ldots, e_t = \{u, v_t\}$ be the edges forming S ($t \leq T$). We may assume that e_1 is a maximum-weight edge (among $e_1, \ldots, e_t$).

Consider the arc set $\{(u, v_1), (v_1, u), (u, v_2), (u, v_3), \ldots, (u, v_t)\}$ (i.e. e_1 generates a pair of opposite arcs while the other edges — just a single one). Taking the union of all these arc sets one gets a T-feasible arc set F obeying $w(F) \geq \sum_{S \in \mathcal{P}} \frac{T+1}{T} w(S) = \mathrm{OPT} \cdot \frac{T+1}{T}$, as claimed. ◀

Applying the correspondence between feasible integer flows in H and T-feasible arc sets and regarding arc weights as costs, a max-weight T-feasible arc set F can be found by a max-cost flow algorithm in strongly-polynomial time, see [18, Sec. 8.4]. (For arc costs $c\colon AH \to \mathbb{Q}$ and a flow f in H, the *cost* of f is $c(f) := \sum_a c(a)f(a)$.)

We turn F into a T-star packing $\mathcal{P}$ obeying $w(\mathcal{P}) \geq \frac{4}{9}w(F)$ as follows. Consider the weakly-connected components of F and perform a case splitting similar to that in the proof of Theorem 3. For each component Q, we extract a T-star packing $\mathcal{P}_Q$ covering some nodes of Q such that $w(\mathcal{P}_Q) \geq \frac{4}{9} w(Q)$ and then take the union $\mathcal{P} := \bigcup_Q \mathcal{P}_Q$.

Case I: Q is a directed out-tree $\mathcal{T}$ rooted at a node r. Call an arc (u, v) in $\mathcal{T}$ *even* (respectively *odd*) if the length of the r–u path in $\mathcal{T}$ is even (respectively odd). Let E^0 (respectively E^1) denote the set of edges (in G) corresponding to even (respectively odd) arcs of $\mathcal{T}$. Sets E^0 and E^1 generate T-star packings $\mathcal{P}^0$ and $\mathcal{P}^1$ in G. Choose from these a packing with the largest weight and denote it by $\mathcal{P}_Q$. Then $w(\mathcal{P}_Q) \geq \frac{1}{2}\left(w(\mathcal{P}^0) + w(\mathcal{P}^1)\right) = \frac{1}{2} w(Q) \geq \frac{4}{9} w(Q)$.

Case II: Q is a directed cycle Ω with a number of out-trees attached to it. Let $g_0, \ldots, g_{l-1}$ be the nodes of Ω (numbered in the order of their appearance) and $\mathcal{T}_0, \ldots, \mathcal{T}_{l-1}$ be the corresponding trees ($\mathcal{T}_i$ is rooted at g_i, $i = 0, \ldots, l-1$).

Subcase II.1: l is even. Choose an arbitrary node r on Ω and label the arcs of Q as *even* and *odd* as in Case I. (Note that for any node v in Q, there is a unique simple r–v path in Q.) This way, a T-star packing $\mathcal{P}_Q$ obeying $w(\mathcal{P}_Q) \geq \frac{1}{2} w(Q) \geq \frac{4}{9} w(Q)$ is constructed.

Subcase II.2: l is odd. We construct a collection of $3l$ packings (each covering a subset of nodes of Q) of total weight at least $\frac{3l-1}{2} w(Q)$. To this aim, label the arcs of $\mathcal{T}_0, \ldots, \mathcal{T}_{l-1}$ as *even* and *odd* like in Case I (starting from their roots). For $i = 0, \ldots, l-1$, let E_i^0 (respectively E_i^1) be the set of edges (in G) corresponding to even (respectively odd) arcs of $\mathcal{T}_i$. Also let $e_i = \{g_i, g_{i+1}\}$ be the i-th edge of Ω (hereinafter indices are taken modulo l). Consider the (edge sets of the) following l packings (taking $i = 0, \ldots, l-1$):

$$\begin{aligned}&\{e_i, e_{i+1}\} \cup \{e_{i+3}, e_{i+5}, \ldots, e_{i+l-2}\} \cup \\ &(E_i^1 \cup E_{i+1}^1 \cup E_{i+2}^1) \cup (E_{i+3}^0 \cup E_{i+4}^1) \cup (E_{i+5}^0 \cup E_{i+6}^1) \cup \ldots \cup (E_{i+l-2}^0 \cup E_{i+l-1}^1).\end{aligned}$$

Also consider the (edge sets of the) following $2l$ packings (taking each value $i = 0, \ldots, l-1$ **twice**):

$$\begin{aligned}&\{e_{i+1}, e_{i+3}, e_{i+5}, \ldots, e_{i+l-2}\} \cup \\ &E_i^0 \cup (E_{i+1}^0 \cup E_{i+2}^1) \cup (E_{i+3}^0 \cup E_{i+4}^1) \cup \ldots \cup (E_{i+l-2}^0 \cup E_{i+l-1}^1).\end{aligned}$$

By a straightforward calculation, one can see that the total weight of these $3l$ packings is

$$\begin{aligned}&\frac{3l-1}{2} \sum_{i=0}^{l} w(e_i) + \frac{3l-1}{2} \sum_{i=0}^{l} w(E_i^0) + \frac{3l+1}{2} \sum_{i=0}^{l} w(E_i^1) \geq \\ &\frac{3l-1}{2} \left(\sum_{i=0}^{l} w(e_i) + \sum_{i=0}^{l} w(E_i^0) + \sum_{i=0}^{l} w(E_i^1) \right) = \frac{3l-1}{2} w(Q).\end{aligned}$$

Choosing a max-weight packing $\mathcal{P}_Q$ among these $3l$ instances, one gets $w(\mathcal{P}_Q) \geq \frac{1}{3l} \cdot \frac{3l-1}{2} w(Q) \geq \frac{4}{9} w(Q)$ (since $l \geq 3$), as claimed.

The above postprocessing converting F into $\mathcal{P}$ can be done in strongly-polynomial time. Together with Lemma 6 this proves the following:

▶ **Theorem 7.** *A $\frac{9}{4}\frac{T}{T+1}$-factor approximation to the edge-weighted T-star packing problem can be found in strongly polynomial time.*

4 Node-Weighted Packings

4.1 General Weights

Now consider a node-weighted counterpart of the problem. Let $w \colon VG \to \mathbb{Q}$ be node weights, and let the *weight* of a T-star packing $\mathcal{P}$ be the sum of weights of nodes covered by $\mathcal{P}$.

Now one cannot freely assume that weights are non-negative. Indeed, removing a node with a negative weight may change the optimum (consider $G = K_{1,T}$, where the weight of the central node is negative while the weights of the others are positive). In fact, for $T \geq 3$ and arbitrary w, we get an NP-hard problem:

▶ **Theorem 8.** *The problem of deciding, for given G, $T \geq 3$, w, and $\lambda \in \mathbb{Q}$, if G contains a T-star packing of node weight at least λ, is NP-hard.*

Proof. Recall (see [11] and [14, Sec.12.3]) that the following *perfect 3-uniform hypergraph matching problem* is NP-hard: given a nonempty finite domain V, a collection of subsets $\mathcal{E} \subseteq 2^V$, where each element $X \in \mathcal{E}$ is of size 3, and an integer μ, decide if V can be covered by at exactly $\mu := |V|/3$ elements of $\mathcal{E}$.

We reduce this problem to node-weighted 3-star packings as follows. Construct a bipartite graph G taking V as the left part. For each $X = \{v_1, v_2, v_3\} \in \mathcal{E}$ add a node X to the right part and connect it to nodes v_1, v_2, v_3 in the left part. The weights of nodes in the left part are set to M, where M is a sufficiently large positive integer; the weights of nodes in the right part are -1.

Each subcollection $\mathcal{E}' \subseteq \mathcal{E}$ obeying $\bigcup \mathcal{E}' = V$ generates a packing $\mathcal{P}$ of 3-stars (with centers located in the right part and leafs — in the left one). Clearly $w(\mathcal{P}) = M \cdot |V| - |\mathcal{E}'|$.

Vice versa, consider a max-weight packing $\mathcal{P}$ of 3-stars. Assuming $\bigcup \mathcal{E} = V$, $\mathcal{P}$ must cover all nodes in the left part of G (since M is large enough). Let $\mathcal{E}'$ be the set of nodes in the right part of G that are covered by $\mathcal{P}$. Then $\bigcup \mathcal{E}' = V$ and $w(\mathcal{P}) = M \cdot |V| - |\mathcal{E}'|$. Therefore V can be covered by μ elements of $\mathcal{E}$ if and only if G admits a 3-star packing of weight at least $\lambda := M \cdot |V| - \mu$. The reduction is complete. ◀

4.2 Non-Negative Weights

If node weights are non-negative then the problem is tractable. Recall the construction of the auxiliary network H and assign non-negative arc costs $c\colon AH \to \mathbb{Q}$ as follows: $c(v^2, t) := w(v)$ for all $v \in VG$ and $c(a) := 0$ for the other arcs a. Then by Theorem 3 computing a max-cost flow in H also solves the maximum weight T-star packing problem. The max-cost flow problem is solvable in strongly-polynomial time (see [6, 7] and also [16, Ch.12] for a survey) but using a general method here is an overkill. Note that the costs are non-zero only on arcs incident to the sink. This makes the problem essentially lexicographic.

In what follows, we employ an equivalent treatment, which involves *multi-terminal* networks. Namely, let Γ be a digraph endowed with arbitrary arc capacities u. Consider a set of *sources* S and a sink t ($S \subseteq V\Gamma$, $t \in V\Gamma$, $t \notin S$). Nodes in $V\Gamma - S - \{t\}$ are called *inner*. The notion of feasible flows (see Definition 4) extends to multi-terminal networks. Sometimes we use the term *S–t flow* to emphasize that f is a multi-source flow.

The *value* of an S–t flow f is $\mathrm{val}(f) := \sum_{s \in S} \mathrm{div}_f(s)$. Also let $w\colon S \to Q_+$ be *weights* of sources. The *weight* of f is defined as $w(f) := \sum_{s \in S} w(s) \,\mathrm{div}_f(s)$. The goal is to find a feasible S–t flow f of maximum weight $w(f)$. When $S = \{s\}$ and $w(s) = 1$, this coincides with the usual max-value flow problem.

Clearly this problem is equivalent to its *multi-sink* counterpart (where weights are assigned to sinks rather than sources). Consider the digraph H constructed in Section 2. Splitting the sink t into n copies (one for each node in VG) and assigning weights to these new sinks appropriately, one reduces the node-weighted star packing problem to the max-weight multi-sink flow problem.

In what follows, we deal with the max-weight multi-source flow problem in Γ. To solve the latter, we present a divide-and-conquer algorithm, which is inspired by [17]. Our flow-based approach, however, is more general and is also much simpler to explain.

For $S', T' \subseteq V\Gamma$, $S' \cap T' = \emptyset$, a subset $X \subseteq V\Gamma$ such that $S' \subseteq X$, $T' \cap X = \emptyset$, is called an *S'–T' cut*. When S' or T' is singleton the notation is abbreviated accordingly. A cut X is called *minimum* (among all S'–T' cuts) if $c(\delta^{\mathrm{out}}(X))$ is minimum. A u-feasible flow f is said to *saturate* X if $f(a) = u(a)$ for all $a \in \delta^{\mathrm{out}}(X)$ and $f(a) = 0$ for all $a \in \delta^{\mathrm{in}}(X)$. In other words, $f(\delta^{\mathrm{out}}(X)) = u(\delta^{\mathrm{out}}(X))$ and $f(\delta^{\mathrm{in}}(X)) = 0$.

Recall that for a u-feasible flow f in a digraph Γ, the *residual graph* $\Gamma_f = (V\Gamma_f := V\Gamma, A\Gamma_f)$ contains *forward* arcs $a = (u, v) \in A\Gamma$, where $f(a) < u(a)$ (endowed with the *residual* capacity $u_f(a) := u(a) - f(a)$), and also *backward* arcs $a^{-1} = (v, u)$, where $a = (u, v) \in A\Gamma$, $f(a) > 0$ (endowed with the *residual* capacity $u_f(a^{-1}) := f(a)$). For a u-feasible flow f is Γ and a u_f-feasible flow g in Γ_f the *sum* $f \oplus g$ is a u-feasible flow in Γ defined by $(f \oplus g)(a) := f(a) + g(a) - g(a^{-1})$ (where terms corresponding to non-existent arcs are assumed to be zero).

W.l.o.g. no arc enters a source and no arc leaves a sink in Γ. Sort the sources in the order of decreasing weight: $w(s_1) \geq w(s_2) \geq \ldots \geq w(s_k)$. For $i = 1, \ldots, k$, define $S_i := \{s_1, \ldots, s_i\}$. We find a feasible S–t flow f and a collection of cuts $X_1, \ldots, X_k$ such that:

(1) (i) $X_1 \subseteq X_2 \subseteq \ldots \subseteq X_k$;
(ii) for $i = 1, \ldots, k$, $X_i \cap S = S_i$, $t \notin X_i$, and f saturates X_i.

▶ **Lemma 9.** *If (1) holds, then f is both a max-weight and a max-value flow.*

Proof. Let $d_i := w(s_i) - w(s_{i+1})$ for $i = 1, \ldots, k-1$ and $d_k := w(s_k)$. For $i = 1, \ldots, k$, define $v_i := \mathrm{div}_f(s_1) + \ldots + \mathrm{div}_f(s_i)$. Applying Abel transformation, one gets $w(f) = d_1 v_1 + \ldots d_k v_k$.

Fix $i = 1, \ldots, k$ and describe f as a sum $f' + f''$, where f' is a feasible $\{s_1, \ldots, s_i\}$–t flow and f'' is a feasible $\{s_{i+1}, \ldots, s_k\}$–t flow (such f', f'' exist due to flow decomposition theorems, see [5]). Clearly $\mathrm{val}(f') = v_i$, therefore $v_i \leq c(\delta^{\mathrm{out}}(X_i))$. Summing over $i = 1, \ldots, k$, we get $w(f) \leq d_1 c(\delta^{\mathrm{out}}(X_1)) + \ldots + d_k c(\delta^{\mathrm{out}}(X_k))$. By (1)(ii), the above inequality holds with equality, hence f is a max-weight flow. Also taking $i = k$ in (1)(ii), we see that X_k is an S–t cut saturated by f. Therefore f is a max-value flow. ◀

It remains to explain how one can find f and X_i obeying (1). Consider an instance $I = (\Gamma, S = \{s_1, \ldots, s_k\}, t)$ (the capacities u and the weights w remain fixed during the whole computation and are omitted from notation). If $k = 1$, then solving I reduces to finding a max-value s_1–t flow f and a minimum s_1–t cut X_1.

Otherwise define $l := \lfloor k/2 \rfloor$, $S^1 := \{s_1, \ldots, s_l\}$, and $S^2 := \{s_{l+1}, s_{l+2}, \ldots, s_k\}$. Compute a max-value S^1–t flow h and the corresponding minimum S^1–t cut Z, which is saturated by h. Since no arc enters a source, we may assume that $Z \cap S = S^1$. To proceed with recursion, construct a pair of problem instances as follows. First, contract $\overline{Z} := V\Gamma - Z$ in Γ into a new sink t^1 and denote the resulting instance by $I^1 := (\Gamma^1 := \Gamma/\overline{Z}, S^1, t^1)$. Second, remove the subset Z in Γ_h (together with the incident arcs) and denote the resulting instance by $I^2 := (\Gamma^2 := \Gamma_h - Z, S^2, t)$.

Let f^1 and f^2 be optimal solutions to I^1 and I^2, respectively, which are found recursively and satisfy (1) (for $f := f^1$, $S := S^1$ and for $f := f^2$, $S := S^2$). Construct an optimal solution to I as follows. First, Z is a minimum S^1–t^1 cut in Γ^1 (since Z is a minimum S^1–t cut in Γ) and by Lemma 9, f^1 is a max-value flow. Hence f^1 saturates Z. Second, f^2 may be regarded as an S^2–t flow in Γ_h. The sum $h \oplus f^2$ forms a u-feasible S–t flow in Γ that also saturates Z. “Glue” f^1 and $h \oplus f^2$ along $\delta^{\text{in}}(Z)$, $\delta^{\text{out}}(Z)$ and construct an S–t flow f in Γ as follows:

$$f(a) := \begin{cases} f^1(a) & \text{for } a \in \gamma(Z), \\ (h \oplus f^2)(a) & \text{for } a \in \gamma(\overline{Z}), \\ u(a) & \text{for } a \in \delta^{\text{out}}(Z), \\ 0 & \text{for } a \in \delta^{\text{in}}(Z). \end{cases}$$

Let $X_1^1, X_2^1, \ldots, X_l^1$ and $X_{l+1}^2, X_{l+2}^2, \ldots, X_k^2$ be the sequence of nested cuts (as in (1)) for f^1 and f^2 (respectively). Then clearly $X_1^1, X_2^1, \ldots, X_l^1, Z \cup X_{l+1}^2, Z \cup X_{l+2}^2, \ldots, Z \cup X_k^2$ and f obey (1). The description of the algorithm is complete.

Let $\Phi(n', m')$ denote the complexity of a max-flow computation in a network with n' nodes and m' arcs. Let the above recursive algorithm be applied to a network with n nodes, m arcs, and k sources. Then its running time $T(n, m, k)$ obeys the recurrence

$$T(n, m, k) = \Phi(n, m) + T(n^1, m^1, \lfloor k/2 \rfloor) + T(n^2, m^2, \lceil k/2 \rceil) + O(n + m),$$

where $n^1 + n^2 = n + 1$, $m^1 + m^2 = m$. For a “natural” time bound Φ this yields $T(n, m, k) = O(\Phi(n, m) \cdot \log k)$ (see [10, Sec. 2.3]).

▶ **Theorem 10.** *In a network with n nodes, m arcs, and k sources a max-weight flow can be found in $O(\Phi(n, m) \cdot \log k)$ time.*

For node-weighted star packings, $\Phi(n, m) = O(m\sqrt{n})$ for the max-flow problems arising during the recursive process (due to results of [2, 9]).

▶ **Corollary 11.** *The node-weighted T-star packing problem with non-negative weights is solvable in $O(m\sqrt{n} \log n)$ time.*

4.3 Node-Weighted Packings of 2-Stars

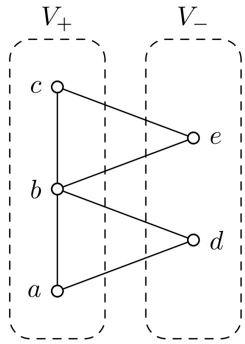
We still have a case where neither a polynomial algorithm nor a hardness result are established. Let $T = 2$ and node weights be arbitrary. Hence T-stars are just paths of length 1 and 2. This case is tractable but the needed machinery is of a bit different nature.

Recall the proof of Theorem 8. The latter fails for $T = 2$ because it shows a reduction from a version of the set cover problem where all subsets are restricted to be of size 1 and 2. The latter set cover problem is equivalent to finding a minimum cardinality *edge cover* in a general (i.e. not necessarily bipartite) graph. Both cardinality and weighted problems regarding edge covers are polynomially solvable (see [16, Ch.27]), so no hardness result can be obtained this way. However, this gives a clue on what techniques may apply here.

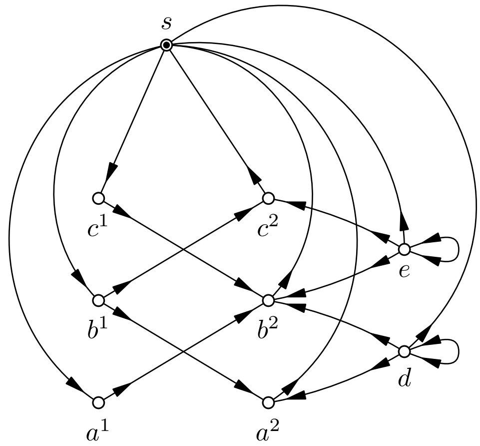
We employ the concept of bidirected graphs, which was introduced by Edmonds and Johnson [3] (more about bidirected graphs can be found in, e.g., [16, Ch. 36].) Recall that in a *bidirected* graph edges of three types are allowed: a usual directed edge, or an *arc*, that leaves one node and enters another one; an edge directed *from both* of its ends; and an edge directed *to both* of its ends. When both ends of an edge coincide, the edge becomes a loop.

The notion of a flow is extended to bidirected graphs in a natural fashion. Namely, let Γ is a bidirected graph whose edges are endowed with integer capacities $u \colon E\Gamma \to \mathbb{Z}_+$ and let s be a distinguished node (a *terminal*). Nodes in $V\Gamma - \{s\}$ are called *inner*.

▶ **Definition 12.** A u-feasible (or just feasible) integer bidirected flow f is a function $f \colon E\Gamma \to \mathbb{Z}_+$ such that: (i) $f(e) \le u(e)$ for each $e \in E\Gamma$; and (ii) $\text{div}_f(v) = 0$ for each inner node v.



(a) Graph G.



(b) Graph H.

Figure 2 Reduction to a bidirected graph.

Here, as usual, $\mathrm{div}_f(v) := f(\delta^{\mathrm{out}}(v)) - f(\delta^{\mathrm{in}}(v))$, where $\delta^{\mathrm{in}}(v)$ denotes the set of edges entering v and $\delta^{\mathrm{out}}(v)$ denotes the set of edges leaving v. It is important to note that a loop e entering (respectively leaving) a node v is counted **two times** in $\delta^{\mathrm{in}}(v)$ (respectively in $\delta^{\mathrm{out}}(v)$) and hence contributes $\pm 2f(e)$ to $\mathrm{div}_f(v)$. Similar to flows in digraphs, $f(\{u,v\})$ is abbreviated to $f(u,v)$.

Consider an undirected graph G endowed with arbitrary node weights $w\colon VG \to \mathbb{Q}$. We reduce the node-weighed 2-star packing problem in G to finding a feasible max-cost integer bidirected flow in an auxiliary bidirected graph. The latter is solvable in strongly polynomial time [16, Ch. 36].

To construct the desired bidirected graph H, denote $V_+ := \{v \in VG \mid w(v) \geq 0\}$ and $V_- := VG \setminus V_+$, Like in Section 2, consider two disjoint copies of V_+ and denote them by V_+^1 and V_+^2. Also add a terminal s and define $VH := V_+^1 \cup V_+^2 \cup V_- \cup \{s\}$.

One may assume that no two nodes in V_- are connected by an edge since these edges may be removed without changing the optimum. For an edge $\{u,v\} \in EG$, $u,v \in V_+$, construct edges $\{u^1, v^2\}$ (leaving u^1 and entering v^2) and $\{v^1, u^2\}$ (leaving v^1 and entering u^2). For an edge $\{u,v\} \in EG$, $u \in V_-$, $v \in V_+$, construct an edge $\{u, v^2\}$ (leaving u^1 and entering v^2). All these bidirected edges are endowed with infinite capacities and zero costs.

For each node $v \in V_+$, add an edge $\{s, v^1\}$ (entering v^1) of capacity 2 and zero cost, and an edge $\{v^2, s\}$ (leaving v^2) of capacity 1 and cost $w(v)$. For each node $v \in V_+$, add a loop $\{v,v\}$ (entering v twice) of capacity 1 and cost $w(v)$ and an edge $\{v,s\}$ (leaving v) of infinite capacity and zero cost. (Since s is a terminal, directions of edges at s are irrelevant.) An example is depicted in Fig. 2.

▶ **Theorem 13.** *The maximum cost of a feasible integer bidirected flow in H coincides with the maximum weight of a 2-star packing in G.*

Proof. We first show how to turn a max-weight 2-star packing $\mathcal{P}$ in G into a feasible integer bidirected flow f in H of cost $w(\mathcal{P})$. Start with $f := 0$. Let S be a star in $\mathcal{P}$. The following cases are possible.

Case I: S covers two nodes, say p and q, and $\{p,q\}$ is the edge of S.

Subcase I.1: $p, q \in V_+$. Increase f by one along the paths (s, p^1, q^2, s) and (s, q^1, p^2, s). This preserves zero divergences at inner nodes and adds $w(p) + w(q) = w(S)$ to $c(f)$.

Subcase I.2: $p \in V_+$, $q \in V_-$. Increase f by one along the path (s, p^2, q, q, s) (where the q, q fragment denotes the loop at q). Divergences at inner nodes are preserved, $c(f)$ is increased by $w(p) + w(q) = w(S)$.

Case II: S covers three nodes, say p, q, and r, and $\{p, q\}, \{q, r\}$ are the edges of S.

Subcase II.1: $p, q, r \in V_+$. Increase f by one along the paths (s, q^1, p^2, s), (s, q^1, r^2, s), and (s, p^1, q^2, s). Divergences at inner nodes are preserved, $c(f)$ is increased by $w(p)+w(q)+w(r) = w(S)$.

Subcase II.2: $p, r \in V_+$ and $q \in V_-$. Increase f by one along the path (s, p^2, q, q, r^2, s) (as above, the q, q fragment is the loop at q). Divergences at inner nodes are preserved, $c(f)$ is increased by $w(p) + w(q) + w(r) = w(S)$.

Since $\mathcal{P}$ is optimal, the other cases are impossible. Applying the above to all $S \in \mathcal{P}$ one gets a feasible integer bidirected flow of cost $w(\mathcal{P})$, as claimed.

For the opposite direction, consider a feasible max-cost integer bidirected flow f in H and construct a 2-star packing $\mathcal{P}$ obeying $w(\mathcal{P}) \geq c(f)$ as follows. Define

$$F_+ := \left\{(u, v) \mid u, v \in V_+,\ f(u^1, v^2) > 0\right\},$$
$$F_- := \left\{(u, v) \mid u \in V_-,\ v \in V_+,\ f(u, v^2) > 0\right\}.$$

Then $F := F_+ \cup F_-$ is a 2-feasible arc set in $\overrightarrow{G}$. (Recall that $\overrightarrow{G}$ is obtained from G by replacing each edge with a pair of opposite arcs.) Indeed, every arc in F leaving a node $u \in V_+$ corresponds to a unit of flow along the edge $\left\{s, u^1\right\}$ and the capacity of the latter is 2. Every arc in F leaving a node $u \in V_-$ corresponds to a unit of flow along the edge $\left\{u, v^2\right\}$, $v \in V_+$, and since the capacity of the loop $\{v, v, \}$ is 1, there can be at most 2 such arcs. Next, if an arc in F enters a node $v \in V_+$ then this arc adds a unit of flow along the edge $\left\{v^2, s\right\}$ (whose capacity is 1). Finally, no arc in F enters a node in V_-.

By Theorem 3, F generates a packing of 2-stars $\mathcal{P}$ in G. We claim that $w(\mathcal{P}) \geq c(f)$. We show that each edge $e \in EH$ with $c(e) > 0$ and $f(e) = 1$ corresponds to a node $v_e \in VG$ covered by $\mathcal{P}$ such that $c(e) = w(v_e)$. Also each node $v \in V_-$ covered by $\mathcal{P}$ corresponds to an edge $e_v \in EH$ with $f(e_v) = 1$ such that $c(e_v) = w(v)$. (The mappings $e \mapsto v_e$ and $v \mapsto e_v$ are injective.) These observations complete the proof of Theorem 13.

For the first part, consider an edge $e = \left\{v^2, s\right\}$, where $f(e) = 1$ and $v \in V_+$. Then v is entered by an arc in F, hence $\mathcal{P}$ covers $v_e := v$. For the second part, consider a node $v \in V_-$ covered by $\mathcal{P}$. Then v must be an endpoint of an arc $a \in F$. No arc in F can enter v (by the construction of F), hence $a = (v, u)$ for $u \in V_+$. Therefore $a \in F_-$ corresponds to the edge $\left\{v, u^2\right\}$. Since $f(v, u^2) > 0$ one has $f(e_v) = 1$, where $e_v := \{v, v\}$ is the loop at v. ◀

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