Report from Dagstuhl Seminar 11411

Computing with Infinite Data: Topological and Logical Foundations

Edited by

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Abstract

There is a large gap between mathematical structures and the structures computer implementations are based on. To stimulate research to overcome this—especially for infinitary structures—highly non-trivial problem the Dagstuhl Seminar 11411 "Computing with Infinite Data: Topological and Logical Foundations" was held. This report collects the ideas that were presented and discussed during the course of the seminar.

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1 Executive Summary

Dieter Spreen

In safety-critical applications it is not sufficient to produce software that is only *tested* for correctness: its correctness ought to be proven formally. This remark also applies to the area of scientific computation. An important example are autopilot systems for aircrafts. The problem is that the current mainstream approach to numerical computing uses programming languages that do not possess sound mathematical semantics. Hence, there is no way to provide formal correctness proofs.

The reason is that on the theoretical side one deals with well-developed analytical theories based on the non-constructive concept of a real number. Implementations, on the other hand, use floating-point realizations of real numbers which do not have a well-studied mathematical structure. Approaches to tackle these problems are currently promoted under the slogan "Computing with Exact Real Numbers".

Well-developed practical and theoretical bases for exact real number computation and, more generally, computable analysis are provided by Scott's Domain Theory and Weihrauch's Type Two Theory of Effectivity (TTE). In both theories real numbers and similar ideal objects are represented by infinite streams of finite objects.

The seminar focused on two problem areas in the realm of computable analysis and computation on infinite streams:

1. Algorithms for stream transforming functions with particular emphasis on (i) logical and category-theoretic methods for the synthesis of provably correct programs, (ii) topological investigations of particular stream representations supporting efficient stream algorithms.
2. Hierarchies and reducibility relations between sets and functions of infinite data as a means of classification. Methods from topology, logic and descriptive set theory were of particular importance in this case.

Infinite streams are infinite words, So there is a close connection to the theory of ω-languages. To study these was a further aim of the seminar.

In the last years much interest in the (logical and topological) structure of the infinite words used to represent continuous data as well as how they code the data space has emerged.

U. Berger and others are developing a constructive theory of digital computation based on co-induction, and are applying it to computable analysis. The aim is to create a mathematical foundation for (lazy) algorithms on analytical data such as real numbers, real functions, compact sets, etc. Since co-induction admits a particularly elegant formalization, this approach is well suited for computer aided modeling and proving in computable analysis. A concrete goal is to use program extraction from proofs as a practical method for obtaining certified programs in computable analysis.

The theory is based on a category of digit spaces. A typical object in this category is a compact metric space with a set of digits, where each digit is a contracting map. A point is then an infinite sequence of function (digit) compositions and hence represented by the corresponding infinite word. Moreover, for any given finite length, the words of that length over the alphabet of digits define a covering of the given space that allows to exactly locate the points represented by infinite words of digits. The set of uniformly continuous functions between such spaces can be characterized by a combined inductive/co-inductive definition that gives rise, via program extraction, to implementations of such functions as non-wellfounded (lazy) trees.

The theory of digit spaces combines known techniques for implementing and verifying stream processing functions (Edalat, Pattinson, Potts, and others) with ideas from co-induction and co-algebra (Jacobs, Rutten, Bertot, Niqui, and others).

H. Tsuiki is pursuing a similar programme, though, from a different perspective. In his case, e.g., every infinite object is represented by exactly one infinite word over $\{0, 1, \perp\}$, where $\perp$ represents "unknown". It turned out that the encoded space has at most topological dimension n, exactly if there are not more than n occurrences of $\perp$ in the corresponding infinite code words.

Besides the problem of how to represent continuous data in an "optimal" way, it is as well an important task to distinguish between computable and non-computable functions and, in the last case, to estimate the degree of non-computability. Most functions are non-computable since they are not even continuous. A somewhat easier and more principal task is in fact to understand the degree of discontinuity of functions. This is mostly achieved by defining appropriate hierarchies and reducibility relations.

In classical descriptive set theory, along with the well-known hierarchies, Wadge introduced and studied an important reducibility relation on Baire space. As shown by van Engelen et al., von Stein, Weihrauch and Hertling, this reducibility of subsets of topological spaces can be generalized in various ways to a reducibility of functions on a topological space. In this way, the degrees of discontinuity of several important computational problems were classified. It turned out that these classifications refine the so called "topological complexity" introduced in the alternative Blum-Shub-Smale approach to computability on the reals, which is used in complexity considerations in computational geometry.

Recently, reducibilities for functions on topological spaces have been used to identify computational relations between mathematical theorems. This programme, started by V. Brattka et al., can be considered as an alternative to reverse mathematics. Whereas reverse mathematics focuses on set existence axioms required to prove certain theorems, the approach pursued in computable analysis is to identify the computational power required to "compute" certain theorems. The approach, with contributions by M. de Brecht, G. Gherardi, A. Marcone, A. Pauly, M. Ziegler and V. Brattka, has led to deep new insights into computable analysis and has revealed close relations to reverse mathematics, but also some crucial differences.

Motivated initially by decidability problems for monadic second-order logic and Church's synthesis problem for switching circuits, researchers from automata theory (Büchi, Trakhtenbrot, Rabin, Wagner and many others) developed the theory of ω-languages which provides a foundation of specification, verification and synthesis of computing systems. Topological investigations in this theory have led to several hierarchies and reducibilities of languages of infinite words and trees. Instead of being just continuous, the reduction functions are now required to be computable by automata of suitable type. The resulting hierarchies (like, for example, the Wagner hierarchy) have sometimes the advantage that their levels are decidable.

Note that even the study of automata on finite words now involves topological methods in the classical form of profinite topology. Recently, deep relations of profinite topology to Stone and Priestly spaces were discovered by Pippenger and developed by Gehrke, Grigorieff and Pin. Again, suitable versions of the Wadge reducibility seem to play an important role in development of this field.

The seminar attracted 51 participants representing 15 countries and 5 continents, and working in fields such as computable analysis, descriptive set theory, exact real number computation, formal language theory, logic and topology, among them 10 young researchers working on their PhD or having just finished it. The atmosphere was very friendly, but discussions were most lively. During the breaks and until late into night, participants gathered in small groups for continuing discussions, communicating new results and exchanging ideas.

The seminar led to new research contacts and collaborations. The participants are invited to submit a full paper for a special issue of *Mathematical Structures in Computer Science*. At least one submission deals with a problem posted in the discussions following a talk.

The great success of the seminar is not only due to the participants, but also to the staff, both in Saarbrücken and Dagstuhl, who always do a great job in making everything run efficient and smoothly. Our thanks extend to both groups!

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3 Overview of Talks

3.1 An injection from the Baire space to natural numbers

Andrej Bauer (University of Ljubljana, SI)

Infinite-time Turing machines by Joel Hamkins provide a realizability model in which there is an embedding of the Baire space of number-theoretic functions into the natural numbers. The model has other strange features, for example every Π^1_1 statement is decidable, but the model is still intuitionistic.

Nevertheless, the model is interesting both for constructive mathematics as a source of counter-examples, and for computable mathematics as a notion of hyper-computation. In fact, there are Type I and Type II models of infinite-time computability.

(Video of the talk is available at http://vimeo.com/30368682.)

3.2 Jumps in the Weihrauch Lattice

Vasco Brattka (University of Cape Town, ZA)

We discuss the algebraic structure of the Weihrauch lattice, in particular the operation of a compositional product and a jump. The jump is supposed to play a similar role a Turing jumps play with respect to Turing degrees.

3.3 Applications of quasi-Polish spaces in computable analysis

Matthew de Brecht (NICT – Kyoto, JP)

We investigate countably based complete quasi-metric spaces, which we call quasi-Polish spaces. We show that this class of spaces is a natural generalization of the class of Polish spaces, but is general enough to include ω-continuous domains and other non-Hausdorff spaces that are important to theoretical computer science. Quasi-Polish spaces can be characterized as the countably based spaces with total admissible representations (with respect to Baire space) and as the spaces that are homeomorphic to the subspace of non-compact elements of an ω-continuous domain. These characterizations suggest that quasi-Polish spaces are important to the study of computable analysis, from both the perspective of Type Two Theory of Effectivity and also for approaches using domain models of spaces. We will also investigate applications of quasi-Polish spaces to the study of degrees of discontinuity of functions between spaces.

3.4 Infinite sets that satisfy the principle of omniscience in all varieties of constructive mathematics

Martin H. Escardó (University of Birmingham, GB)

In a minimalistic setting for constructive mathematics, without assuming Brouwerian axioms such as continuity, bar induction or fan theorem, we show that there are plenty of infinite sets that satisfy the omniscience principle.

3.5 Kolmogorov complexity and the geometry of Brownian motion

Willem L. Fouché (UNISA – Pretoria, ZA)

We discuss the geometry of Brownian motions which are encoded by Kolmogorov-Chaitin random reals (complex oscillations). We thus interpret Kolmogorov-Chaitin complexity in the context of the geometry of Brownian motion. We outline an effective framework for countable dense random sets of reals.

3.6 The effective theory of equivalence relations

Sy David Friedman (Universität Wien, AT)

Effectiveness has played a major role in the development of modern descriptive set theory. Nonetheless, this theme is not emphasized in the current and extremely active area of definable equivalence relations, perhaps due to the fact that there have been so many striking discoveries to be made in the non-effective theory. In this talk I discuss effectively-Borel (= Hyp) reducibility between Hyp equivalence relations on Baire space. The key result that facilitates this study concerns Hyp Wadge-reducibility: There are nonempty Hyp sets of reals A and B such that no Hyp function maps A into B or B into A. From this one can show that there are many pairwise Hyp-incomparable Hyp equivalence relations with only 2 classes and that the Silver and Harrington-Kechris-Louveau dichotomies from the classical theory are not fully effective. Many interesting open questions remain.

3.7 Computing invariant measures and pseudorandom points

Stefano Galatolo (University of Pisa, IT)

Joint work of Galatolo, Stefano; Mathieu Hyorup; Cristobal Rojas; Isaia Nisoli
Main reference Stefano Galatolo, Mathieu Hoyrup, Cristóbal Rojas, "Dynamical systems, simulation, abstract computation," arXiv:1101.0833v2 [math.DS]
URL http://arxiv.org/abs/1101.0833v2

After recalling some basic notions of ergodic theory, we will consider the problem of (more or less abstract) computation of physical invariant measures of dynamical system.

Those measures contain information on several aspects of the statistical behavior of the dynamics of the system.

We will see that in many interesting situations the physical measure is computable but there are cases of computable systems having no computable invariant measures.

We will apply these results to the problem of the existence of pseudorandom points for the dynamics. These are computable points whose dynamics has a typical statistical behavior. This problem is related to the abstract simulability of the system.

We will also see how to solve some implementation problems in the computation of invariant measures and some experiments in some nontrivial system.

3.8 Choice operators and the Bolzano-Weierstrass Theorem

Guido Gherardi (University of Bologna, IT)

Joint work of Brattka, Vasco; Gherardi, Guido; Marcone, Alberto;
Main reference V. Brattka, G. Gherardi, A. Marcone, "The Bolzano-Weierstrass Theorem is the Jump of Weak König's Lemma," arXiv:1101.0792v2 [math.LO]
URL http://arxiv.org/abs/1101.0792v2

The computational complexity of the Bolzano-Weierstrass Theorem operator in different computable metric spaces is investigated.

Its relationships with Closed Choice and Compact Choice operators are analyzed. It is in particular shown that in every computable metric space the Bolzano–Weierstrass operator is strongly Weihrauch equivalent to the derivative of Compact Choice operator.

3.9 Effective theory on arbitrary Polish spaces

Vassilios Gregoriades (TU Darmstadt, DE)

Main reference V. Gregoriades, "Effective Theory on arbitrary Polish spaces," 8th Panhellenic Logic Symposium, Ioannina, Greece, July 4–8, 2011.

We present some results regarding the class of Polish spaces which admit a recursive presentation. In particular we show that -contrary to the classical case- a recursively presented Polish space which is an uncountable set need not be effectively-Borel isomorphic to the Baire space. In order to do this we define a constructive scheme for Polish spaces by assigning to every tree T on ω a Polish space N_T which is recursively presented in T.

The effective structure of N_T depends on the combinatorial properties of T, so that for various choices of a recursive tree T the spaces N_T are not effectively-Borel isomorphic to each other. On the other hand every recursively presented Polish space is up to effective-Borel isomorphism one of the spaces N_T. This leads to the natural question of studying the classes of effective-Borel isomorphism of the spaces N_T. Some preliminary results will be exhibited.

3.10 Functionals using bounded information

Serge Grigorieff (LIAFA, CNRS & Université Paris 7)

Joint work of Grigorieff, Serge; Valarcher, Pierre

Let us define the size of a basic clopen set $[u] = \{f \in \mathbb{N}^{\mathbb{N}} \mid f \textit{ extends } u\}$ (where u is a partial function $\mathbb{N} \to \mathbb{N}$ with finite domain) as the cardinality of the domain of u. Continuity of a functional $\Phi : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}$ requires that there exists some covering of $\mathbb{N}^{\mathbb{N}}$ by basic clopen sets on which Φ is constant. Consider the following conditions on a continuous Φ.

(1_k) *(Φ is locally constant on a covering by clopens of size k)*
There is a covering of $\mathbb{N}^{\mathbb{N}}$ by basic open sets with size at most k on which Φ is constant.

(2_k) *(Φ is locally constant on a covering by disjoint clopens of size k)*
There is a covering of $\mathbb{N}^{\mathbb{N}}$ by pairwise disjoint basic open sets with size at most k on which Φ is constant.

(3_k) *(Φ uses k-bounded deterministic information)*
There exists an algorithm, using some function $\mathbb{N} \to \mathbb{N}$ as oracle, which, for every $f \in \mathbb{N}^{\mathbb{N}}$, asks for only k values of f to compute $\Phi(f)$.

(4_k) *(Φ can be computed by some Gurevich Abstract State Machine)*
There exist an integer a and $k-1$ functions $(\omega_i : \mathbb{N}^i \to \mathbb{N})_{0<i<k}$ such that there is an algorithm, using these functions as oracles, which, for every $f \in \mathbb{N}^{\mathbb{N}}$, computes $\Phi(f)$ using only k values of f at

$$a,\ \omega_1(f(a)),\ \omega_2(f(a), f(\omega_1(f(a))),\\ \omega_3(f(a), f(\omega_1(f(a))), f(\omega_2(f(a), f(\omega_1(f(a)))))), \ldots$$

Implications $(4_k) \Rightarrow (3_k) \Rightarrow (2_k) \Rightarrow (1_k)$ are straightforward.
We prove $(1_k) \Rightarrow (4_{k^2})$ (k^2 is optimal already for $(1_k) \Rightarrow (2_{k^2})$) and an effective version $(1_k^{\textit{effective}}) \Rightarrow (4_{B(k)}^{\textit{effective}})$ for some primitive recursive B.

3.11 Types for programming with infinite data

Peter G. Hancock (The University of Strathclyde – Glasgow, GB)

It seems to me worthwhile to investigate the type structure needed to program with infinite data. Programs that implement continuous functions on ultrametric spaces are themselves infinite data of "higher order". Some kind of exponential or arrow type is lurking in the vicinity, perhaps even in broad daylight.

3.12 Countably presentable locales are spatial (proved with a generalization of the Baire Category Theorem)

Reinhold Heckmann (AbsInt – Saarbrücken, DE)

We present the category of countably presentable locales, i.e. locales whose frame of opens has a presentation with countably many generators and countably many relations. All such locales are spatial, which can be proved without choice (but with excluded middle) using a generalization of the Baire Category Theorem.

3.13 Complexity issues for preorders on finite labeled forests

Peter Hertling (Universität der Bundeswehr – München, DE)

Joint work of Hertling, Peter; Selivanov, Victor;
Main reference P. Hertling and V.L. Selivanov, "Complexity Issues for Preorders on Finite Labeled Forests," in: B. Löwe, D. Normann, I. N. Soskov, A. A. Soskova (eds) 7th Conference on Computability in Europe, CiE 2011, Proceedings, LNCS, vol. 6735, Springer, pp. 112–121, 2011.
URL http://dx.doi.org/10.1007/978-3-642-21875-0_12

We prove that three preorders on the finite k-labeled forests are polynomial time computable. Together with an earlier result of the first author, this implies polynomial-time computability for an important initial segment of the corresponding degrees of discontinuity of k-partitions on the Baire space.

Furthermore, we show that on omega-labeled forests the first of these three preorders is polynomial time computable as well while the other two preorders are NP-complete.

3.14 Generalized geometric theories and set-generated classes

Hajime Ishihara (JAIST – Nomi, JP)

We introduce generalized geometric theories over a set S and their models which are subsets of S.

Then we show that the class M of models of a generalized geometric theory is set-generated in CZF, that is, there exists a subset G of M such that each set a in M is the union of subsets of a in G.

We also present some applications to algebra, topology, formal topology and Sambin's theory of basic pairs.

3.15 On function spaces and polynomial-time computability

Akitoshi Kawamura (University of Tokyo, JP)

In Computable Analysis, elements of uncountable spaces, such as the real line $\mathbb{R}$, are represented by functions on strings (or equivalently as infinite strings) and fed to the Turing machine as oracles. To obtain reasonable notions of computability and complexity, it is thus important that we choose the "right" representation (encoding) of the spaces being considered.

Let's say we have already agreed upon representations γ and δ of spaces X and Y (that are admissible with the topologies of X and Y). How would we represent the space $\mathrm{C}[X \to Y]$ of continuous functions from X to Y? It is known that there is a natural representation $[\gamma \to \delta]$ of $\mathrm{C}[X \to Y]$ which is characterized by the property that it is the poorest representation that makes function evaluation computable.

Is there a representation with a similar property also at the level of polynomial-time computability (as introduced by Ko and Friedman and extended by Kawamura and Cook)? In this talk I will point out that there is such a nice representation for the space $\mathrm{C}[\mathbb{R} \to \mathbb{R}]$, but it is not clear how generally a similar construction is possible. I would like to leave this question to open discussion at the seminar.

3.16 Counterexamples in computable continuum theory

Takayuki Kihara (Tohoku University, JP)

We provide several pathological examples of tree-like co-c.e. continua fulfilling certain incomputability properties:

(1) there is a computable planar dendrite which is not approximable from the inside by a co-c.e. tree; (2) there is a co-c.e. planar dendrite which is not approximable from the inside by a computable dendrite; (3) there is a computable contractible planar dendroid which is not approximable from the inside by a co-c.e. dendrite; (4) there is a co-c.e. contractible planar dendroid which has no computable point.

As corollaries:

(1) there is a planar computable closed set which is [computable path]-connected, but not [co-c.e. arc]-connected; (2) there is a contractible and locally contractible planar co-c.e. closed set which is not approximable from the inside by a connected computable closed set; (3) there is a contractible planar computable closed set which is not approximable from the inside by a connected and locally connected co-c.e. closed set; (4) there is a contractible planar co-c.e. closed set which has no computable point.

(This is the solution to Question of le Roux-Ziegler, and it is also a new non-basis result on Π^0_1 classes.)

3.17 Boolean algebras of regular languages

Anton Konovalov (A. P. Ershov Institute – Novosibirsk, RU)

Joint work of Konovalov, Anton; Selivanov, Victor

We characterize up to isomorphism the Boolean algebras of regular languages and of regular aperiodic languages, and show decidability of classes of regular languages related to these characterizations.

3.18 Reachability in control polynomial dynamical systems

Margarita Korovina (The University of Manchester, UK and IIS SB RAS Novosibirsk, RU)

Joint work of Korovina, Margarita; Vorobjov, Nicolai

A fundamental problem in the design of biological, chemical or physical processes is to automatically synthesise models from performance specifications. In general, by practical and theoretical reasons, it is highly nontrivial to achieve. However, in some cases, given by partial designs, it could be possible to automatically complete modelling in order to get desired properties. In this talk we discuss synthesis of finite control strategies to meet reachability and time requirements for partial designs given by controlled polynomial dynamical systems.

A controlled polynomial dynamical system is defined by a polynomial depending on control parameters. The choice of a parameter determines a certain motion. In general case, in order to achieve reachability or time requirements it is necessary to switch between motions corresponding to various control parameters at certain points of time. We focus on the following problem. For a partial design, given by CPDS, determine whether there exist finite sequences of time points and control parameters that guide the system from an initial state to a desired state. If the answer is positive then a finite control strategy is automatically synthesised. In the general class of o-minimal dynamical systems this problem is undecidable. Indeed, it has been shown that the reachability problem is already undecidable for three-dimensional piecewise constant derivative systems and two-dimensional o-minimal dynamical systems with non-determinism. Moreover, for one-dimensional o-minimal dynamical systems with non-determinism, the problem remains open. In this paper we show that it is possible to find finite control strategies for the certain broad class of one-dimensional controlled polynomial dynamical systems. The key results are summarised next. Firstly, we prove that for this class of one-dimensional controlled polynomial dynamical systems the existence of a finite control strategy is decidable and construct an polynomial complexity algorithm which synthesises finite control strategies. Secondary, for the algorithm we show an upper bound on the numbers of switches in finite control strategies. Finally, we prove that finite control strategies generated by the algorithm are time-optimal.

3.19 The Bolzano-Weierstrass principle and the cohesive principle

Alexander Kreuzer (TU Darmstadt, DE)

Let BW be the usual Bolzano-Weierstrass principle, i.e. the statement that each bounded sequence of real numbers contains a subsequence converging at the rate 2^{-n}. Let BW_{weak} be the statement that each bounded sequence of reals contains a -possibly slowly- converging subsequence, i.e. a subsequence converging but possibly without any computable rate of convergence.

We show that BW is instancewise equivalent to WKL for Σ^0_1-trees and that BW_{weak} is instancewise equivalent to the cohesive principle.

Thus the degrees $d >> 0'$ (i.e. the degrees d that contain an infinite branch of each $0'$-computable 0/1-tree) are exactly those degrees that contain for each computable sequence (x_n) a subsequence converging at the rate 2^{-n}. In particular, there is a degree d that is low over $0'$ that contains a solution of each computable instance of BW.

Using the classification of the cohesive principle of Jockush and Stephan one obtains that a slowly converging subsequence of (x_n) is computable in a degree d that is low_2, i.e. $d'' = 0''$, and thus that BW_{weak} does not compute $0'$.

We also comment on the strength of Bolzano-Weierstrass principle for weak compactness on the Hilbert space ℓ_2.

3.20 Some remarks related to Katětov's construction

Hans-Peter Albert Kuenzi (University of Cape Town, ZA)

If one deletes in the usual definition of a metric space the symmetry condition, one obtains the concept of a quasi-metric space. (A more precise definition will be given in the talk.) We discuss an asymmetric approach to M. Katětov's functions (compare with his well-known article "On universal metric spaces").

Our approach turned out to be very useful in our recent work on hyperconvexity (joint work with E. Kemajou and O.O. Otafudu) and on universality (joint work with M. Sanchis) in quasi-metric spaces.

3.21 Infinite sequential Nash equilibrium

Stephane Le Roux (TU Darmstadt, DE)

Generalisation of Martin's Theorem for games with many agents (instead of two) and many outcomes (instead of two) and Nash equilibrium (instead of winning strategy).

3.22 Reduction games and reducibilities for sets of reals

Luca Motto Ros (Universität Freiburg, DE)

The Wadge hierarchy has been generalized in many directions. One of these directions is to enlarge the class of reducing functions from the set of continuous functions to some natural class $\mathcal{F}$, usually called *reducibility*. Examples of these reducibilities are the classes D_α of those f such that $f^{-1}(D) \in \mathbf{\Delta}^0_\alpha$ for every $D \in \mathbf{\Delta}^0_\alpha$.

Given such an $\mathcal{F}$ and two sets $A, B \subseteq {}^\omega\omega$, we say that A is $\mathcal{F}$-reducible to B ($A \leq_\mathcal{F} B$ in symbols) just in case there is $f \in \mathcal{F}$ such that $f^{-1}(B) = A$. The preorder induced by $\leq_\mathcal{F}$ on the quotient of $\mathcal{P}({}^\omega\omega)$ with respect to the equivalence relation induced by $\leq_\mathcal{F}$ is called *$\mathcal{F}$-hierarchy*. The structure of each of the D_α-hierarchies, $\alpha < \omega_1$, has been completely determined under a certain weakening of the Axiom of Determinacy, called Semi-Liner Ordering principle for continuous functions, and it turns out to be isomorphic to the Wadge one. However, while in the cases $\alpha = 1, 2$ the proof of this fact follows from a characterization of the functions in D_α in terms of games, in the other cases the original proof is of a topological nature and heavily relies on the structure of the Wadge degrees.

In this talk I will introduce the concept of a reduction game, and show that many reducibilities can be characterized in terms of these games. This approach has two advantages:
- one can use reduction games to have a more "combinatorial" proof of the fact that the D_α-hierarchies are isomorphic to the Wadge one;
- unlike in the original proof, the axioms needed for this alternative proof are potentially weaker than the one used to determine the Wadge hierarchy.

3.23 Some steps towards a verified real number arithmetic

Norbert T. Müller (Universität Trier, DE)

Computing with real numbers requires special care about the validity of the found results.

Quite often, billions of operations can be performed without big problems concerning the precision. But there are also important examples were already less that 100 dependend arithmetic operations lead to grossly wrong results when performed with the usual 64bit floating point numbers.

In the talk we present an approach where exact real arithmetic is used, so rounding errors are completely avoided. Currently, we are working on tools that aim at the use of interactive proofs systems like COQ for the formal verification of the (partial) correctness of the software package.

3.24 On separation question for tree languages

Damian Niwinski (University of Warsaw, PL)

Joint work of Niwinski, Damian; Arnold, Andre; Michalewski, Henryk

Once we know that a class C is different from co-C, we may ask a more subtle question: can any two disjoint sets in C be "approximated" by some disjoint super-sets in co-C ?

In the classical hierarchies (Borel, projective, Wadge...), typically one of the two dual classes on each level enjoys this property. We pursue the question for the Rabin-Mostowski index hierarchy of alternating automata on infinite trees.

The property turns out to fail for all Sigma classes, however it is open if it holds for the Pi classes for levels > 2.

The result for trees comes through an analogous result for words, more precisely, for the index hierarchy of deterministic automata on infinite words. In this case we solve the problem completely: the approximation (separation) property holds for Pi classes and fails for Sigma classes. As a by-product we discover a simplification of the Arnold's proof of the strictness of the Rabin-Mostowski index hierarchy. More specifically: the use of the Banach Fixed-Point Theorem can be replaced by an explicit construction of a fixed point.

3.25 Conway's surreal numbers as an inductive-inductive definition

Fredrik Nordvall Forsberg (University of Wales – Swansea, GB)

The class of surreal numbers contains both the real numbers and the class of ordinals. We show how they can be naturally represented in Martin-Löf type theory by an inductive-inductive definition, and take the opportunity to introduce the principle of such definitions.

3.26 A finitisation of the infinite Ramsey's Theorem

Paulo Oliva (Queen Mary University of London, GB)

Alexander Kreuzer and Ulrich Kohlenbach have recently shown how the Erdös/Rado proof of the infinite Ramsey theorem can be formalised using "weak" König's lemma for Σ^0_1 definable trees. One can view their proof as a combination of three major principles: (1) Π_1 countable choice, (2) weak König's lemma, and (3) the infinite pigeon-hole principle. In this talk we see how each of these three principles has a neat computational interpretation via the product of selection functions. A combination of these three applications of the products of selection functions gives a construction that witnesses the no- counterexample (meta-stability) version of the infinite Ramsey's theorem.

3.27 The intermediate value theorem is not idempotent

Arno Pauly (University of Cambridge, GB)

As an example for both results and proof-techniques in computable reverse mathematics, a proof is given that the intermediate value theorem is not idempotent. This means that we it is impossible to solve two instances of the intermediate value theorem using a solution of a single instance together with computable means.

The proof techniques we use apply to choice principles in general, these are multi-valued functions mapping negative information about closed sets to members of the sets. As choice principles seem to be ubiquitous in computable reverse mathematics, the techniques are promising to be useful in many more cases.

3.28 Logic, duality and Pervin spaces

Jean-Eric Pin (University Paris-Diderot, FR)

In a recent paper, we proved that any lattice of [regular] languages can be defined by a set of [profinite] "equations". This result applies to any set of languages defined by a reasonable fragment of logic.

This result involves the description of the Stone dual of a lattice of languages of A*. For the Boolean algebra of all regular languages, the dual space is the completion of A* for a certain metric. What about the general case? Completions of quasi-uniformities do the job, but actually only a very special case is needed: the Pervin spaces. Turning to Pervin spaces simplifies a number of results and leads to an alternative point of view on Stone's duality.

3.29 Computable curves

Robert Rettinger (FernUniversität in Hagen, DE)

Several computability notions of curves in $\mathbb{R}^2$ will be discussed. We put the emphasis on open problems on this topic.

3.30 The extensional versus the intensional hierarchy over the reals

Matthias Schröder (Unibw – München, DE)

In functional programming, there are essentially two approaches to computability on the real numbers. The *extensional* approach assumes an idealistic functional language containing the real numbers as an own datatype. The *intensional* approach uses data structures of ordinary functional languages and encodes real numbers as streams using the signed-digit representation.

It is known that both approaches yield the same classes of Type-1 and Type-2 functionals over the reals. This has been shown by Bauer, Escardó and Simpson. Whether this is also the case for functionals of type $n \geq 3$ was an open problem for a long time.

In this talk I will prove the non-coincidence of the hierarchies from level 3 on. I will do this by using a result by Normann. He came up with a purely topological condition on the Kleene-Kreisel functionals over the natural numbers which is equivalent to the Coincidence Problem. So I will show that the Kleene-Kreisel space $\mathbb{N}^{\mathbb{N}^{\mathbb{N}}}$ does not satisfy this topological property.

3.31 Induction in algebra

Peter Schuster (University of Leeds, GB)

Many a concrete theorem of abstract algebra admits a short and elegant proof by contradiction but with Zorn's Lemma (ZL). A few of these theorems have recently turned out to follow in a direct and elementary way from the Principle of Open Induction distinguished by Raoult. A proof of the latter kind may be obtained systematically from a proof of the former sort, and the tree one can grow alongside the induction encodes the computation corresponding to the theorem. If the theorem has finite input data, then a finite partial order carries the required instance of induction, which thus is constructively provable. The ideal objects characteristic of any invocation of ZL are eliminated, and it is made possible to pass from classical to intuitionistic logic. This approach is intended as a contribution to a partial realisation of Hilbert's Programme, and was motivated by related work of Berger, Coquand and by the rise of dynamical and logical approaches to algebra.

3.32 Simultaneous inductive/coinductive definition of continuous functions

Helmut Schwichtenberg (Universität München, DE)

When extracting computational content from proofs in constructive analysis it can be helpful to use simultaneous inductive/coinductive definitions of (uniformly) continuous real functions. The talk reports on an attempt to design the underlying theory, based on recent work of Ulrich Berger.

3.33 Computing solution operators of boundary probems for systems of PDE

Svetlana Selivanova (Sobolev Institute of Mathematics – Novosibirsk, RU)

We discuss possibilities of applications of numerical analysis methods to proving computability (in the sense of the TTE approach) of the solution operators for boundary probems for systems of PDE.

3.34 Δ^0_α-Reductions in quasi-Polish spaces

Victor Selivanov (A. P. Ershov Institute – Novosibirsk, RU)

There are several directions to generalize the classical Wadge reducibility on the Baire space, in particular to a wider class of reducing functions or to more complicated topological spaces. For a space X and a poinclass $C \subseteq P(X)$, C-reducibility is the preorder on $P(X)$ corresponding to many-one eductions by functions on X such that the preimage of any set in C is again in C. In a series of papers. A. Andretta, D. Martin and L.M. Ros have shown that, under suitable set-theoretic assumptions, the structure of C-degrees in the Baire space is isormorphic to the structure of Wadge degrees, where C is the class of Borel sets or is a level of the Borel hierarchy. P. Hertling has shown that the structure of Wadge degrees in the space of reals is much more complicated than the structure of Wadge degrees in the Baire space.

We show that for many C-reducibilities (this applies e.g. to the case when C is any infinite level of the Borel hierarchy) the structure of C-degrees in any uncountable quasi-Polish space X is isormorphic to the structure of Wadge degrees in the Baire space. This immediately follows from the following extension and refinement of a classical fact: any two uncountable quasi-Polish spaces X,Y are C-isomorphic, where $C(X)$ is the class of sets of finite Borel rank in X, and C-isomorhism is a bijection between X and Y which preserves the classes $C(X)$ and $C(Y)$ in both directions. Quasi-Polish spaces is a natural class of spaces (countably-based completely quasi-metrizable spaces) containing all Polish spaces and all omega-continuous domains.

3.35 Joint topologies for finite and infinite words

Ludwig Staiger (Martin-Luther-Universität Halle-Wittenberg, DE)

Infinite words are often considered as limits of finite words. As topological methods have been proved to be useful in the theory of omega-languages it seems to be providing to include finite and infinite words into one (topological) space. The attempts so far have their drawbacks.

Therefore, in the present paper we investigate the possibility to join separate topologies on the space of finite words with a topology in the space of infinite words via a natural mapping. A requirement in this linking of topologies consists in the compatibility of topological properties (openness, closedness etc) of images with pre-images and vice versa.

Here we choose the natural CANTOR topology for infinite words and the delta-limit as linking mapping, and we show that several natural topologies on the space of finite words prove to be compatible with the topology of the CANTOR space. It is interesting to observe that besides the well-known prefix topology there are at least two more whose origin is from language theory, from the construction of centers and super-centers of languages.

These center- and supercenter-topologies on the space of finite words, fit into the class of L-topologies investigated by Prodinger. Moreover they exhibit special properties within the classes of topologies compatible with the CANTOR topology.

3.36 Turing machines on represented sets, a model of computation for analysis

Nazanin Tavana-Roshandel (Amir Kabir University of Technology – Teheran, IR)

Joint work of Tavana-Roshandel, Nazanin; Weihrauch, Klaus

We introduce a new type of generalized Turing machines (GTMs), which is intended as a tool for the mathematician who studies computability in Analysis.

In a single tape cell a GTM can store a symbol, a real number, a continuous real function or a probability measure, for example. The model is based on TTE, the representation approach for computable analysis. As a main result we prove that the functions that are computable via given representations are closed under GTM programming. This generalizes the well known fact that these functions are closed under composition. The theorem allows to speak about objects themselves instead of names in algorithms and proofs. By using GTMs for specifying algorithms, many proofs become more rigorous and also simpler and more transparent since the GTM model is very simple and allows to apply well-known techniques from Turing machine theory. We also show how finite or infinite sequences as names can be replaced by sets (generalized representations) on which computability is already defined via representations.

This allows further simplification of proofs. All of this is done for multi-functions, which are essential in Computable Analysis, and multi- representations, which often allow more elegant formulations. As a byproduct we show that the computable functions on finite and infinite sequences of symbols are closed under programming with GTMs. We conclude with examples of application.

3.37 "Good" strategies in infinite games

Wolfgang Thomas (RWTH Aachen, DE)

We discuss the algorithmic synthesis of winning strategies in regular infinite games, focusing on the following aspects:

(1) a new approach to connect the logical format of winning conditions (requirements) and of winning strategies,
(2) quantitative refinements concerning properties of infinite plays (e.g. the amount of nondeterminism that can be realized in a strategy, the amount of lookahead that can be granted to the opponent without affecting the possibility to win),
(3) the finite number of moves that suffices to decide the winner of a game.

3.38 A stream program that takes margin in recursive calls.

Hideki Tsuiki (Kyoto University, JP)

Joint work of Tsuiki, Hideki; Yamada, Shuji
Main reference Hideki Tsuiki, Shuji Yamada, "On Finite-time Computability Preserving Conversions," J. UCS 15(6): 1365–1380 (2009).

In this talk, we present a new kind of recursively-defined function on infinite sequences. We introduce the notion of a finite-time computable function and then a finite-time computability preserving conversion (ftcp in short) as a function which preserve finite-time computability. Then, we show that a ftcp function f can be written as

$$f(x) = (g\,x){+}{+}(drop(mu\,x)(f(tail\,x)))$$

for a computable function g which takes an infinite list and returns a finite list, and a computable function mu which takes an infinite list and returns a number. That is, the computation of f proceeds as producing by itself some part of the output $(g\,x)$ and then makes a recursive call with the tail, and takes $(mu\,x)$ number of output as a kickback and outputs only the rest.

We show that a ftcp function can be represented as an extended sliding block function, and that it is suffix-identity preserving.

3.39 Computing with infinite data in Lucid

Bill Wadge (University of Victoria, CA)

The dataflow language Lucid, invented by the author and E. A. Ashcroft around 1976, was one of the first to incorporate infinite data.

Originally, the infinite data took the form of streams of finite objects indexed by the natural numbers. Programmers could think of these streams as being generated incrementally in a dataflow network. This modest feature proved surprisingly powerful when combined with recursively defined stream functions ("filters") because the corresponding dataflow network itself grows (incrementally). The next simple step was to add so-called "space" (and other) dimensions. A variable could, for example, depend on one time and two space dimensions and be thought of as a stream of infinite matrices. We give examples of useful programs using these possibilities.

Two problems arose, however, which still await complete solutions. One is the problem of caching values of variables in the presences of large numbers of dimensions – when searching for a cached value, how do we know which coordinate values are relevant to our search? The second is the semantics of local (in terms of scope) dimensions. In particular, when local dimensions are combined with filter recursion, it appears to entail the existence of infinitely many simultaneaously active dimensions.

We will discuss some of the current approaches (due to the author, A. Faustini, J. Plaice and others) to solving these serious problems.

3.40 Uniform polynomial-time maximization of univariate analytic functions

Martin Ziegler (TU Darmstadt, DE)

Joint work of Kawamura, Akitoshi; Müller, Norbert; Rettinger, Robert; Rösnick, Carsten

Ko and Friedman have shown in the 1980ies that the maximum and the integral of a smooth (i.e. infinitely differentiable) polynomial-time computable real function is in general again polytime-computable iff P equals NP and #P, respectively. For polytime- computable analytic functions f, on the other hand, both maximum and integral are polytime-computable [Ko'91, Section 6.2], [Müller'87] — nonuniformly, i.e. for fixed f:

One reason being that a satisfactory uniform complexity theory of real operators has only been devised recently by Kawamura and Cook (2010).

But secondly, the known algorithms actually implicitly exploit 'knowing' many unspecified parameters of the fixed function to be maximized.

We present an (almost) uniform algorithm for calculating the maximum of a given analytic function on a given Jordan domain in polynomial time.

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