Cost-Parity and Cost-Streett Games*

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— Abstract —

We consider two-player games played on finite graphs equipped with costs on edges and introduce two winning conditions, cost-parity and cost-Streett, which require bounds on the cost between requests and their responses. Both conditions generalize the corresponding classical ω-regular conditions as well as the corresponding finitary conditions.

For cost-parity games we show that the first player has positional winning strategies and that determining the winner lies in $\mathbf{NP} \cap \mathbf{coNP}$. For cost-Streett games we show that the first player has finite-state winning strategies and that determining the winner is **EXPTIME**-complete. This unifies the complexity results for the classical and finitary variants of these games. Both types of cost games can be solved by solving linearly many instances of their classical variants.

1   Introduction

In recent years, boundedness problems arose in topics pertaining to automata and logics leading to the development of novel models and techniques to tackle these problems. Although in general undecidable, many boundedness problems for automata turn out to be decidable if the acceptance condition can refer to boundedness properties of variables, but the transitions cannot access variable values. A great achievement was made by Hashiguchi [15] who proved decidability of the star-height problem by reducing it to a boundedness problem for a certain type of finite automaton and by solving this problem. This led the path to recent developments towards a general theory of bounds in automata and logics, comprising automata and logics with bounds [1, 3], satisfiability algorithms for these logics [2, 4], and regular cost-functions [10].

In this work, we consider boundedness problems in turn-based two-player graph games of infinite duration. Using the acceptance condition of the automata models of [3] (namely ωB- and ωS automata) yields games that are equivalent to ω-regular games. Hence, we take a different route and introduce cost-parity and cost-Streett conditions which generalize the (classical) ω-regular parity- respectively Streett condition, as well as the finitary parity-respectively finitary Streett condition [8]. While both finitary variants strengthen the classical conditions by adding bounds, the complexity of solving these games diverges: (in the state of the art) finitary parity games are simpler than parity games, while finitary Streett games are harder than Streett games. Indeed solving finitary parity games can be carried out in polynomial time [8], while no polynomial-time algorithm for parity games is yet known, and

the decision problem is in **NP** ∩ **coNP**. The situation is reversed for Streett games, since solving them is **coNP**-complete [13] while solving finitary Streett games is **EXPTIME**-complete. The latter result is shown in unpublished work by Chatterjee, Henzinger, and Horn: by slightly modifying the proof of **EXPTIME**-hardness of solving request-response games presented in [9] they prove **EXPTIME**-hardness of solving finitary Streett games.

A cost-parity game is played on an arena whose vertices are colored by natural numbers, and where traversing an edge incurs a non-negative cost. Player 0 wins a play if there is a bound b such that almost all odd colors (which we think of as requests) are followed by a larger even color (which we think of as responses) that is reached with cost at most b. The definition of (cost-) Streett games goes along the same lines, but the requests and responses are independent and not hierarchically ordered as in parity conditions. The cost of traversing an edge can be used to model the consumption of a resource. Thus, if Player 0 wins a play she can achieve her goal along an infinite run with bounded resources. On the other hand, Player 1's objective is to exhaust the resource, no matter how big the capacity is.

We show that cost-parity games enjoy two nice properties of parity and finitary parity games: Player 0 has memoryless winning strategies, and determining the winner lies in **NP** ∩ **coNP**. Furthermore, we show that solving cost-parity games can be algorithmically reduced to solving parity games, which allows to solve these games almost as efficiently as parity games. We then consider cost-Streett games and prove that Player 0 has finite-state winning strategies, and that determining the winner is **EXPTIME**-complete. Our complexity results unify the previous results about finitary parity and Streett games and the results about their classical variants.

To obtain our results, we present an algorithm to solve cost-parity games that iteratively computes the winning region of Player 0 employing an algorithm to solve parity games. As a by-product, we obtain finite-state winning strategies for Player 0. We improve this by showing how to transform a finite-state winning strategy into a positional winning strategy. This construction relies on so-called scoring functions (which are reminiscent of the simulation of alternating tree-automata by nondeterministic automata presented in [18]) and presents a general framework to turn finite-state strategies into positional ones, which we believe to be applicable in other situations as well. Finally, we present an algorithm that solves cost-Streett games by solving Streett games. Here, we show the existence of finite-state winning strategies for Player 0 in cost-Streett games.

In our proof, we reduce games with boundedness winning conditions to games with ω-regular winning conditions. This is reminiscent of the solution of the domination problem for regular cost-functions on finite trees [11]. In contrast to this work, which is concerned with proving decidability, we are interested in efficient algorithms. Hence, we need a more sophisticated reduction and a careful analysis of the memory requirements.

Adding quantitative requirements to qualitative winning conditions has been an active field of research during the last decade. Just recently, there has been a lot of interest in so-called energy games, whose winning conditions are boundedness requirements on the consumption of resources. Solving energy games with multiple resources is in general intractable [14] while so-called consumption games, a subclass of energy games, are shown to be tractable in [5]. Furthermore, energy parity games, whose winning conditions are a conjunction of an (single resource) energy and a parity condition, can be solved in **NP** ∩ **coNP** and Player 0 has positional winning strategies [7]. Although the first two results are similar to our results on cost-parity games, the energy parity condition does not relate the energy consumption to the parity condition. In contrast, the costs in cost-parity games give a qualitative measure of the satisfaction of the parity condition.

2   Definitions

Infinite Games.   A (cost) arena $\mathcal{A} = (V, V_0, V_1, E, \mathrm{Cst})$ consists of a finite, directed graph (V, E), a partition $\{V_0, V_1\}$ of V, and an edge-labeling $\mathrm{Cst}\colon E \to \{\varepsilon, i\}$. An edge with label i is called increment-edge, edges labeled by ε are called accordingly ε-edges. We extend the edge-labeling to a cost function over finite and infinite paths obtained by counting the number of increment-edges traversed along the path. A play in $\mathcal{A}$ starting in $v \in V$ is an infinite path $\rho = \rho_0\rho_1\rho_2\cdots$ such that $\rho_0 = v$. To avoid the nuisance of dealing with finite plays, we assume every vertex to have an outgoing edge.

A game $\mathcal{G} = (\mathcal{A}, \mathrm{Win})$ consists of an arena $\mathcal{A}$ and a set $\mathrm{Win} \subseteq V^\omega$ of winning plays for Player 0. The set of winning plays for Player 1 is $V^\omega \setminus \mathrm{Win}$. A strategy for Player i is a mapping $\sigma\colon V^*V_i \to V$ such that $(v, \sigma(wv)) \in E$ for all $wv \in V^*V_i$. We say that σ is positional if $\sigma(wv) = \sigma(v)$ for every $wv \in V^*V_i$. A play $\rho_0\rho_1\rho_2\ldots$ is consistent with σ if $\rho_{n+1} = \sigma(\rho_0\cdots\rho_n)$ for every n with $\rho_n \in V_i$. A strategy σ for Player i is a winning strategy from a set of vertices $W \subseteq V$ if every play that starts in some $v \in W$ and is consistent with σ is won by Player i. The winning region $W_i(\mathcal{G})$ of Player i in $\mathcal{G}$ is the set of vertices from which Player i has a winning strategy. We say that a strategy is uniform, if it is winning from all $v \in W_i(\mathcal{G})$. We always have $W_0(\mathcal{G}) \cap W_1(\mathcal{G}) = \emptyset$. On the other hand, if $W_0(\mathcal{G}) \cup W_1(\mathcal{G}) = V$, then we say that $\mathcal{G}$ is determined. All games we consider in this work are determined. Solving a game amounts to determining its winning regions.

A memory structure $\mathcal{M} = (M, \mathrm{Init}, \mathrm{Upd})$ for an arena $(V, V_0, V_1, E, \mathrm{Cst})$ consists of a finite set M of memory states, an initialization function $\mathrm{Init}\colon V \to M$, and an update function $\mathrm{Upd}\colon M \times V \to M$. The update function can be extended to $\mathrm{Upd}^*\colon V^+ \to M$ in the usual way: $\mathrm{Upd}^*(\rho_0) = \mathrm{Init}(\rho_0)$ and $\mathrm{Upd}^*(\rho_0 \ldots \rho_n\rho_{n+1}) = \mathrm{Upd}(\mathrm{Upd}^*(\rho_0 \ldots \rho_n), \rho_{n+1})$. A next-move function (for Player i) $\mathrm{Nxt}\colon V_i \times M \to V$ has to satisfy $(v, \mathrm{Nxt}(v, m)) \in E$ for all $v \in V_i$ and all $m \in M$. It induces a strategy σ for Player i with memory $\mathcal{M}$ via $\sigma(\rho_0 \ldots \rho_n) = \mathrm{Nxt}(\rho_n, \mathrm{Upd}^*(\rho_0 \ldots \rho_n))$. A strategy is called finite-state if it can be implemented by a memory structure.

An arena $\mathcal{A} = (V, V_0, V_1, E, \mathrm{Cst})$ and a memory structure $\mathcal{M} = (M, \mathrm{Init}, \mathrm{Upd})$ for $\mathcal{A}$ induce the expanded arena $\mathcal{A} \times \mathcal{M} = (V \times M, V_0 \times M, V_1 \times M, E', \mathrm{Cst}')$ where $((v, m), (v', m')) \in E'$ if and only if $(v, v') \in E$ and $\mathrm{Upd}(m, v') = m'$, and $\mathrm{Cst}'((v, m)(v', m')) = \mathrm{Cst}(v, v')$. Every play ρ in $\mathcal{A}$ has a unique extended play $\rho' = (\rho_0, m_0)(\rho_1, m_1)(\rho_2, m_2)\ldots$ in $\mathcal{A} \times \mathcal{M}$ defined by $m_0 = \mathrm{Init}(\rho_0)$ and $m_{n+1} = \mathrm{Upd}(m_n, \rho_{n+1})$, i.e., $m_n = \mathrm{Upd}^*(\rho_0 \cdots \rho_n)$. Note that every infix of ρ and the corresponding infix of ρ' have the same cost.

A game $\mathcal{G} = (\mathcal{A}, \mathrm{Win})$ is reducible to $\mathcal{G}' = (\mathcal{A}', \mathrm{Win}')$ via $\mathcal{M}$, written $\mathcal{G} \leq_\mathcal{M} \mathcal{G}'$, if $\mathcal{A}' = \mathcal{A} \times \mathcal{M}$ and every play ρ in $\mathcal{G}$ is won by the player who wins the extended play ρ' in $\mathcal{G}'$, i.e., $\rho \in \mathrm{Win}$ if and only if $\rho' \in \mathrm{Win}'$.

▶ **Lemma 1.** *Let $\mathcal{G}$ be a game with vertex set V and $W \subseteq V$. If $\mathcal{G} \leq_\mathcal{M} \mathcal{G}'$ and Player i has a finite-state winning strategy for $\mathcal{G}'$ from $\{(v, \mathrm{Init}(v)) \mid v \in W\}$, then she also has a finite-state winning strategy for $\mathcal{G}$ from W.*

Let $\mathcal{A} = (V, V_0, V_1, E, \mathrm{Cst})$ and let $i \in \{0, 1\}$. The i-attractor of $F \subseteq V$ in $\mathcal{A}$, denoted by $\mathrm{Attr}_i^\mathcal{A}(F)$, is the set of vertices from which Player i has a strategy such that every play that starts in this set and is consistent with the strategy visits F.

Cost-Parity Games.   Let $\mathcal{A} = (V, V_0, V_1, E, \mathrm{Cst})$ be an arena and let $\Omega\colon V \to \mathbb{N}$ be a coloring of its vertices. In all games we are about to define, we interpret the occurrence of a color as request, which has to be answered by visiting a vertex of larger even color at a later position. By imposing conditions on the responses, we obtain three different types of

winning conditions. To simplify our notation, let $\text{Ans}(c) = \{c' \in \mathbb{N} \mid c' \geq c \text{ and } c' \text{ even}\}$ be the set of colors that answer a request of color c. Note that $\text{Ans}(c) \subseteq \text{Ans}(c')$ for $c \geq c'$ and $c \in \text{Ans}(c)$ if c is even.

The parity winning condition, denoted by $\text{Parity}(\Omega)$, requires that almost all requests are eventually answered. Equivalently, $\rho \in \text{Parity}(\Omega)$ if and only if the maximal color that occurs infinitely often in ρ is even. The finitary parity condition [8] is a strengthening of the parity condition; it requires the existence of a bound b such that almost all requests are answered within b steps. Formally, given a play $\rho = \rho_0\rho_1\rho_2\cdots$ and $k \in \mathbb{N}$, we define

$$\text{Dist}(\rho, k) = \inf\{k' - k \mid k' \geq k \text{ and } \Omega(\rho_{k'}) \in \text{Ans}(\Omega(\rho_k))\}$$

where we use $\inf \emptyset = \infty$. Hence, $\text{Dist}(\rho, k)$ is the number of steps between the request at position k and its earliest response. The finitary parity winning condition is $\text{FinParity}(\Omega) = \{\rho \in V^\omega \mid \limsup_{k\to\infty} \text{Dist}(\rho, k) < \infty\}$.

Note that both the parity and the finitary parity condition do not depend on the cost function. Finally, by not bounding the number of steps between the requests and their responses, but by bounding the *cost* between requests and responses, we obtain the cost-parity condition. Given a play $\rho = \rho_0\rho_1\rho_2\cdots$ and $k \in \mathbb{N}$, we define the cost-of-response by

$$\text{Cor}(\rho, k) = \inf\{\text{Cst}(\rho_k \cdots \rho_{k'}) \mid k' \geq k \text{ and } \Omega(\rho_{k'}) \in \text{Ans}(\Omega(\rho_k))\} \ .$$

The cost-parity winning condition is $\text{CostParity}(\Omega) = \{\rho \in V^\omega \mid \limsup_{k\to\infty} \text{Cor}(\rho, k) < \infty\}$.

▶ **Remark 2.** $\text{FinParity}(\Omega) \subseteq \text{CostParity}(\Omega) \subseteq \text{Parity}(\Omega)$.

A game $\mathcal{G} = (\mathcal{A}, \text{CostParity}(\Omega))$ is called cost-parity game. In a similar way, we define parity and finitary parity games. Note that if $\mathcal{A}$ contains no increment-edges, then we have $\text{CostParity}(\Omega) = \text{Parity}(\Omega)$, and if $\mathcal{A}$ contains no ε-edges, then $\text{CostParity}(\Omega) = \text{FinParity}(\Omega)$. Hence, cost-parity games generalize both parity and finitary parity games.
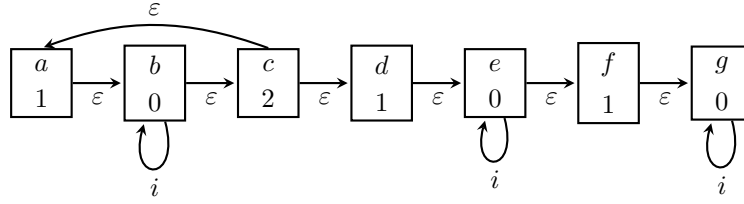
Since cost-parity conditions can be shown to be on the third level of the Borel hierarchy, we obtain the following result as a consequence of the Borel determinacy theorem [17].

▶ **Remark 3.** Cost-parity games are determined.

Fix a play ρ. We say that a request at position k is answered with cost c, if $\text{Cor}(\rho, k) = c$. Note that a request at a position k with an even color is answered with cost 0. Furthermore, we say that a request at position k is unanswered with cost c, if there is no position $k' \geq k$ such that $\Omega(\rho_{k'}) \in \text{Ans}(\Omega(\rho_k))$, but we have $\text{Cst}(\rho_k\rho_{k+1}\cdots) = c$, i.e., there are exactly c increment-edges after position k. Finally, we say that a request at position k is unanswered with cost ∞, if there is no position $k' \geq k$ such that $\Omega(\rho_{k'}) \in \text{Ans}(\Omega(\rho_k))$ and we have $\text{Cst}(\rho_k\rho_{k+1}\cdots) = \infty$, i.e., there are infinitely many increment-edges after position k. We say that a request is unanswered, if it is unanswered with cost in $\mathbb{N} \cup \{\infty\}$. We often use the following equivalence.

▶ **Remark 4.** A play ρ has only finitely many unanswered requests if and only if $\rho \in \text{Parity}(\Omega)$.

▶ **Example 5.** Consider the cost-parity game depicted in Figure 1 where all vertices belong to V_1, and the label of a vertex denotes its name (in the upper part) and its color (in the lower part). Player 1 wins from $\{a, b, c\}$ by requesting color 1 at vertex a infinitely often and staying at vertex b longer and longer, but also visiting c infinitely often (and thereby answering the request). Note that this strategy is not finite-state. Indeed, one can easily prove that Player 1 does not have a finite-state winning strategy for this game. Player 0 wins from every other vertex, since Player 1 can raise only finitely many requests from these vertices, albeit these requests are unanswered with cost ∞.

■ **Figure 1** A cost-parity game.

3   Solving Cost-Parity Games

In this section, we show how to determine the winning regions in a cost-parity game. Our algorithm proceeds in two steps: in Subsection 3.1, we show that it suffices to solve games whose winning condition is a strengthening of the cost-parity condition, the so-called bounded cost-parity condition. Then, in Subsection 3.2, we show how to solve bounded cost-parity games by solving ω-regular games. The main result of this section is the following theorem. Here, n is the number of vertices, m is the number of edges, and d is the number of colors in the cost-parity game.

▶ **Theorem 6.** *Given an algorithm that solves parity games in time $T(n, m, d)$, there is an algorithm that solves cost-parity games in time $O(n \cdot T(d \cdot n, d \cdot m, d + 2))$.*

3.1   From Cost-Parity Games to Bounded Cost-Parity Games

The cost-parity condition can be rephrased as follows: $\rho \in \text{CostParity}(\Omega)$ if and only if

> there exists a $b \in \mathbb{N}$ such that all but finitely many requests are answered or unanswered with cost less or equal than b and there are only finitely many unanswered requests.

Note that this allows a finite number of unanswered requests with cost ∞. By disallowing this, we obtain a strengthening of the cost-parity condition. Formally, we define the bounded cost-parity condition $\text{BndCostParity}(\Omega)$ to be the set of plays ρ satisfying the following:

> there exists a $b \in \mathbb{N}$ such that every request is answered or unanswered with cost less or equal than b and there are only finitely many unanswered requests.

A game $(\mathcal{A}, \text{BndCostParity}(\Omega))$ is called bounded cost-parity game.

▶ **Example 7.** Consider the game in Figure 1, this time with the bounded cost-parity condition: Player 1 wins from every vertex but g by moving to g and then staying there ad infinitum. Every such play contains a request of color 1 that is unanswered with cost ∞.

Since bounded cost-parity conditions are on the third level of the Borel hierarchy, bounded cost-parity games are determined. Furthermore, the bounded cost-parity condition is indeed a strengthening of the cost-parity condition.

▶ Remark 8. $\text{BndCostParity}(\Omega) \subseteq \text{CostParity}(\Omega)$.

The following lemma implies that being able to solve bounded cost-parity games suffices to solve cost-parity games.

▶ **Lemma 9.** *Let $\mathcal{G} = (\mathcal{A}, \text{CostParity}(\Omega))$ and let $\mathcal{G}' = (\mathcal{A}, \text{BndCostParity}(\Omega))$.*
1. $W_0(\mathcal{G}') \subseteq W_0(\mathcal{G})$.
2. *If $W_0(\mathcal{G}') = \emptyset$, then $W_0(\mathcal{G}) = \emptyset$.*

Proof. 1. This follows directly from Remark 8: a winning strategy for Player 0 in $\mathcal{G}'$ from v is also a winning strategy for her in $\mathcal{G}$ from v.

2. Due to determinacy, if $W_0(\mathcal{G}') = \emptyset$, then we have $W_1(\mathcal{G}') = V$, i.e., from every vertex v, Player 1 has a winning strategy τ_v. Consider a play consistent with τ_v starting in v: either, for every b, there is a request that is not answered with cost at most b, or the maximal color seen infinitely often is odd (i.e., there are infinitely many unanswered requests).

We define a strategy τ for Player 1 as follows: it is guided by a vertex v_{cur} and a counter b_{cur}. Assume a play starts in vertex v. Then, we initialize v_{cur} by v and b_{cur} by 1. The strategy plays the strategy $\tau_{v_{\mathrm{cur}}}$ until a request is not answered with cost b_{cur}. If this is the case, then v_{cur} is set to the current vertex, b_{cur} is incremented, and τ plays according to $\tau_{v_{\mathrm{cur}}}$ (forgetting the history of the play constructed so far).

We show that τ is winning (in the cost-parity game) from every vertex, which implies $W_0(\mathcal{G}) = \emptyset$. Let ρ be a play that is consistent with τ and distinguish two cases: if the counter is incremented infinitely often, then ρ is winning for Player 1 in the cost-parity game. On the other hand, if b_{cur} is incremented only finitely often (say to value b), then there is a suffix ρ' of ρ with some first vertex v that is consistent with the strategy τ_v. Since the counter is not incremented during ρ', every request in ρ' is either answered or unanswered with cost at most b. As ρ' is nevertheless winning for Player 1, the maximal color seen infinitely often during ρ' is odd. As ρ' and ρ only differ by a finite prefix, the maximal color seen infinitely often during ρ is the same as in ρ' and therefore odd. Thus, ρ is winning for Player 1 in $\mathcal{G}$. ◀

To conclude this subsection, we show how Lemma 9 can be used to solve cost-parity games, provided we are able to solve bounded cost-parity games (which is the subject of the next subsection). Let $\mathcal{A}$ be an arena and Ω a coloring of its vertices, and let $\mathcal{G} = (\mathcal{A}, \mathrm{CostParity}(\Omega))$. The following algorithm proceeds by iteratively removing parts of $\mathcal{A}$ that are included in the winning region of Player 0 in $\mathcal{G}$. In each step, one computes the winning region of the current arena w.r.t. the bounded cost-parity condition. Due to Lemma 9.1, this is included in the winning region of Player 0 in $\mathcal{G}$. Furthermore, the attractor of this region also belongs to the winning region in $\mathcal{G}$, since the cost-parity condition is prefix-independent. This continues until Player 0's winning region of the current arena w.r.t. the bounded cost-parity condition is empty. In this situation, Player 1 wins everywhere w.r.t. the cost-parity condition in the current arena due to Lemma 9.2. Since Player 0 cannot leave this region (in the original arena $\mathcal{A}$), it is also winning for Player 1 in the original game $\mathcal{G}$.

In Section 4, we prove that Player 0 has uniform positional winning strategies for bounded cost-parity games. Composing such strategies for the regions X_j with positional attractor strategies, we obtain a positional winning strategy for Player 0 in a cost-parity game. In Example 5, we have already seen that Player 1 needs infinite memory to win.

Algorithm 1 A fixed-point algorithm for cost-parity games.

$j \leftarrow 0; W_j \leftarrow \emptyset; \mathcal{A}_j \leftarrow \mathcal{A}$
repeat
$\quad j \leftarrow j + 1$
$\quad X_j \leftarrow W_0(\mathcal{A}_{j-1}, \mathrm{BndCostParity}(\Omega))$
$\quad W_j \leftarrow W_{j-1} \cup \mathrm{Attr}_0^{\mathcal{A}_{j-1}}(X_j)$
$\quad \mathcal{A}_j \leftarrow \mathcal{A}_{j-1} \setminus \mathrm{Attr}_0^{\mathcal{A}_{j-1}}(X_j)$
until $X_j = \emptyset$
return W_j

3.2   From Bounded Cost-Parity Games to ω-regular Games

Next, we show how to solve bounded cost-parity games. Let $\mathcal{A}$ be an arena. In this subsection, we assume that no vertex of $\mathcal{A}$ has both incoming increment- and ε-edges. This can be achieved by subdividing every increment-edge $e = (v, v')$: we add a new vertex $\mathrm{sub}(e)$ and replace e by $(v, \mathrm{sub}(e))$ (which is an increment-edge) and by $(\mathrm{sub}(e), v')$ (which is an ε-edge). Now, only the newly added vertices have incoming increment-edges, but they do not have incoming ε-edges. Furthermore, it is easy to see that Player i wins from v in the original game if and only if she wins from v in the modified game (where we color $\mathrm{sub}(e)$ by $\Omega(v')$).

Assuming this convention, we say that a vertex is an increment-vertex, if it has an incoming increment-edge (which implies that all incoming edges have an increment). Let F be the set of increment-vertices and denote the set of plays with finite cost by $\mathrm{coBüchi}(F)$. Furthermore, let $\mathrm{RR}(\Omega)$ denote the set of plays ρ in which every request is answered (no matter at which cost). Using these definitions, let

$$\mathrm{PCRR}(\Omega) = \big(\mathrm{Parity}(\Omega) \cap \mathrm{coBüchi}(F)\big) \ \cup \ \mathrm{RR}(\Omega) \ .$$

As $\mathrm{PCRR}(\Omega)$ is ω-regular, a game with winning condition $\mathrm{PCRR}(\Omega)$ is determined and both players have uniform finite-state winning strategies [6]. Furthermore, $\mathrm{PCRR}(\Omega)$ is weaker than $\mathrm{BndCostParity}(\Omega)$.

▶ **Remark 10.** $V^\omega \setminus \mathrm{PCRR}(\Omega) \subseteq V^\omega \setminus \mathrm{BndCostParity}(\Omega)$.

Note that the converse implication is false, as the request-response condition does not bound the cost between requests and their responses. However, every finite-state winning strategy bounds the distance between requests and their responses, and thereby also the cost.

▶ **Lemma 11.** *Let* $\mathcal{G} = (\mathcal{A}, \mathrm{BndCostParity}(\Omega))$, *and let* $\mathcal{G}' = (\mathcal{A}, \mathrm{PCRR}(\Omega))$. *Then,* $W_i(\mathcal{G}) = W_i(\mathcal{G}')$ *for* $i \in \{0, 1\}$.

Proof. We have $W_1(\mathcal{G}') \subseteq W_1(\mathcal{G})$ due to Remark 10. Thus, it suffices to show $W_0(\mathcal{G}') \subseteq W_0(\mathcal{G})$. So, let σ be a uniform finite-state winning strategy for Player 0 for $\mathcal{G}'$ (which exists, since $\mathrm{PCRR}(\Omega)$ is ω-regular). We argue that σ is also a uniform winning strategy for Player 0 for $\mathcal{G}$: let ρ be consistent with σ, which implies $\rho \in \mathrm{PCRR}(\Omega)$.

If ρ satisfies $\mathrm{Parity}(\Omega)$ and has only finitely many increments, then every request is answered or unanswered with bounded cost and there are only finitely many unanswered requests, i.e., ρ is winning for Player 0 in $\mathcal{G}$. To conclude, we consider the case $\rho \in \mathrm{RR}(\Omega) \setminus (\mathrm{Parity}(\Omega) \cap \mathrm{coBüchi}(F))$. Since σ is a finite-state winning strategy, there is a bound b (which only depends on the size of σ and the size of the arena) such that every request in ρ is answered with cost at most b (if not, then there would also be a play consistent with σ with an unanswered request and with cost ∞. This play would not satisfy $\mathrm{PCRR}(\Omega)$, which yields the desired contradiction). Hence, ρ is winning for Player 0 in $\mathcal{G}$.   ◀

So, to determine the winning regions of a bounded cost-parity game it suffices to solve an ω-regular game with winning condition $\mathrm{PCRR}(\Omega)$. This condition can be reduced to a parity condition using a *small* memory structure that keeps track of the largest unanswered request and goes to a special state $\bot$ if this request is answered. Using this memory structure, the $\mathrm{RR}(\Omega)$-condition reduces to a Büchi condition requiring the state $\bot$ to be visited infinitely often. We are now left with the union of a Büchi condition and an intersection of a parity and a co-Büchi condition. First, by coloring the increment-vertices of $\mathcal{A}$ by an odd color that is larger than $\max \Omega(V)$ (and leaving the colors of all other vertices unchanged), we obtain a new parity condition that is equivalent to the intersection of $\mathrm{Parity}(\Omega)$ and $\mathrm{coBüchi}(F)$.

Using a similar construction, we can turn the union of the new parity condition and of the Büchi condition into an equivalent parity condition. Thus, to determine the winning regions of a game with winning condition $\mathrm{PCRR}(\Omega)$, if suffices to solve a linearly larger parity game. This also concludes the proof of Theorem 6: Algorithm 1 terminates after at most n iterations and solves a *small* parity game in each iteration.

Furthermore, from the previous observations we obtain an upper bound on the memory requirements for both players, which we improve in the next section.

▶ **Remark 12.** In every bounded cost-parity game, both players have uniform finite-state winning strategies with $\ell + 1$ memory states, where ℓ is the number of odd colors in $\Omega(V)$.

The winning regions of parity games can be determined in non-deterministic polynomial time by guessing both regions $W_i(\mathcal{G})$ and positional strategies σ_i for both players and then verifying in (deterministic) polynomial time whether σ_i is a uniform winning strategy for Player i from $W_i(\mathcal{G})$. Algorithm 1 solves cost-parity games by solving at most n parity games, which have at most $(\ell + 1) \cdot n$ vertices, $(\ell + 1) \cdot m$ edges, and $d + 2$ colors, where n, m, d, and ℓ denote the number of vertices, edges, colors, and odd colors in the cost-parity game. Thus, the algorithm runs in non-deterministic polynomial time. Together with a dual argument this implies the following result.

▶ **Theorem 13.** *The following problems are in* $\mathbf{NP} \cap \mathbf{coNP}$:
1. *Given a bounded cost-parity game $\mathcal{G}$ and a vertex v, is $v \in W_0(\mathcal{G})$?*
2. *Given a cost-parity game $\mathcal{G}$ and a vertex v, is $v \in W_0(\mathcal{G})$?*

4 Half-Positional Determinacy of (Bounded) Cost-Parity Games

Next, we show that Player 0 has positional winning strategies in (bounded) cost-parity games.

▶ **Theorem 14.** *In cost-parity games, Player 0 has uniform positional winning strategies.*

As we have already explained while proving the correctness of Algorithm 1, the existence of positional winning strategies for bounded cost-parity games implies Theorem 14. Thus, the remainder of this section is devoted to proving the following lemma.

▶ **Lemma 15.** *In bounded cost-parity games, Player 0 has uniform positional winning strategies.*

For bounded cost-parity games we have already proved the existence of uniform finite-state winning strategies (see Remark 12). Hence, it remains to show how to eliminate the memory. To this end, we define a so-called scoring function for bounded cost-parity games that measures the quality of a play prefix (from Player 0's vantage point) by keeping track of the largest unanswered request, the number of increment-edges traversed since it was raised, and how often each odd color was seen since the last increment-edge.

In the following, fix an arena $(V, V_0, V_1, E, \mathrm{Cst})$ and a coloring function $\Omega \colon V \to \mathbb{N}$. Let $\Omega(V) \subseteq \{0, \ldots, d\}$, where we assume d to be odd, and define $\ell = \frac{d+1}{2}$ to denote the number of odd colors in $\{0, \ldots, d\}$. The (score-) sheet of a play prefix w, denoted by $\mathrm{Sh}(w)$, is a vector $(c, n, s_d, s_{d-2}, \ldots, s_1) \in \mathbb{N}^{2+\ell}$ containing the following information:

- c denotes the largest unanswered request in w, i.e., the largest odd color in w that is not followed by a color in $\mathrm{Ans}(c)$.
- n denotes the cost of the suffix starting with the *first* unanswered occurrence of c in w.
- $s_{c'}$ (here, c' is odd) denotes the number of times c' occurred in w since the first unanswered occurrence of c, since the last increment-edge was traversed, or since the last time a color larger than c' occurred in w, depending on which happened last.

Finally, we use the empty sheet $\perp$ for play prefixes without unanswered requests.

The reversed ordering of the score values $s_d, \ldots, s_1$ in the sheets is due to the max-parity condition, in which larger colors are more important than smaller ones. This is reflected by the fact that we compare sheets in the lexicographical order induced by $<$ on its components and add $\perp$ as minimal element. For example, $(3, 3, 0, 1, 1) < (3, 3, 1, 0, 7)$ and $\perp < s$ for every sheet $s \neq \perp$. As usual, we write $s \leq s'$, if $s = s'$ or $s < s'$. We say that a sheet $(c, n, s_d, \ldots, s_1)$ is bounded by $b \in \mathbb{N}$, if we have $n \leq b$ and $s_c \leq b$ for every c. Also, $\perp$ is bounded by every b. The following lemma shows that Sh is a congruence w.r.t. $\leq$. Here $\mathrm{Lst}(x)$ denotes the last vertex of the non-empty finite play x.

▶ **Lemma 16.** *If* $\mathrm{Lst}(x) = \mathrm{Lst}(y)$ *and* $\mathrm{Sh}(x) \leq \mathrm{Sh}(y)$*, then* $\mathrm{Sh}(xv) \leq \mathrm{Sh}(yv)$ *for every* $v \in V$*.*

We begin the proof of Lemma 15 by noting that the sheets of a play ρ being bounded is a sufficient condition for ρ satisfying the bounded cost-parity condition.

▶ **Lemma 17.** *If there exists a bound b such that the sheets of all prefixes of a play ρ are bounded by b, then* $\rho \in \mathrm{BndCostParity}(\Omega)$*.*

The next lemma shows that finite-state winning strategies uniformly bound the sheets.

▶ **Lemma 18.** *Let σ be a uniform finite-state winning strategy σ (of size m) for Player 0 in a bounded cost-parity game $\mathcal{G} = (\mathcal{A}, \mathrm{BndCostParity}(\Omega))$. Furthermore, let ρ be starting in $W_0(\mathcal{G})$ and be consistent with σ. Then, the sheets of all prefixes of ρ are bounded by $m \cdot |V|$.*

Now we are able to prove our main technical result of this section: Player 0 has a uniform positional winning strategy in every bounded cost-parity game (and therefore also in every cost-parity game).

Proof of Lemma 15. Fix some uniform finite-state winning strategy σ' for Player 0 in a bounded cost-parity game $\mathcal{G}$. For every $v \in W_0(\mathcal{G})$, let P_v denote the set of play prefixes that begin in $W_0(\mathcal{G})$, are consistent with σ', and end in v. Due to Lemma 18, the sheets of the prefixes in P_v are bounded by some b. Thus, for every v the set $\{\mathrm{Sh}(w) \mid w \in P_v\}$ is finite. Hence, there exists a play prefix $\max_v \in P_v$ such that $\mathrm{Sh}(w) \leq \mathrm{Sh}(\max_v)$ for every $w \in P_v$.

We define a uniform positional strategy σ by $\sigma(wv) = \sigma'(\max_v)$ and claim that it is a uniform winning strategy for $\mathcal{G}$. An inductive application of Lemma 16 shows that we have $\mathrm{Sh}(\rho_0 \cdots \rho_n) \leq \mathrm{Sh}(\max_{\rho_n})$ for every n and every play ρ that is consistent with σ. Hence, the sheets of ρ are bounded by b, which implies $\rho \in \mathrm{BndCostParity}(\Omega)$ due to Lemma 17. ◀

5 Cost-Streett Games

In this section, we introduce cost-Streett games which generalize both Streett games and finitary Streett games [8]. We first present an algorithm to solve these games following the same ideas as in the previous sections, and prove **EXPTIME**-completeness of the corresponding decision problem. From our algorithm, we deduce finite-state determinacy for Player 0, while Player 1 needs infinite memory in general.

To simplify our notation, let $[d] = \{0, \ldots, d-1\}$. A d-dimensional cost-arena has the form $\mathcal{A} = (V, V_0, V_1, E, (\mathrm{Cst}_\ell)_{\ell \in [d]})$ where the first four components are as usual and where each Cst_ℓ is a mapping from E to $\{\varepsilon, i\}$. Again, each of these functions induces a cost on (infixes of) plays, denoted by Cst_ℓ as well. Let $\Gamma = (Q_\ell, P_\ell)_{\ell \in [d]}$ be a collection of (Streett) pairs of subsets of V, i.e., $Q_\ell, P_\ell \subseteq V$. We define

$$\mathrm{StCor}_\ell(\rho, k) = \begin{cases} 0 & \text{if } \rho_k \notin Q_\ell, \\ \inf\{\mathrm{Cst}_\ell(\rho_k \cdots \rho_{k'}) \mid \rho_{k'} \geq \rho_k \text{ and } \rho_{k'} \in P_\ell\} & \text{if } \rho_k \in Q_\ell, \end{cases}$$

where we use $\inf \emptyset = \infty$, and $\mathrm{StCor}(\rho, k) = \max\{\mathrm{StCor}_\ell(\rho, k) \mid \ell \in [d]\}$. Using this, we define the following three winning conditions:

- the (classical) Streett condition $\mathrm{Streett}(\Gamma)$ requires for every ℓ that P_ℓ is visited infinitely often if Q_ℓ is visited infinitely often.
- the request-response condition $\mathrm{RR}(\Gamma)$ requires for every ℓ that every visit to Q_ℓ is answered by a later visit to P_ℓ.
- the cost-Streett condition $\mathrm{CostStreett}(\Gamma) = \{\rho \mid \limsup_{k \to \infty} \mathrm{StCor}(\rho, k) < \infty\}$.

A cost-Streett game $(\mathcal{A}, \Gamma)$ consists of a d-dimensional arena and a collection Γ of d Streett pairs. Streett and request-response games are defined accordingly. As for cost-parity games, a cost-Streett game in which every edge is an increment-edge (w.r.t. all Cst_ℓ) is a finitary Streett game and a cost-Streett game in which every edge is an epsilon-edge (w.r.t. all Cst_ℓ) is a Streett game. Furthermore, just as classical Streett games subsume parity games, cost-Streett games subsume cost-parity games.

▶ **Remark 19.** Cost-Streett games are determined.

Our main theorem is proved along the same lines as Theorem 6.

▶ **Theorem 20.** *To solve a cost-Streett game with n vertices, it suffices to solve n Streett games which are exponentially larger (but only in the number of Streett pairs).*

We say that the requests at position k are answered with cost c, if $\mathrm{StCor}(\rho, k) = c$; that the requests are unanswered with cost c, if $\mathrm{StCor}(\rho, k) = \infty$, but there are at most c many increment-edges after position k (w.r.t. all Cst_ℓ such that $\rho_k \in Q_\ell$); and that the requests are unanswered with cost ∞, if $\mathrm{StCor}(\rho, k) = \infty$ and there are infinitely many increment-edges after position k (w.r.t. some Cst_ℓ such that $\rho_k \in Q_\ell$).

As in the case of cost-parity games, we begin by introducing a strengthening of the cost-Streett condition. The bounded cost-Streett condition, denoted by $\mathrm{BndCostStreett}(\Gamma)$, is the set of plays ρ that satisfy the following condition:

there exists a b such that all requests are answered or unanswered with cost at most b, and there are only finitely many unanswered requests.

▶ **Lemma 21.** *Let $\mathcal{G} = (\mathcal{A}, \mathrm{CostStreett}(\Gamma))$, and let $\mathcal{G}' = (\mathcal{A}, \mathrm{BndCostStreett}(\Gamma))$.*
1. $W_0(\mathcal{G}') \subseteq W_0(\mathcal{G})$.
2. *If $W_0(\mathcal{G}') = \emptyset$, then $W_0(\mathcal{G}) = \emptyset$.*

The proof is analogously to the one for Lemma 9. Also, Algorithm 1 (where X_j is now Player 0's winning region in the bounded cost-Streett game) works for this pair of winning conditions as well. To solve bounded cost-Streett games, we again assume for every ℓ that no vertex has both an incoming increment-edge (w.r.t. Cst_ℓ) and an incoming epsilon-edge (again, w.r.t. Cst_ℓ). Assuming this, let F_ℓ denote the vertices with incoming increment-edges w.r.t. Cst_ℓ. Then, $\mathrm{coBüchi}(F_\ell) = \{\rho \mid \mathrm{Cst}_\ell(\rho) < \infty\}$ is the set of plays with finitely many increment-edges w.r.t. Cst_ℓ. Finally, we define the ω-regular condition

$$\mathrm{SCRR}(\Gamma) = \bigcap_{\ell \in [d]} \left[\left(\mathrm{Streett}(Q_\ell, P_\ell) \cap \mathrm{coBüchi}(F_\ell) \right) \cup \mathrm{RR}(Q_\ell, P_\ell) \right] .$$

▶ **Lemma 22.** *Let $\mathcal{G} = (\mathcal{A}, \mathrm{BndCostStreett}(\Gamma))$, and let $\mathcal{G}' = (\mathcal{A}, \mathrm{SCRR}(\Gamma))$. Then, $W_i(\mathcal{G}) = W_i(\mathcal{G}')$ for $i \in \{0, 1\}$.*

The proof is again analogously to the one for Lemma 11 and relies on finite-state determinacy of ω-regular games. To solve $(\mathcal{A}, \mathrm{SCRR}(\Gamma))$ one can reduce this game to a

classical Streett game with $2d$ Streett pairs using a memory structure of size 2^d to keep track of open requests. Hence, using the algorithm for Streett games from [19], we can solve the resulting game in exponential time (in the size of $(\mathcal{A}, \mathrm{SCRR}(\Gamma))$), even though the arena is of exponential size (but only in d). Together with the **EXPTIME**-hardness of solving finitary Streett games[1], which are a special case, we obtain the following result.

▶ **Theorem 23.** *The problem "Given a cost-Streett game $\mathcal{G}$ and a vertex v, is $v \in W_0(\mathcal{G})$?" is* **EXPTIME***-complete.*

Furthermore, the reduction described above (and the memory requirements for Streett games [12]) yields upper bounds on the memory requirements in (bounded) cost-Streett. Player 1 needs in general infinite memory in cost-Streett games.

▶ **Corollary 24.**

- *Player* 0 *has finite-state winning strategies of size* $2^d \cdot (2d)!$ *in bounded cost-Streett games and cost-Streett games.*
- *Player* 1 *has finite-state winning strategies of size* 2^d *in bounded cost-Streett games.*

6 Conclusion

We introduced infinite games with cost conditions, generalizing both classical conditions and finitary conditions. For cost-parity games, we proved half-positional determinacy and that solving these games is not harder than solving parity games. Furthermore, the corresponding decision problem is in **NP ∩ coNP**. For cost-Streett games, we showed that Player 0 has finite-state winning strategies and that solving these games is not harder than solving finitary Streett games and can be done by solving linearly many (classical) Streett games of exponential size (in the number of Streett pairs). Our results unify the previous results on both classical and finitary variants. Table 1 sums up all these results.

winning condition	computational complexity	memory Player 0	memory Player 1
parity	**NP ∩ coNP**	positional	positional
finitary parity	**PTIME**	positional	infinite
cost-parity	**NP ∩ coNP**	positional	infinite
Streett	**coNP**-complete	finite	positional
finitary Streett	**EXPTIME**-complete	finite	infinite
cost-Streett	**EXPTIME**-complete	finite	infinite

■ **Table 1** Overview of results

There are at least three directions to extend these results: first, our winning conditions do not cover all acceptance conditions (for automata) discussed in [3, 20]. In ongoing research, we investigate whether our techniques are applicable to these more expressive conditions and to winning conditions specified in weak-MSO with the unbounding quantifier [2, 4]. Second, one could consider infinite arenas, e.g., configuration graphs of pushdown systems. This is already open for finitary games and requires new ideas, since our techniques rely heavily on the finiteness of the arena. Finally, one could add decrement-edges.

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[1] Shown in unpublished work by Chatterjee, Henzinger, and Horn, obtained by slightly modifying the proof of **EXPTIME**-hardness of solving request-response games [9].

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