Knowledge Spaces and the Completeness of Learning Strategies*

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Abstract

We propose a theory of learning aimed to formalize some ideas underlying Coquand's game semantics and Krivine's realizability of classical logic. We introduce a notion of knowledge state together with a new topology, capturing finite positive and negative information that guides a learning strategy. We use a leading example to illustrate how non-constructive proofs lead to continuous and effective learning strategies over knowledge spaces, and prove that our learning semantics is sound and complete w.r.t. classical truth, as it is the case for Coquand's and Krivine's approaches.

1998 ACM Subject Classification F.4.1 Mathematical Logic; F.1.2 [Theory of Computation]: Modes of Computation; Interactive and reactive computation; I.2.6 [Artificial Intelligence]: Learning Induction.

Keywords and phrases Classical Logic, Proof Mining, Game Semantics, Learning, Realizability.

Digital Object Identifier 10.4230/LIPIcs.CSL.2012.77

1 Introduction

Several methods have been proposed to give a recursive interpretation of non-recursive constructions of mathematical objects, whose existence and properties are classically provable. A non-exhaustive list includes the continuation-based approach initiated by Griffin [10], the game theoretic semantics of classical arithmetic by Coquand [5] and Krivine's realizability of classical logic [12].

As observed by Coquand, there is a common informal idea underlying the different approaches, which is learning. With respect to the dialogic approach, learning consists into interpreting the strategy for the defender of a statement against the refuter by a strategy guiding the interaction between a learning agent and the "world", representing what can be experienced by direct computation.

Under the influence of Gold's ideas [8, 9] and of Hayashi's Limit Computable Mathematics [11] we have proposed a formal theory of "learning" and "well founded limits" in [2]. In the theory the goal of the learning process is to find an evidence, or a witness as it is usually called, of the truth of some given sentence, which is the "problem" that the learning strategy solves. Such an evidence is always tentative, since it could be attained only in the ideal

* This work was partially supported by PRIN project n. 2008H49TEH.

Computer Science Logic 2012 (CSL'12).
Editors: Patrick Cégielski, Arnaud Durand; pp. 77–91
Leibniz International Proceedings in Informatics
LIPICS Schloss Dagstuhl – Leibniz-Zentrum für Informatik, Dagstuhl Publishing, Germany

limit. The task of the learning strategy is to tell how to react to the discovery that the current guess is actually wrong, and this is done on the basis of the knowledge collected in the learning process, which includes all the "counterexamples" that have been seen up to the time.

Here we propose the same idea, but in the different perspective of topological spaces and continuous maps. We assume having an ideal object being the result of some non-effective mental construction and satisfying some decidable property. We say that this construction may be learned w.r.t. to the property if we may find a "finite approximation" of the construction which still satisfies the property. In particular given a classical proof of an existential statement, we see the computational content of the proof as the activity of guessing more and more about the object (individual) the sentence is about, without ever obtaining a full information; in a discrete setting such as natural numbers, the approximation is actually a different object, which we think as "close" to the ideal one, the "limit", only with respect to some given property which both satisfy.

When reasoning about ideal objects, we deal with descriptions rather than with the objects themselves. While learning of an ideal object we have step by step certain amounts of knowledge, which consist of pieces of evidence (e.g. decidable statements): therefore we topologize *states of knowledge* to express the idea that a continuous strategy only depends on finite positive and negative information to yield finite approximations of the ideal limit. We call *interactive realizer* any continuous function of states of knowledge that roughly tells which are the further guesses to improve the given knowledge, and how to react to the discovery of negative evidences (e.g. counterexamples to certain assumptions). We claim that an interactive realizer corresponds to a lambda term with continuations in Griffin's work, to a classical realizer in Krivine's sense and to a winning strategy in the sense of Coquand: however, we will not support this claim here.

We call a *model* any perfect (usually infinite) knowledge state. The main result of this paper is that any object that can be ideally learned in a model can be effectively learned in a finite state of knowledge approximating the model, and this state is found by means of a realizer. Since models represent classical truth, the completeness theorem can be read as stating that the learning semantics for classical proofs is complete w.r.t. classical truth, namely Tarskian truth, as it is the case for Coquand's dialogic semantics of proofs and for Krivine's classical realizability, and that the learning process is indeed effective.

The paper is organized as follows. In §2 we introduce a motivating example, which is used throughout the paper. In §3 we define the state knowledge and topology. In §4 the concept of relative truth is introduced to define sound, complete and model knowledge states. Finally in §5 we define interactive realizability and prove the completeness theorem. Due to space restrictions, proofs of technical lemmas have been omitted.

Related works

The suggestion by Coquand that the dialogic interpretation of classical proofs could be seen as learning of some abstract entities can be found in [4], a preliminary version of [5]. The idea has been illustrated by means of a suggestive example by von Plato in [14]. Beside Krivine's [12], Miquel's work in [13] illustrates in detail the behavior of classical realizability of existential statements (also in comparison to Friedman's method), and we strongly believe that the construction is a learning process in the sense of the present paper.

Learning in the limit of undecidable properties and ideal entities comes from Gold's work [8, 9], and has been recently rediscovered by Hayashi e.g. in [11]. We have investigated the concept of learning in the limit incorporating Coquand's ideas in [2], although in a combinatory rather than topological perspective. We have further elaborated the concept of learnable in the limit in [3], where a "solution" in terms of the following §5 is called "individual", since we identify the ideal limit with the map generating its approximations from states of knowledge. The formal definition of the state topology, however, is new with the present work, as well as the treatment of the general, non-monotonic case. Also the concept of "interactive realizability" has been introduced in [3], but in the simpler case of monotonic learning. Essentially the same construction as in [3] is used in [1] to define a realizability interpretation of **HA** plus excluded middle restricted to Σ^0_1-formulas. It turns out that interactive realizability is a generalisation of Kleene's realizability, and this motivates the terminology.

2 Solving problems by learning

To illustrate the idea of learning strategies, either monotonic or non-monotonic, we propose an example suggested by Coquand and developed by Fridlender in [6]. Let f_1, f_2 be total functions over $\mathbb{N}$. Fix some integer $k > 0$ and consider the statements: "there is an increasing sequence of k integers which is weakly increasing w.r.t. f_1" and "there is an increasing sequence of k integers which is weakly increasing w.r.t. f_1 and f_2". Formally:

$$\exists x_1 \ldots \exists x_k.\, x_1 < \cdots < x_k \;\wedge\; f_1(x_1) \leq \cdots \leq f_1(x_k) \tag{1}$$

$$\exists x_1 \ldots \exists x_k.\, x_1 < \cdots < x_k \;\wedge\; f_1(x_1) \leq \cdots \leq f_1(x_k) \;\wedge\; f_2(x_1) \leq \cdots \leq f_2(x_k) \tag{2}$$

We look at these statements as the *problems* of finding a k-tuple $n_1 < \cdots < n_k$ witnessing their truth. We begin by observing that these statements can be proved classically as follows. For any $f : \mathbb{N} \to \mathbb{N}$ and $A \subseteq \mathbb{N}$ we say that "n is a local minimum of f w.r.t. A" (shortly n is an f, A-minimum) if n is the minimum of f in $A \cap [n, \infty[$. Formally:

$$f, A\text{-min}(n) \Leftrightarrow \forall y \in A.\; n < y \Rightarrow f(n) \leq f(y).$$

Observe that the predicate $f, A\text{-min}(n)$ is undecidable in general, even if f is recursive and A decidable. The statement $\neg f, A\text{-min}(n)$ is classically equivalent to $\exists y \in A.\; n < y \wedge f(n) > f(y)$. For any $f : \mathbb{N} \to \mathbb{N}$ and infinite $A \subseteq \mathbb{N}$, we denote by

$$A_f = \{a \in A \mid f, A\text{-min}(a)\} \subseteq A$$

the set of all f, A-minima. We now study the set A_f.

▶ **Lemma 1.** *For any $f : \mathbb{N} \to \mathbb{N}$ and infinite $A \subseteq \mathbb{N}$, the set $A_f = \{a \in A \mid f, A\text{-min}(a)\}$ of f, A-minima is infinite, and f is monotonic over A_f.*

Proof. Toward a contradiction suppose that A_f is finite, possibly empty. Then there exists $a_0 = \min(A \setminus A_f)$, as A is infinite. By definition of a_0 for all $a \in A$ with $a \geq a_0$ we have $\neg f, A\text{-min}(a)$, that is: there is some $a' > a$, $a' \in A$ such that $f(a) > f(a')$. If we choose $a = a_0$ we deduce that there exists some $a_1 \in A$ such that $a_0 < a_1$ and $f(a_0) > f(a_1)$, and so on. By iterating the reasoning we get an infinite sequence $a_0 < a_1 < a_2 < \cdots$ such that $f(a_i) > f(a_{i+1})$ for all $i \in \mathbb{N}$, which is on turn an infinite descending chain in $\mathbb{N}$, that is an absurdity. ∎

▶ **Theorem 2.** *Both statements (1) and (2) are classically provable.*

Proof. In Lemma 1.2 take $A = \mathbb{N}$: then f_1 is monotonic over the infinite set A_{f_1}. If we take the first k elements of A_{f_1} we have an increasing sequence of k integers whose values are weakly increasing under f_1, establishing (1).

To prove (2) we use use again Lemma 1.2 taking $A = \mathbb{N}_{f_1}$, which we know to be infinite by the same lemma; then f_1, f_2 are both monotonic over the infinite set $A_{f_2} = (\mathbb{N}_{f_1})_{f_2}$. ■

The proof of Lemma 1 and its use in the proof of Theorem 2 are non-constructive, as they rely on the "computation" of the minimum of f in A for certain f and A. In order to compute the minimum of f we need, in general, to know infinitely many values of f. However, this proof may be interpreted by a computation as soon as we only require finitely many $n_1, \ldots, n_k$ that satisfy (1) or (2): $n_1, \ldots, n_k$ may be found using a finite knowledge about f.

Indeed the proof of Lemma 1 can be turned into an effective strategy to learn a solution to problem (1), assuming that f_1 is recursive. The basic remark is that we do not actually need to know the infinitely many elements of $\mathbb{N}_{f_1}$, nor we have to produce some n which belongs to $\mathbb{N}_{f_1}$ *beyond any doubt*. We can approximate the infinite set $\mathbb{N}_{f_1}$ by some finite set B, whose elements are not necessarily in $\mathbb{N}_{f_1}$, rather they are just $f_1, \mathbb{N}$-minima *as far as we know*. B is a kind of hypothesis about $\mathbb{N}_{f_1}$.

More precisely, we will find a set B such that $f_1, B\text{-min}(n)$ for all $n \in B$. This property is trivially true for any singleton set, say $\{0\}$. In the general case, if B has k elements or more, then by definition of f_1, B-min the set B is a solution to (1). Otherwise we take any $m > \max(B)$ and try adding m to B. Since we cannot decide whether any n is a local minimum of f_1, we are not allowed to increase B to $B \cup \{m\}$, because it could be the case that $f_1(n) > f_1(m)$ for some $n \in B$. Rather we update B, by removing all $n \in B$ such that $f_1(n) > f_1(m)$:

$$B' = \{n \in B \mid f_1(n) \leq f_1(m)\} \cup \{m\},$$

The new set B' includes m and satisfies the invariant property of containing only f_1, B'-minima. The cardinality of B' is not necessarily greater than that of B, so that we need an argument to conclude that, starting from the singleton $\{0\}$ and iterating the step from B to B', the learning agent will eventually reach a k-element set with the required property.

The termination argument in the case works as follows. Although the sequence of sets is not increasing w.r.t. inclusion, the knowledge that some elements are *not* local minima of f_1 grows monotonically, since more and more pairs n, m are found such that $f_1(n) > f_1(m)$. From this remark, one can prove by a fixed-point argument (over a suitable topology) that the growth of knowledge eventually ends, which implies that a set B with k elements will be found after finitely many steps. In this case we speak of *monotonic learning*.

In order to include an example of non-monotonic learning, we assume that both f_1 and f_2 are recursive, and we outline an effective computation approximating an initial segment of $(\mathbb{N}_{f_1})_{f_2}$, solving problem (2). The informal interpretation we include here will be formalized using the notion of layered valuation.

As it happens in the classical proof of Theorem 2, we iterate the same method used for problem (1), and build a $C \subseteq B$ of f_2, C-minima, where B is the current approximation of the infinite set $\mathbb{N}_{f_1}$. In doing so the learning agent assumes that B is a subset of $\mathbb{N}_{f_1}$ (though he cannot be certain of this), and that all elements of C are f_2-minima w.r.t. $\mathbb{N}_{f_1}$ (again an uncertain belief). At each step, the learner takes some $m \in B$, such that $m > \max(C)$, and manages to add m to C, possibly by removing some of its elements, by computing:

$$C' = \{p \in C \mid f_2(p) \leq f_2(m)\} \cup \{m\} \subseteq B.$$

This is only possible when such an m exists in B: if not the algorithm generating B has to be resumed to get a larger set containing an element greater than $\max(C)$. But since B does not grow monotonically, elements of C will be dropped while computing C' also because they are no longer in B. This makes the convergence proof much harder. Indeed the knowledge accumulated while building B takes the simple form of (sets of statements) $f_1(n) > f_1(m)$, and it grows monotonically; on the contrary the "knowledge" gathered while computing C consists of more complex statements of the form: $m \in B \wedge f_2(n) > f_2(m)$, with B changing non-monotonically during the computation of C. This knowledge is the conjunction of an hypothesis $m \in B$ and a fact, $f_2(n) > f_2(m)$. This second layer of of knowledge, mixing hypothesis and fact, does not grow monotonically, because any hypothesis $m \in B$ may turn out to be false: therefore it is unsafe, yet it guides the construction of C. In this case we speak *non-monotonic learning*. Non-monotonic learning is the more general form of learning.

3 States of knowledge and their topology

There are three kinds of entities in learning: questions, answers and states of knowledge. The main concern are states of knowledge, which on turn are certain sets of answers. Answers are viewed as atomic objects, since their internal structure is immaterial. Questions instead are represented indirectly by equivalence classes of answers, each to be thought of as the set of alternative, incompatible choices for an answer to the same question.

▶ **Definition 3** (Knowledge Structure and State of Knowledge)**.** A *knowledge structure* $(\mathbb{A}, \sim)$ consists of a a non-empty at most countable set $\mathbb{A}$ of *answers* and an equivalence relation $\sim\, \subseteq \mathbb{A} \times \mathbb{A}$. As a topological space, $\mathbb{A}$ is equipped with the discrete topology.

The set $\mathbb{Q} = \mathbb{A}/_\sim$ of equivalence classes $[x]$ w.r.t. $\sim$ is the set of *questions*, and it is equipped with the discrete topology.

A subset $X \subseteq \mathbb{A}$ is a *state of knowledge* if for all $x \in \mathbb{A}$ the set $X \cap [x]$ is either empty or a singleton. We denote by $\mathbb{S}$ the set of knowledge states and by $\mathbb{S}_{\mathit{fin}}$ the subset of finite elements of $\mathbb{S}$.

If $x \sim y$ then x, y are two answers to the same question. The equivalence class $[x]$ abstractly represents the question answered by x. Two answers $x, y \in \mathbb{A}$ are *compatible*, written $x\#y$, if they are not different answers to the same question:

$$x\#y \;\Leftrightarrow\; x = y \vee x \not\sim y.$$

Example 1 Let us reconsider the example in §2. A knowledge structure $(\mathbb{A}_0, \sim_0)$ for learning a solution to (1) can be defined by taking $\mathbb{A}_0 = \{(n,m) \in \mathbb{N} \times \mathbb{N} \mid n < m\}$, where we interpret a pair (n,m) as the statement: "m is a counterexample to f_1, $\mathbb{N}$-min(n)", and more precisely as the formula:

$$n < m \wedge f_1(n) > f_1(m).$$

If we think of m as the answer to the question about n, we obtain the definition of the relation $(n,m) \sim_0 (n',m')$ by $n = n'$.

A knowledge structure $(\mathbb{A}_1, \sim_1)$ for learning a solution to (2) can be defined by taking $\mathbb{A}_1 = \mathbb{A}_0$ and $\sim_1 = \sim_2$. $(\mathbb{A}_1, \sim_1)$ has not the same intended meaning as $(\mathbb{A}_0, \sim_0)$: an answer $(n,m) \in \mathbb{A}_1$ is interpreted by the statement:

$$n < m \wedge f_2(n) > f_2(m) \wedge n, m \in \mathbb{N}_{f_1}.$$

Finally we set $\mathbb{A}_2 = \mathbb{A}_0 \uplus \mathbb{A}_1 = \{(i,n,m) \mid i \in \{0,1\} \ \& \ (n,m) \in \mathbb{A}_i\}$, namely the disjoint union of $\mathbb{A}_0$ and $\mathbb{A}_1$ and we define $(i,n,m) \sim_2 (j,n',m')$ if and only if $i = j$ and $n = n'$ (that is $(n,m) \sim_i (n',m')$).

Let $\mathbb{S}_2$ be the knowledge space associated to $\mathbb{A}_2$ and $X \in \mathbb{S}_2$ be any state of knowledge. Then we interpret $(0,n,m) \in X$ by "the agent knows at X that $n < m$ and $f_1(n) > f_1(m)$", hence that n is not in $\mathbb{N}_{f_1}$. We interpret: for all $p \in \mathbb{N}$, $(0,n,p) \notin X$ by "the agent knows at X of no p such that $n < p$ and $f_1(n) > f_1(p)$", hence he believes that n is in $\mathbb{N}_{f_1}$. We write $(\mathbb{N}_{f_1})^X = \{n \in \mathbb{N} \mid \forall p \in \mathbb{N}.(0,n,p) \notin X\}$ for the set of n which the agent believes to be in $\mathbb{N}_{f_1}$ at X. In the same way we write $((\mathbb{N}_{f_1})_{f_2})^X = \{n \in (\mathbb{N}_{f_1})^X \mid \forall p \in \mathbb{N}.(1,n,p) \notin X\}$ for the set of n which the agent believes to be in $(\mathbb{N}_{f_1})_{f_2}$ at X. We will see in the following sections that $(\mathbb{A}_2, \sim_2)$ is a knowledge structure apt to learn (2).

The "state of knowledge" of a finite agent should be finite; for the sake of the theory we also consider infinite states of knowledge, which are naturally approximated by finite ones in a sense to be made precise by a topology. Let us define a query map $\mathsf{q} : \mathbb{S} \times \mathbb{Q} \to \mathcal{P}_{\mathit{fin}}(\mathbb{A})$ by $\mathsf{q}(X,[x]) = X \cap [x]$. Then $\mathsf{q}(X,[x])$ is either a singleton $\{y\}$, meaning that y is the answer at X to the question $[x]$, or the empty set, meaning that the agent knows at X of no answer to $[x]$, and he assumes that there is none. Take the discrete topology over $\mathcal{P}_{\mathit{fin}}(\mathbb{A})$; we then consider the smallest topology over $\mathbb{S}$ making q continuous.

▶ **Definition 4** (State Topology). The *state topology* $(\mathbb{S}, \Omega(\mathbb{S}))$ is generated by the sub-basics A_x, B_x, with $x \in \mathbb{A}$:

$$\begin{aligned} A_x &= \{X \in \mathbb{S}_{\mathit{fin}} \mid x \in X\} = \{X \in \mathbb{S}_{\mathit{fin}} \mid X \cap [x] = \{x\}\}, \\ B_x &= \{X \in \mathbb{S}_{\mathit{fin}} \mid X \cap [x] = \emptyset\}. \end{aligned}$$

A_x is the set of all states X such that $\mathsf{q}(X,[x]) = \{x\}$, which means that at state X the answer $x \in \mathbb{A}$ has been selected to the question $[x] \in \mathbb{Q}$; on the other hand if $X \in B_x$, that is $\mathsf{q}(X,[x]) = \emptyset$, then at X the learning agent knows of no answer to the question $[x]$. Let X, Y, Z range over $\mathbb{S}$, and s, t over $\mathbb{S}_{\mathit{fin}}$. By definition, a basic open of $\Omega(\mathbb{S})$ has the shape:

$$\mathcal{O}_{U,V} = \bigcap_{x \in U} A_x \cap \bigcap_{y \in V} B_y,$$

for finite $U, V \subseteq \mathbb{A}$. If $\neg x \# y$, that is $x \sim y$ and $x \neq y$, then $A_x \cap A_y = \emptyset$, because if $x, y \in X$ then X is inconsistent, so that $\emptyset$ is a basic open. On the other hand if $x \sim y$ then $B_x = B_y$. Therefore without loss of generality we assume U, V to be consistent, and that all basic opens of $\Omega(\mathbb{S})$ are of the form $\mathcal{O}_{s,t}$, for some $s, t \in \mathbb{S}_{\mathit{fin}}$. Summing up, assume that $s = \{x_1, \ldots, x_n\}$ and $t = \{y_1, \ldots, y_m\}$. Then $X \in \mathcal{O}_{s,t}$ means that the agent at X knows the answers $x_1, \ldots, x_n$ to the questions $[x_1], \ldots, [x_n]$ (a finite positive information), while he knows of no answer to the questions $[y_1], \ldots, [y_m]$, and assumes that there is none (a finite negative information).

The state topology is distinct from, yet strictly related to, several well-known topologies. $\Omega(\mathbb{S})$ is discrete if and only if $\mathbb{Q}$ is finite (there are finitely many equivalence classes). $\Omega(\mathbb{S})$ is homeomorphic to the product space $\Pi_{[x] \in \mathbb{Q}}([x] \uplus 1)$ of the discrete topologies over $[x] \uplus 1$, where 1 is any singleton (representing the "undefined" answer to the question $[x]$) and $\uplus$ is disjoint union. Every state topology is homeomorphic to some subspace of the Baire topology over $\mathbb{N}^{\mathbb{N}}$. The state topology is totally disconnected and Hausdorff: it is compact (that is, is a Stone space) if and only if all equivalence classes are finite. If all equivalence classes in $\mathbb{Q}$ are singletons (that is, if $\sim$ is the equality relation on $\mathbb{A}$) and $\mathbb{Q}$ is infinite then $\Omega(\mathbb{S})$ is homeomorphic to the Cantor space $2^{\mathbb{N}}$. If all equivalence classes in $\mathbb{Q}$ are infinite and $\mathbb{Q}$ is infinite, then $\Omega(\mathbb{S})$ is homeomorphic to the whole Baire space.

A clopen is an open and closed set; hence clopens are closed under complement, finite unions and intersections. There are significative examples of clopen sets in $\mathbb{S}$.

▶ **Lemma 5** (Sub-basic opens of State Topology are clopen). *Assume that $x \in \mathbb{A}$, $f : \mathbb{S} \to I$ is continuous, I is a discrete space and $J \subseteq I$. Then:*

1. *B_x is clopen*
2. *A_x is clopen*
3. *$f^{-1}(J)$ is clopen.*

▶ Remark. As a consequence of Lemma 5.1 we have that the predicate $n \in \mathbb{N}^X_{f_1}$ is continuous in X, since $\{X \mid n \in \mathbb{N}^X_{f_1}\} = B_{(0,n,n+1)}$ which is a clopen set, and indeed $n \notin \mathbb{N}^X_{f_1}$ if and only if $X \in \bigcup_{m>n} A_{(0,n,m)}$, namely the complement of $B_{(0,n,n+1)}$. A similar remark holds for $n \in (\mathbb{N}_{f_1})^X_{f_2}$.

It is instructive to compare the state topology to Scott and Lawson topologies over $\mathbb{S}$. First observe that $\mathbb{S}$ is a poset by subset inclusion, and it is downward closed. It follows that $(\mathbb{S}, \cap, \subseteq)$ is an inf-semilattice with bottom $\emptyset$. $\mathbb{S}$ is closed under arbitrary but non-empty inf, as the empty inf, namely the whole set $\mathbb{A}$, is not consistent in general. Indeed $\mathbb{S}$ is not closed under union, unless the compatibility relation is the identity. We say that X and Y are compatible w.r.t. inclusion, if $X \subseteq Z \supseteq Y$ for some $Z \in \mathbb{S}$. Clearly the union of a family $\mathcal{U} \subseteq \mathbb{S}$ belongs to $\mathbb{S}$ if and only if all elements of $\mathcal{U}$ are pairwise compatible sets. By this $\mathbb{S}$ is closed under directed sups, so it is a coherence space in the sense of Girard and a cpo, which in fact has compacts $K(\mathbb{S}) = \mathbb{S}_{\mathit{fin}}$ and it is algebraic.

It follows that the *Scott topology* over the cpo $(\mathbb{S}, \subseteq)$ is determined by taking all the A_x with $x \in \mathbb{A}$ as sub-basics. On the other hand any B_x is a Scott-closed set, since its complement $\mathbb{S} \setminus B_x$ is equal to the union $\bigcup\{A_y \mid y \in [x]\}$ of Scott opens. However B_x is not Scott-open because it is not upward closed.

Recall that (see [7]) the *lower topology* over a poset is generated by the complements of principal filters; the *Lawson topology* is the smallest refinement of both the lower and the Scott topology. In case of the cpo $(\mathbb{S}, \subseteq)$ the Lawson topology is generated by the sub-basics:

$$\overline{X\uparrow} = \{Y \in \mathbb{S} \mid X \not\subseteq Y\} \quad \text{and} \quad s\uparrow = \{Y \in \mathbb{S} \mid s \subseteq Y\},$$

for $X \in \mathbb{S}$ and $s \in \mathbb{S}_{\mathit{fin}}$, representing the negative and positive information respectively.

The state topology includes the Lawson topology, and in general it is finer than that. The next lemma tells that all Lawson opens are open in the state topology, but if some equivalence class $[x]$ is infinite then some open of the state topology is not open in the Lawson topology. Recall that $\Omega(\mathbb{S})$ denotes the family of open sets in the state topology.

▶ **Lemma 6.**

1. *All basic opens of Lawson topology are in $\Omega(\mathbb{S})$.*
2. *For all $x \in \mathbb{A}$, B_x is Lawson-open if and only if $[x]$ is finite.*

As an immediate consequence of Lemma 6 we have the following.

▶ **Theorem 7** (State versus Lawson Topology). *The state topology $\Omega(\mathbb{S})$ refines the Lawson topology over the cpo $(\mathbb{S}, \subseteq)$, and they coincide if and only if $[x]$ is finite for all $x \in \mathbb{A}$.*

4 Relative truth and layered states

Answers to a question can be either true or false. In the perspective of learning we think of truth values with respect to the actual knowledge that a learning agent can have at some stage of the process, so that we relativize the valuation of the answers to the knowledge states. Furthermore the example of learning the solution to problem (2) in §2 shows that there can be dependencies among answers in a state of knowledge. We formalize this by means of a stratification into levels of the set of answers. In the example of §1 we only need levels 0 and 1. In the definition, however, we allow any number of levels, even transfinite.

Denote with *Ord* the class of ordinals. Let us assume the existence of a map $\mathsf{lev} : \mathbb{A} \to \mathit{Ord}$, associating to each answer x the ordinal $\mathsf{lev}(x)$, and such that any two answers to the same question are of the same level. If $X \in \mathbb{S}$ and $\alpha \in \mathit{Ord}$ we write $X \upharpoonright \alpha = \{x \in X \mid \mathsf{lev}(x) < \alpha\}$. We can now make precise the notions of level and of truth of an answer w.r.t. a knowledge state. Let us denote with $2 = \{\mathsf{true}, \mathsf{false}\}$ the set of truth values.

▶ **Definition 8** (Layered Valuations)**.** A *layered knowledge structure* is any tuple $(\mathbb{A}, \sim, \mathsf{lev}, \mathsf{tr})$, with $(\mathbb{A}, \sim)$ a knowledge structure, $\mathsf{lev} : \mathbb{A} \to \mathit{Ord}$ and $\mathsf{tr} : \mathbb{A} \times \mathbb{S} \to 2$ two maps such that:

1. *Two answers to the same question have the same level*:

$$\forall x, y \in \mathbb{A}.(x \sim y \Rightarrow \mathsf{lev}(x) = \mathsf{lev}(y))$$

2. tr *is continuous*, by taking $\mathbb{A}$ and 2 with the discrete topology, $\mathbb{S}$ with the state topology $\Omega(\mathbb{S})$, and $\mathbb{A} \times \mathbb{S}$ with the product topology;
3. tr *is layered:* $\forall x \in \mathbb{A}, X \in \mathbb{S}.\mathsf{tr}(x, X) = \mathsf{tr}(x, X \upharpoonright \mathsf{lev}(x))$

Set $\mathsf{T}_x = \mathsf{tr}(x)^{-1}(\{\mathsf{true}\}) = \{X \in \mathbb{S} \mid \mathsf{tr}(x, X) = \mathsf{true}\}$, and similarly $\mathsf{F}_x = \mathsf{tr}(x)^{-1}(\{\mathsf{false}\})$. When tr is a layered valuation, if $X \in \mathsf{T}_x$ or $X \in \mathsf{F}_x$, we say that x is true or false w.r.t. X respectively. By definition, the truth of x w.r.t. X depends only on the answers of lower level than $\mathsf{lev}(x)$; it follows that

$$\mathsf{lev}(x) = 0 \Rightarrow \mathsf{tr}(x, X) = \mathsf{tr}(x, \emptyset)$$

that is, the truth value of answers of level 0 is absolute, and it depends just on the choice of tr. Level 0 answers play the role of "facts". On the contrary answers of level greater than 0 are better seen as empirical hypothesis, that are considered true as far as they are not falsified by answers of lower level.

Example 2 Continuing example 1, let us set $\mathsf{lev}(i, n, m) = i$. Then we define the meaning of the answers in $\mathbb{A}_2$ via the mapping tr by putting: $\mathsf{tr}((0, n, m), X) = \mathsf{true}$ if and only if $f_1(n) > f_1(m)$, and: $\mathsf{tr}((1, n, m), X) = \mathsf{true}$ if and only if $n, m \in (\mathbb{N}_{f_1})^X$ and $f_2(n) > f_2(m)$. Recall that $(\mathbb{N}_{f_1})^X$ is the set of n such that $(0, n, p) \notin X$ for all p, and that if $(i, n, m) \in \mathbb{A}_2 = \mathbb{A}_0 \uplus \mathbb{A}_1$ then $n < m$.

Clearly tr is layered since $\mathsf{tr}((i, n, m), X)$ does not depend on X when $i = 0$, while when $i = 1$ it depends on $X \upharpoonright 1 = \{(i, n, m) \in X \mid i = 0\}$, which is morally $\mathbb{A}_0 \cap X$.

To see that tr is continuous let us observe that $\mathsf{tr}((0, n, m), X)$ is the constant function w.r.t. X, and that $\mathsf{tr}((1, n, m), X) = \mathsf{true}$ iff and only if $f_2(n) > f_2(m)$ and $n, m \in (\mathbb{N}_{f_1})^X$, and the set of X for which this is true is a clopen as a consequence of Lemma 5.

From now on, we assume that some layered knowledge structure $(\mathbb{A}, \sim, \mathsf{lev}, \mathsf{tr})$ has been fixed, with some level map lev and some continuous layered truth predicate tr. We now introduce the set $\mathtt{S}$ of *sound* knowledge states (those from which nothing should be removed), the set $\mathtt{C}$ of *complete* knowledge states (those to which nothing should be added), the set $\mathtt{M}$

of *model* states (the "perfect" states, those from which nothing should be removed and to which nothing should be added).

▶ **Definition 9** (Sound and Complete States)**.** Let $X \in \mathbb{S}$, $x \in \mathbb{A}$. Then:
1. X is *sound* if $\forall x \in \mathbb{A}.\ x \in X \Rightarrow \mathsf{tr}(x, X) = \mathsf{true}$;
2. X is *complete* if $\forall x \in \mathbb{A}.\ X \cap [x] = \emptyset \Rightarrow \mathsf{tr}(x, X) = \mathsf{false}$;
3. X is a *model* if it is sound and complete.

We call $\mathtt{S}$, $\mathtt{C}$ and $\mathtt{M}$ the sets of sound, complete and model states respectively.

A state of knowledge X is sound if all the answers it contains are true w.r.t. X itself; X is complete if no answer which is true w.r.t. X and compatible with the answers in X can be consistently added to X; hence X is a model if it is made of answers true w.r.t. X and it is maximal. We think of a model X as a perfect representation of the world. For instance with respect to the examples in §2 and §3, if X is a model then the sets $(\mathbb{N}_{f_1})^X$ and $(\mathbb{N}_{f_1})^X_{f_2}$ are equal to $(\mathbb{N}_{f_1})$ and $(\mathbb{N}_{f_1})_{f_2}$ respectively, that is the beliefs of the agent perfectly agree with absolute truth.

In spite of this interpretation, models are far from being unique even w.r.t. a fixed map tr. Two models can include two different answers to the same question, because a question can have many true answers, while w.r.t. any state of knowledge each question is associated to a memory cell having room for a single answer.

Let us define $\mathtt{S}_x = \{X \in \mathbb{S} \mid x \in X \Rightarrow \mathsf{tr}(x, X) = \mathsf{true}\}$, or equivalently $\mathtt{S}_x = (\mathbb{S} \setminus A_x) \cup \mathsf{T}_x$; $\mathtt{C}_x = \{X \in \mathbb{S} \mid X \cap [x] = \emptyset \Rightarrow \mathsf{tr}(x, X) = \mathsf{false}\}$, that is $\mathtt{C}_x = (\mathbb{S} \setminus B_x) \cup \mathsf{F}_x$, and $\mathtt{M}_x = \mathtt{S}_x \cap \mathtt{C}_x$. Clearly we have $\mathtt{S} = \bigcap_{x \in \mathbb{A}} \mathtt{S}_x$, $\mathtt{C} = \bigcap_{x \in \mathbb{A}} \mathtt{C}_x$ and $\mathtt{M} = \bigcap_{x \in \mathbb{A}} \mathtt{M}_x$.

From a topological viewpoint, it is interesting to observe that all the above subsets of $\mathbb{S}$ are closed in $\Omega(\mathbb{S})$, while some of them are clopen.

▶ **Lemma 10.** *For all $x \in \mathbb{A}$, $\mathsf{T}_x, \mathsf{F}_x, \mathtt{S}_x, \mathtt{C}_x, \mathtt{M}_x$ are clopen in $\Omega(\mathbb{S})$. $\mathtt{S}, \mathtt{C}, \mathtt{M}$ are closed in $\Omega(\mathbb{S})$.*

It is immediate that sound sets exist, as well as complete ones: trivial examples are $\emptyset$ which is vacuously sound, and any set X including one answer x for each equivalence class $[x] \in \mathbb{Q}$, which is vacuously complete but not necessarily sound. Here is a non-trivial though simple example of these concepts.

Example 3 Suppose that $f_1(0) = 2 = f_1(n)$ for all $n > 2$, and that $f_1(1) = 1$ and $f_1(2) = 0$. If we consider the states over $\mathbb{A}_0$ only, and the restriction to $\mathbb{A}_0$ of the mapping tr in Example 2, we have the models $\{(0, 0, 1), (0, 1, 2)\}$ and $\{(0, 0, 2), (0, 1, 2)\}$. Any subset of these sets is sound, while $\{(0, n, n + 1) \mid n \in \mathbb{N}\}$ is complete but not sound.

It is not obvious, however, that models exist in general.

▶ **Theorem 11** (Existence of Models)**.** *For every layered knowledge structure* $(\mathbb{A}, \sim, \mathsf{lev}, \mathsf{tr})$ *and space of knowledge* $\mathbb{S}$ *over it, there exists a model* $X \in \mathbb{S}$*.*

Proof. Fix a layered valuation tr, and an arbitrary indexing $x_0, x_1, \ldots$ of the countable set $\mathbb{A}$. For each $x \in \mathbb{A}$ and $Y \in \mathbb{S}$ set:

$$\gamma(x, Y) = \begin{cases} \{x_i\} & \text{if } i \text{ is the minimum index } j \text{ s.t.} \\ & x_j \in [x] \wedge \mathsf{tr}(x_j, Y) = \mathsf{true}, \text{ if it exists} \\ \emptyset & \text{otherwise} \end{cases}$$

Now define inductively for each $\alpha \in \mathit{Ord}$:

$$X_\alpha = \bigcup \{\gamma(x, X_{<\alpha}) \mid \mathsf{lev}(x) = \alpha\} \quad \text{where} \quad X_{<\alpha} = \bigcup_{\beta < \alpha} X_\beta$$

In words, X_α is obtained by choosing an answer x', if any, for each equivalence class $[x]$ with $\mathsf{lev}(x) = \alpha$, such that x' is true w.r.t. all the choices made at previous stages $\beta < \alpha$. Since $x_i \in [x]$ implies that $\mathsf{lev}(x_i) = \mathsf{lev}(x)$, X_α is made of answers of level α.

Then we prove that $X = \bigcup_{\alpha \in Ord} X_\alpha$ is a model. First by construction X_α is consistent for all α, because it contains at most one answer for each equivalence class; this implies that X is consistent, since two answers in the same equivalence class are in the same X_α. Second, if $x \in X \upharpoonright \alpha$ then $x \in X_{<\alpha}$, so that:

$$\mathsf{tr}(x, X) = \mathsf{tr}(x, X \upharpoonright \mathsf{lev}(x)) = \mathsf{tr}(x, X_{<\mathsf{lev}(x)}) = \mathsf{true}$$

and X is sound. Finally, for all $x \in \mathbb{A}$, if $X \cap [x] = \emptyset$ then

$$\begin{array}{rcl} X \cap [x] = \emptyset & \Rightarrow & X_{\mathsf{lev}(x)} \cap [x] = \emptyset \\ & \Rightarrow & \forall x' \in [x].\ \mathsf{tr}(x', X_{<\mathsf{lev}(x)}) = \mathsf{false} \\ & \Rightarrow & \mathsf{tr}(x, X_{<\mathsf{lev}(x)}) = \mathsf{false} \\ & \Rightarrow & \mathsf{tr}(x, X) = \mathsf{false} \end{array}$$

by $\mathsf{tr}(x, X) = \mathsf{tr}(x, X_{<\mathsf{lev}(x)})$. Therefore X is complete and hence a model.

■

The construction of Theorem 11 is not effective, even when the layered knowledge structure is recursive. Assume that $\gamma \in Ord$ is the number of levels of the knowledge structure. If we look closely to the proof, we see that we defined a model by some $\Delta^0_{1+\gamma}$-predicate. In particular, if there are infinitely many levels, then the definition is not an arithmetical predicate. We claim that the recursive complexity in our result is optimal: for any γ there is some recursive layered knowledge structure with γ levels, whose models are all (the extensions of) $\Delta^0_{1+\gamma}$-complete predicates, and therefore are never $\Delta^0_{1+\delta}$-predicate, for any $\delta < \gamma$. In general models are not recursive sets, and *a fortiori* are not finite.

5 Interactive Realizability

Given a layered knowledge structure $(\mathbb{A}, \sim, \mathsf{lev}, \mathsf{tr})$, the goal of a learning process is to reach some sound $X \in \mathbb{S}$ which is sufficiently large to compute a solution to the problem at hand, e.g. a k-tuple $n_1, \dots, n_k$ of natural numbers witnessing the truth of (1) or of (2) in Section 2. To make this precise, we formally define what does it mean that a *problem* $P \subseteq \mathbb{N}$ has a solution α relative to a state X. Informally, we require that $\alpha(X)$ is a number continuously depending on a knowledge state X, which satisfies P whenever X is a model. In the terminology of [3] α is an "individual".

▶ **Definition 12** (Solution of a Problem w.r.t. a Knowledge Structure)**.** Let $(\mathbb{A}, \sim, \mathsf{lev}, \mathsf{tr})$ be a layered knowledge structure and $\mathbb{S}$ its space of states of knowledge. Given a continuous $\alpha : \mathbb{S} \to \mathbb{N}$ (where $\mathbb{N}$ is a discrete space) a predicate $P \subseteq \mathbb{N}$ (a *problem*), and $X \in \mathbb{S}$, we define:

1. $X \models_{\mathbb{A}} \alpha : P \Leftrightarrow \alpha(X) \in P$,
2. $\models_{\mathbb{A}} \alpha : P \Leftrightarrow \forall X \in \mathbb{S}.\ X$ is a model $\Rightarrow X \models_{\mathbb{A}} \alpha : P$.

When $\models_{\mathbb{A}} \alpha : P$ we say that α is a *solution* of P w.r.t. $(\mathbb{A}, \sim)$.

We shall omit the subscript $\mathbb{A}$ in $\models_{\mathbb{A}}$ when $\mathbb{A}$ is understood.

Example 4 Let $(\mathbb{A}_2, \sim_2)$ be the knowledge structure defined in example 1 in §3, and $\mathbb{S}_2$ its knowledge space. Fix $k \in \mathbb{N}$; writing $\langle n_1, \dots, n_k \rangle$ for the code number of the k-tuple

$n_1, \ldots, n_k$ we define the "problem" P_2:

$$P_2 = \{\langle n_1, \ldots, n_k \rangle \mid \bigwedge_{i<k} (n_i < n_{i+1} \ \wedge \ f_1(n_i) \le f_1(n_{i+1}) \ \wedge \ f_2(n_i) \le f_2(n_{i+1}))\},$$

P_2 is the set of all (coding of) k-tuple witnessing that (1) and (2) in §2 are true. Now for any $X \in \mathbb{S}$ define:

$$\alpha_2(X) = \min\{\langle n_1, \ldots, n_k \rangle \mid n_1 < \cdots < n_k \ \wedge \ n_1, \ldots, n_k \in (\mathbb{N}_{f_1})^X_{f_2}\}$$

where min is understood as the lexicographic ordering of the k-tuples. By definition the mapping α_2 picks the first k elements in the set $(\mathbb{N}_{f_1})^X_{f_2}$ in increasing order. α_2 is no dummy search procedure, is a reading primitive that assumes that X has been given. α_2 is always defined because $\mathbb{N}^X_{f_1}$ and $(\mathbb{N}_{f_1})^X_{f_2}$ are infinite for every $X \in \mathbb{S}_2$. Indeed this can be proved by a relativization to X of the argument of Lemma 1: if $n \notin \mathbb{N}^X_{f_1}$ then there exists $m \in \mathbb{N}$ s.t. $n < m$ but $f_1(n) > f_1(m)$ in the knowledge state X, namely we have $(0, n, m) \in X$. Were $\mathbb{N}^X_{f_1}$ finite, we would be able to find infinitely many such m forming an infinite increasing chain, and so an infinite descending chain via f_1. Similarly one proves that $(\mathbb{N}_{f_1})^X_{f_2}$ is infinite (quantifying over $\mathbb{N}^X_{f_1}$ in place of $\mathbb{N}$ and coding the counterexamples known at X by $(1, n, m) \in X$).

We show that α_2 is continuous. Let $\alpha_2(X) = \langle n_1, \ldots, n_k \rangle$: then $n_i \in (\mathbb{N}_{f_1})^X_{f_2}$ for all $i \le k$, and and for all $m < n_k$ with $m \ne n_1, \ldots, n_k$, we have $m \notin (\mathbb{N}_{f_1})^X_{f_2}$. Conversely one can check that for all $Y \in \mathbb{S}$, if $\bigwedge_{i \le k} n_i \in (\mathbb{N}_{f_1})^Y_{f_2}$ and $\bigwedge_{m<n_k, m \ne n_1, \ldots, n_k} m \notin (\mathbb{N}_{f_1})^Y_{f_2}$ then $\alpha_2(Y) = \langle n_1, \ldots, n_k \rangle$. But since we know that the predicate $n \in (\mathbb{N}_{f_1})^Y_{f_2}$ is continuous in Y (see the remark after Lemma 5), the last condition defines a finite intersection of clopens, which is clopen.

We show that α_2 is a solution of P_2, that is, that $\models \alpha_2 : P_2$. Let X be a model: then we have $(\mathbb{N}_{f_1})^X_{f_2} = (\mathbb{N}_{f_1})_{f_2}$. Since $\alpha(X) = \langle n_1, \ldots, n_k \rangle \in (\mathbb{N}_{f_1})^X_{f_2}$, we deduce $\alpha_2(X) \in (\mathbb{N}_{f_1})_{f_2}$, that is, that f_1 and f_2 are weakly increasing w.r.t. $n_1, \ldots, n_k$. Thus, $\alpha_2(X) \in P_2$.

A solution α is some way to produce an inhabitant $\alpha(X) \in P$ out of any model X. A learning strategy for a problem P admitting a solution α w.r.t. $(\mathbb{A}, \sim, \mathsf{lev}, \mathsf{tr})$ is ideally a search procedure of some model $X \in \mathbb{S}$. But models are in general infinite and non-recursive states of knowledge: to make learning effective we rely on the continuity of α which implies that if $\alpha(X) = n \in P$ for some model X there exists a finite $s \subseteq X$ such that $\alpha(s) = n$.

We describe the search of such finite sound approximations of a model X via certain continuous functions r over $\mathbb{S}$. The function r such that for any sound X (not necessarily a model) the set $r(X)$ is a finite set of answers that are not in X but are compatible with the answers in X and true w.r.t. X, and if $r(X) = \emptyset$ then $\alpha(X) \in P$. When such a function exists for given P and α we say that it is a *realizer* and that the solution α is realized by r.

▶ **Definition 13** (Realizers and Zeros). A *realizer* is a continuous map $r : \mathbb{S} \to \mathcal{P}_{\mathit{fin}}(\mathbb{A})$ such that for all $X \in \mathbb{S}$ and all $x \in r(X)$:

1. $X \cap [x] = \emptyset$,
2. $\mathsf{tr}(x, X) = \mathsf{true}$.

We denote by $\mathcal{R}$ the set of realizers. Finally we say that $X \in \mathbb{S}$ is a *zero* of $r \in \mathcal{R}$ if $r(X) = \emptyset$.

We can see a realizer r as the essential part of a learning strategy, which tries to update the current state of knowledge. This is obtained by evaluating $r(X)$ to get a finite set of new answers by which X could be soundly extended. To see this let $\mathsf{new} : \mathbb{S} \times \mathcal{P}_{\mathit{fin}}(\mathbb{A}) \to \mathcal{P}_{\mathit{fin}}(\mathbb{A})$ be defined by:

$$\mathsf{new}(X, U) = \{x \in U \mid X \cap [x] = \emptyset \wedge \mathsf{tr}(x, X) = \mathsf{true}\}.$$

The function new is continuous, where $\mathcal{P}_{\mathit{fin}}(\mathbb{A})$ and $\mathbb{S} \times \mathcal{P}_{\mathit{fin}}(\mathbb{A})$ are taken with discrete and product topology respectively.

Say that an *operator* over $\mathbb{S}$ is any continuous map $r : \mathbb{S} \to \mathcal{P}_{\mathit{fin}}(\mathbb{A})$, and call $Op_{\mathbb{S}}$, or simply Op, the set of operators over $\mathbb{S}$. For $r \in Op$ we set: $\widehat{r}(X) = \mathsf{new}(X, r(X))$. Note that the set of realizers is the subset of operators such that $r = \widehat{r}$.

Example 5 We propose a realizer solving the problem (1), expressed by the predicate:

$$P_1 = \{\langle n_1, \dots, n_k\rangle \mid \bigwedge_{i<k} (n_i < n_{i+1} \ \wedge \ f_1(n_i) \le f_1(n_{i+1}))\}.$$

We first define a function $\beta_1(X, \langle n_1, \dots, n_h\rangle)$ extending any list $\langle n_1, \dots, n_h\rangle$ with $h \le k$ to a solution of (1):

$$\beta_1(X, \langle n_1, \dots, n_h\rangle) = \begin{cases} \langle n_1, \dots, n_h\rangle & \text{if } h = k \\ \beta_1(X, \langle n_1, \dots, n_h, m\rangle) & \text{where } m \text{ is the minimum s.t.} \\ & m \in \mathbb{N}^X_{f_1} \text{ and} \\ & m > n_h \text{ if } h > 0. \end{cases}$$

A function solving (1) may then be defined by $\alpha_1(X) = \beta_1(X, \langle\,\rangle)$. We claim that $\models \alpha_1 : P_1$. Indeed β_1 is continuous w.r.t. X since $m \in \mathbb{N}^X_{f_1}$ is equivalent to $X \notin B_{(0,m,m+1)}$ which is a clopen by 5.1, so that α_1 is continuous. Further, if X is a model, then $\alpha_1(X) \in P_1$. We define a realizer r_1 looking for some X such that $\alpha_1(X)$ solves P_1. $r_1(X)$ takes any knowledge state X, and adds to it the first counterexample to (1) we may find in the list $\langle n_1, \dots, n_k\rangle$ generated by α_1, unless $\alpha_1(X)$ solves P_1. We define r_1 in two steps: first, we define a map g_1 finding the first counterexample to (1) in a given list:

$$g_1(\langle n_1, \dots, n_k\rangle) = \begin{cases} \{(0, n_i, n_{i+1}) \mid 1 \le i < k \text{ min. s.t.} & \\ \qquad\qquad f_1(n_i) > f_1(n_{i+1})\} & \text{if } i \text{ exists} \\ \emptyset & \text{otherwise.} \end{cases}$$

Then we define the realizer by composing g_1 with the output of α_1: $r_1(X) = g_1(\alpha_1(X))$. r_1 is a realizer, because $r_1(X)$ always outputs atoms not in X. Indeed, if $\alpha_1(X) = \langle n_1, \dots, n_k\rangle \in \mathbb{N}^X_{f_1}$, and if $r_1(X)$ output the atom $f_1(n_i) > f_1(n_{i+1})$, then $(f_1(n_i) > f_1(n_{i+1})) \notin X$ by definition of $\langle n_1, \dots, n_k\rangle \in \mathbb{N}^X_{f_1}$. We may use $r_1(X)$ to extend X until we find some X such that $r_1(X) = \emptyset$. Whenever $r_1(X) = \emptyset$ we have $g_1(\alpha_1(X)) = \emptyset$, hence $\alpha_1(X)$ solves (1) by definition of g_1.

To step from P_1 to P_2, namely to problem (2), we just replace $\mathbb{N}^X_{f_1}$ by $\mathbb{N}^X_{f_1,f_2}$, namely:

$$\beta_2(X, \langle n_1, \dots, n_h\rangle, k) = \begin{cases} \langle n_1, \dots, n_h\rangle & \text{if } h = k \\ \beta_2(X, \langle n_1, \dots, n_h, m\rangle, k) & \text{where } m \text{ is the minimum s.t.} \\ & m \notin (\mathbb{N}_{f_1})^X_{f_2} \\ & \text{and } m > n_h \text{ if } h > 0. \end{cases}$$

and

$$g_2(\langle n_1, \dots, n_k\rangle) = \begin{cases} \{(0, n_i, n_{i+1}) \mid 1 \le i < k \text{ min. s.t.} & \\ \quad f_1(n_i) > f_1(n_{i+1}) \vee f_2(n_i) > f_2(n_{i+1})\} & \text{if } i \text{ exists} \\ \emptyset & \text{otherwise.} \end{cases}$$

We have that $\alpha_2(X) = \beta_2(X, \langle\,\rangle, k)$, where α_2 is from example 4. Now let us define $r_2(X) = g_2(\beta_2(X, \langle\,\rangle, k)) = g_2(\alpha_2(X))$. Then we can show that r_2 is a realizer looking for some X such that $\alpha_2(X) \in P_2$ just as in the case of r_1 above.

If X is a model and $r \in \mathcal{R}$ then $r(X) = \emptyset$ by definition; on the other hand if $\models_{\mathbb{A}} \alpha : P$ then $\alpha(s) = \alpha(X) = n \in P$ for some finite $s \subseteq X$. Since r is continuous, the condition that reveals that the approximation s of X is good enough to compute an $n \in P$ is that $r(s) = r(X) = \emptyset$. This suggests that a constructive way to meet the requirement about models in the definition of $\models_{\mathbb{A}} \alpha : P$ is to ask that sound zeros of a realizer r are enough to find inhabitants of P via α, and then look for finite sound zeros of r. We turn this into the following definition.

▶ **Definition 14** (Interactive Realizability). Let $\alpha : \mathbb{S} \to \mathbb{N}$ be continuous (w.r.t. the discrete topology over $\mathbb{N}$), and $P \subseteq \mathbb{N}$ a predicate:

1. $r \in \mathcal{R}$ *interactively realizes* P w.r.t. α, written $r \vdash \alpha : P$, if and only if:

   $$\forall X \in \mathbb{S}.\ X \text{ sound zero of } r \Rightarrow \alpha(X) \in P$$

2. P is *interactively realizable* w.r.t. α, written $\vdash \alpha : P$, if and only if:

   $$\exists r \in \mathcal{R}.\ r \vdash \alpha : P.$$

If P_1 and P_2 are the predicates defined in examples 4 and 5, α_1, α_2 their respective solutions and r_1, r_2 the realizers from example 5; then we claim (without proof) that $r_i \vdash \alpha_i : P_i$ for both $i = 1, 2$.

The main result of the paper is that the apparently stronger $r \vdash \alpha : P$ for some $r \in \mathcal{R}$ is equivalent to $\models \alpha : P$. That is, whenever $\alpha : P$ is valid then it is interactively learnable, and we have some strategy to find some finite X such that $\alpha(X) \in P$.

Before we establish the existence of sound finite zeros of any $r \in \mathcal{R}$. This is a non trivial fact because, whenever we add to some state Y (no matter whether finite or not) a $y \in r(Y)$, we know that $\mathsf{tr}(z, Y) = \mathsf{true}$ for all $z \in Y$, but we do not know about the value of $\mathsf{tr}(z, Y \cup \{y\})$, so that $Y \cup \{y\}$ is not necessarily sound. Moreover it is not true that if $s \subseteq X$ and X is sound then s is sound.

Example 6 Let us redefine f_1 by $f_1(0) = 10, f_1(1) = 30, f_1(2) = 20$ and define $f_2(0) = 20, f_2(1) = 10, f_2(2) = 20$. Also we let $x = (1, 0, 1)$ meaning that $0 \notin (\mathbb{N}_{f_1})_{f_2}$, and $y = (0, 1, 2)$ meaning that $1 \notin \mathbb{N}_{f_1}$. Then $\mathsf{tr}(x, \{x\}) = \mathsf{true}$ since at $\{x\}$ it is likely that $0 \notin (\mathbb{N}_{f_1})_{f_2}$ because of the counterexample in the point 1; but $\mathsf{tr}(x, \{x, y\}) = \mathsf{false}$ because the discovery that $1 \in \mathbb{N}_{f_1}$ contradicts counterexample on point 1.

We prove below that finite sound zeros exist for all $r \in \mathcal{R}$ and that these are finite approximations of sound states of knowledge which are themselves zeros of r, hence in particular of models.

▶ **Lemma 15.** *If $X \in \mathbb{S}$ is sound, $s \in \mathbb{S}_{fin}$ is a finite state such that $s \subseteq X$, then there exists a finite sound $t \in \mathbb{S}_{fin}$ such that $s \subseteq t \subseteq X$.*

We are now in place to conclude the proof that every realizer has a finite sound zero. We do not provide an effective method to find some, but we claim that we can obtain it by a suitable sequence of answers added by the realizer and of removal of answers.

▶ **Theorem 16** (Existence of Sound and Finite Zeros of Realizers). *If $r \in \mathcal{R}$, then there exists a finite sound zero $t \in \mathbb{S}_{fin}$ of r.*

Proof. Models exist by Theorem 11 and they are sound by definition, hence $r(X) = \emptyset$ for some sound $X \in \mathbb{S}$ since $r \in \mathcal{R}$. By continuity there is a basic open $\mathcal{O}_{s_0,t_0}$ such that $X \in \mathcal{O}_{s_0,t_0}$ and $r(\mathcal{O}_{s_0,t_0}) = \emptyset$. This implies that $s_0 \subseteq X$ and $X \cap t_0 = \emptyset$, so that a fortiori any finite $t \in \mathbb{S}_{fin}$ such that $s_0 \subseteq t \subseteq X$ satisfies $t \cap t_0 = \emptyset$ and therefore $t \in \mathcal{O}_{s_0,t_0}$, i.e. it is a zero of r. By Lemma 15 there exists a sound t among them, which is the desired finite sound zero of r. ∎

We come now to the completeness theorem. Our thesis is that interactive realizability is complete in the sense that if $\alpha(X) \in P$ for all models X, then we may replace the model X by the finite sound zeros of a suitable realizer $r \in \mathcal{R}$.

▶ **Theorem 17.** *(Completeness of Realization) For any continuous $\alpha : \mathbb{S} \to \mathbb{N}$ and predicate $P \subseteq \mathbb{N}$: $\vdash \alpha : P \Leftrightarrow \models \alpha : P$.*

Proof.

($\Rightarrow$) If $X \in \mathbb{S}$ is a model then X is a sound zero of any realizer by Definition 13; hence if $r \vdash \alpha : P$ for some $r \in \mathcal{R}$ we immediately have $\alpha(X) \in P$, i.e. $X \models \alpha : P$ for arbitrary model X.

($\Leftarrow$) We have to show that, if $\models \alpha : P$, namely if $\alpha(X) \in P$ for $X \in \mathtt{M}$, then there exists an $r \in \mathcal{R}$ such that $r \vdash \alpha : P$. We establish the contrapositive:

$$\alpha(X) \notin P \Rightarrow X \text{ not sound } \vee \ r(X) \neq \emptyset$$

for some realizer r and arbitrary $X \in \mathbb{S}$.

If $\alpha(X) \notin P$ then, by the hypothesis, $X \notin \mathtt{M}$, hence $X \notin \mathtt{S}$ or $X \notin \mathtt{C}$. By definition of $\mathtt{S}$ and $\mathtt{C}$, this implies that $\exists x \in \mathbb{A}.\ X \notin \mathtt{S}_x \vee X \notin \mathtt{C}_x$. Fix an enumeration $x_0, x_1, \dots$ of the countable set $\mathbb{A}$. Let us define $r : \mathbb{S} \to \mathcal{P}_{fin}(\mathbb{A})$ by:

$$r(X) = \begin{cases} \emptyset & \text{if } \alpha(X) \in P \\ \{x_i\} & \text{where } i = \min\{j \in \mathbb{N} \mid X \in \mathbb{S} \setminus \mathtt{M}_{x_j}\}, \text{ else.} \end{cases}$$

Then r is a total function since if $\alpha(X) \notin P$ then $X \in \mathbb{S} - \mathtt{M}$ so that $\{x_j \in \mathbb{A} \mid X \in \mathbb{S} \setminus \mathtt{M}_{x_j}\} \neq \emptyset$. If r is continuous then $\widehat{r}(X) = \mathsf{new}(X, r(X))$ because $X \in \mathbb{S} \setminus \mathtt{M}_{x_j}$ implies $X \in \mathbb{S} \setminus \mathtt{C}_{x_j}$, and consequently, r is a realizer. We have $\widehat{r} \vdash \alpha : P$. Indeed, assume for contradiction that $\alpha(X) \notin P$, $X \in \mathtt{S}$ and $\widehat{r}(X) = \emptyset$. Then $r(X) = \{x_i\}$ and $X \in \mathbb{S} \setminus \mathtt{M}_{x_i} = (\mathbb{S} \setminus \mathtt{S}_{x_i}) \cup (\mathbb{S} \setminus \mathtt{C}_{x_i})$. Since $X \in \mathtt{S} \subseteq \mathtt{S}_{x_i}$, then $X \in \mathbb{S} \setminus \mathtt{C}_{x_i}$. We conclude that $\widehat{r}(X) = \{x_i\}$, contradiction.

To see that r is continuous it suffices to check that both $r^{-1}(\emptyset)$ and $r^{-1}(\{x\})$ (for any $x \in \mathbb{A}$) are opens in $\Omega(\mathbb{S})$. Now $r(X) = \emptyset$ if and only if $\alpha(X) \in P$, that is $X \in \alpha^{-1}(P)$ which is clopen by Lemma 10. On the other hand $X \in r^{-1}(\{x\})$ if and only if:

$$\exists i.\ x_i = x \ \wedge \ X \in (\mathbb{S} \setminus \mathtt{M}_{x_i}) \wedge \ \forall j < i.\, X \in \mathtt{M}_{x_j}.$$

This is equivalent to $X \in \mathtt{M}_{x_0} \cap \ldots \cap \mathtt{M}_{x_{i-1}} \cap (\mathbb{S} \setminus \mathtt{M}_{x_i})$ which, by Lemma 10, is a finite intersection of clopens, hence a clopen itself. ∎

6 Concluding remarks and further work

We have defined the notions of state of knowledge and of state topology. We have then redefined in the more general setting of non-monotonic learning, the concepts of individual (here called "solution") and of interactive realizer that we treated elsewhere, proving the completeness of learnability w.r.t. validity, which is the counterpart of classical truth in the present setting.

The definitions and results obtained are aimed at the development of a full theory of learning strategies and of their convergence properties, which is work in progress. We also observe that the solution and the realizer illustrated in the examples of §5 are crude simplifications of the learning strategy implicit in the example of §2, which is capable of using the counterexamples in a more ingenuous and efficient way. The investigation of the interpretation of classical proofs in terms of learning strategies is a natural further step, extending the work we have done in the monotonic case.

Since learning strategies working with finite approximations are effective (and indeed we have shown that finite and sound knowledge states exist and suffice), a question of efficiency of the algorithms one extracts from proofs with our method is naturally there, together with the analysis of suitable data structures representing time and logical dependancies, which are essential to complete the present approach.

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