Faster Deciding MSO Properties of Trees of Fixed Height, and Some Consequences

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Abstract

We prove, in the universe of trees of bounded height, that for any MSO formula with m variables there exists a set of kernels such that the size of each of these kernels can be bounded by an elementary function of m. This yields a faster MSO model checking algorithm for trees of bounded height than the one for general trees. From that we obtain, by means of interpretation, corresponding results for the classes of graphs of bounded tree-depth (MSO_2) and shrub-depth (MSO_1), and thus we give wide generalizations of Lampis' (ESA 2010) and Ganian's (IPEC 2011) results. In the second part of the paper we use this kernel structure to show that FO has the same expressive power as MSO_1 on the graph classes of bounded shrub-depth. This makes bounded shrub-depth a good candidate for characterization of the hereditary classes of graphs on which FO and MSO_1 coincide, a problem recently posed by Elberfeld, Grohe, and Tantau (LICS 2012).

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1 Introduction

First order (FO) and monadic second-order (MSO) logics play undoubtedly crucial role in computer science. Besides traditional tight relations to finite automata and regular languages, this is also witnessed by their frequent occurrence in the so called *algorithmic metatheorems* which have gained increasing popularity in the past few years. The term algorithmic metatheorem commonly refers to a general algorithmic toolbox ready to be applied onto a wide range of problems in specific situations, and MSO or FO logic is often used in the expression of this "range of problems".

One of the perhaps most celebrated algorithmic metatheorems (and the original motivation for our research) is Courcelle's theorem [3] stating that every graph property ϕ expressible in the MSO_2 *logic of graphs* (allowing for both vertex and edge set quantifiers) can be decided in linear FPT time on graphs of bounded tree-width. Courcelle, Makowsky, and Rotics [4] then have analogously addressed a wider class of graphs, namely those of bounded clique-width, at the expense of restricting ϕ to MSO_1 *logic* (i.e., with only vertex set quantification). Among other recent works on algorithmic metatheorems we just briefly mention two survey articles by Kreutzer [16] and by Grohe–Kreutzer [15], and an interesting recent advance by Dvořák, Král', and Thomas [7] showing linear-time FPT decidability of FO model checking on the graphs of "bounded expansion".

Returning back to Courcelle's theorem [3] and closely related [1, 4], it is worth to remark that a solution can be obtained via interpretation of the respective graph problem into an

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MSO formula over coloured trees (which relates the topic all the way back to Rabin's S2S theorem [21]). However, a drawback of these metatheorems is that, when their runtime is expressed as $\mathcal{O}\big(f(\phi, width(G)) \cdot |G|\big)$, this function f grows asymptotically as $2^{2^{\cdot^{\cdot^{width(G)}}}}\Big\}a$ where the height a depends on ϕ, precisely on the quantifier alternation depth of ϕ (i.e., f is a *non-elementary function* of the parameter ϕ). The latter is not surprising since Frick and Grohe [11, 10] proved that it is not possible to avoid a non-elementary tower of exponents even in deciding MSO properties on all trees or coloured paths (unless P=NP).

Given the importance of Courcelle's and other related algorithmic metatheorems, it is a bit of surprise that apparently no research paper tackled this "nonelementary exponential tower" issue of deciding graph MSO properties until recently: The first step in this direction occurred in a 2010 ESA paper by Lampis [17], giving an FPT algorithm for MSO_2 model checking on graphs of bounded vertex cover with only a double-exponential parameter dependence. Ganian [13] then analogously addressed MSO_1 model checking problem on graphs of bounded so-called twin-cover (much restricting bounded clique-width).

MSO on trees of bounded height

Frick–Grohe's negative result leaves main room for possible improvement on suitably restricted subclass(es) of all coloured trees, namely on those avoiding long paths. In this respect, our first result here (Theorem 3.2 and Corollary 3.3) gives a new algorithm for deciding MSO properties ϕ of rooted m-coloured trees T of fixed height d. This algorithm uses so called kernelization—which means it efficiently reduces the input tree into an equivalent one of elementarily bounded size, leading to an FPT algorithm with runtime

$$\mathcal{O}\left(|V(T)|\right) + 2^{2^{\cdot^{\cdot^{\mathcal{O}(m|\phi|^2)}}}}\Big\}d+1 .$$

Informally, our algorithm "trades" quantifier alternation of ϕ for bounded height of the tree. Hence there is nothing interesting brought for all trees, while on the other hand, our algorithm presents an improvement over the previous on the trees of height $\leq d$ for every fixed value d. We refer to Section 3.1 for details and exact expression of runtime.

In a more general perspective, our algorithm can be straightforwardly applied to any suitable "depth-structured" graph class via efficient interpretability of logic theories. This includes the aforementioned results of Lampis [17] and Ganian [13] as special cases. We moreover extend the algorithm (Theorem 3.4) to testing MSO_2 properties on all graphs of *tree-depth* $\leq d$ (see Definition 2.1) in elementary FPT, covering a much wider graph class than that of bounded vertex cover. This in Section 3.2 concludes the first half of our paper.

Expressive power of FO and MSO

Secondly, the existence of an (elementarily-sized) kernel for MSO properties ϕ of trees of fixed height d (Theorem 3.2) is interesting on its own. Particularly, it immediately implies that any such MSO sentence ϕ can be equivalently expressed in FO on the trees of height d (simply testing the finitely many bounded-size kernels for which ϕ is true, Theorem 4.1). This brings us to the very recent paper of Elberfeld, Grohe, and Tantau [9] who proved that FO and MSO_2 have equal expressive power on the graphs of bounded tree-depth. Their approach is different and uses a constructive extension of Feferman–Vaught theorem for unbounded partitions. We can now similarly derive the result from Theorem 3.2, as in the tree case.

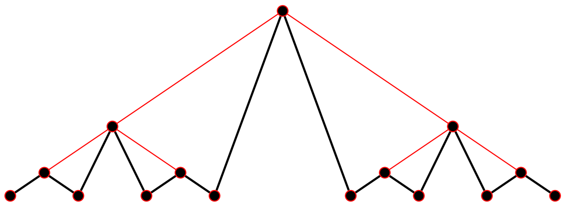

Figure 1 The path of length 14 has tree-depth $3 + 1 = 4$ since it is contained in the closure of the depicted (red) tree of height 3. It can be proved that this is optimal.

Going a step further, we actually half-answer the main open question posted in [9]; what characterizes the hereditary graph classes on which the expressive powers of FO and MSO_1 coincide? We use Theorem 3.2 and the new notions of [14] to prove that FO and MSO_1 coincide (Theorem 4.3) on all graph classes of bounded so called *shrub-depth* (see Definition 2.3). Unfortunately, due to lack of a suitable "forbidden substructure" characterization of shrub-depth, we are not yet able to prove the converse direction, but we conjecture that a hereditary class on which FO and MSO_1 coincide must have bounded shrub-depth (Conjecture 4.4). This conjecture is also supported by the following claim in [14]; a graph class $\mathcal{C}$ has an MSO_1 interpretation in the class of coloured trees of height $\leq d$ iff $\mathcal{C}$ is of shrub-depth $\leq d$.

2 Preliminaries

We assume standard terminology and notation of graph theory, see e.g. Diestel [5]. Due to limited space, we refer there [5] for the standard definition of tree-width $tw(G)$.

For an introduction to parameterized complexity we suggest [6]. Now we just recall that a problem $\mathcal{P}$ with an input $\langle x, k\rangle \in \Sigma^* \times \mathbb{N}$ is *fixed parameter tractable*, or FPT, if it admits an algorithm in time $\mathcal{O}\big(f(k) \cdot |x|^{\mathcal{O}(1)}\big)$ where f is an arbitrary computable function. It is known that $\mathcal{P}$ is in FPT if, and only if, it has a *kernel*, i.e., every instance $\langle x, k\rangle$ can be in polynomial time transformed to an equivalent instance $\langle x', k'\rangle$ such that $\langle x, k\rangle \in \mathcal{P} \iff \langle x', k'\rangle \in \mathcal{P}$ and $|\langle x', k'\rangle| \leq g(k)$ for some computable g.

Measuring depth of graphs

Our paper deals with some not-so-known decompositions of graphs, too. The first one is related to tree-decompositions of low depth.

▶ **Definition 2.1** (Tree-depth [18]). The *closure* $cl(F)$ of a rooted forest F is the graph obtained from F by adding from each node all edges to its descendants. The *tree-depth* $td(G)$ of a graph G is one more than the smallest height (distance from the root to all leaves) of a rooted forest F such that $G \subseteq cl(F)$.

Note that tree-depth is always an upper bound for tree-width. Some useful properties of it can be derived from the following asymptotic characterization: If L is the length of a longest path in a graph G, then $\lceil \log_2(L + 2)\rceil \leq \mathrm{td}(G) \leq L + 1$. See Figure 1. For a simple proof of this, as well as for a more extensive study of tree-depth, we refer the reader to [19, Chapter 6]. Particularly, it follows that $td(G)$ can be approximated up to an exponential error by a depth-first search, and furthermore computed exactly in linear FPT using the tree-width algorithm of Bodlaender [2].

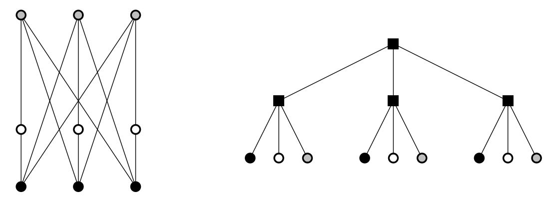

Figure 2 The graph obtained from $K_{3,3}$ by subdividing a matching belongs to $\mathcal{TM}_3(2)$. The respective tree model is depicted on the right.

Besides tree-width, another useful width measure of graphs is *clique-width*; defined for a graph G as the smallest number of labels $k = cw(G)$ such that G can be constructed using operations to create a new vertex with label i, take the disjoint union of two labeled graphs, add all edges between vertices of label i and label j, and relabel all vertices with label i to have label j. Similarly as tree-depth is related to tree-width, there exists a very new notion of *shrub-depth* [14] which is (in a sense) related to clique-width, and which we explain next.

▶ **Definition 2.2** (Tree model [14])**.** We say that a graph G has a *tree model of m colours and depth $d \geq 1$* if there exists a rooted tree T (of height d) such that

i. the set of leaves of T is exactly $V(G)$,
ii. the length of each root-to-leaf path in T is exactly d,
iii. each leaf of T is assigned one of m colours (this is not a graph colouring, though),
iv. and the existence of a G-edge between $u, v \in V(G)$ depends solely on the colours of u, v and the distance between u, v in T.

The class of all graphs having a tree model of m colours and depth d is denoted by $\mathcal{TM}_m(d)$.

For instance, $K_n \in \mathcal{TM}_1(1)$ or $K_{n,n} \in \mathcal{TM}_2(1)$. Definition 2.2 is further illustrated in Figure 2. It is easy to see that each class $\mathcal{TM}_m(d)$ is closed under complements and induced subgraphs, but neither under disjoint unions, nor under subgraphs. One can also routinely verify that each class $\mathcal{TM}_m(d)$ is of bounded clique-width. The depth d of a tree model can be seen as a generalization of the aforementioned tree-depth parameter, and for that reason it is useful to work with a more streamlined notion which only requires a single parameter. To this end we introduce the following (and we refer to [14] for additional details):

▶ **Definition 2.3** (Shrub-depth [14])**.** A class of graphs $\mathcal{G}$ has *shrub-depth* d if there exists m such that $\mathcal{G} \subseteq \mathcal{TM}_m(d)$, while for all natural m it is $\mathcal{G} \not\subseteq \mathcal{TM}_m(d-1)$.

Note that Definition 2.3 is asymptotic as it makes sense only for infinite graph classes. Particularly, classes of shrub-depth 1 are known as the graphs of bounded *neighbourhood diversity* in [17], i.e., those graph classes on which the twin relation on pairs of vertices (for a pair to share the same set of neighbours besides this pair) has a finite index.

MSO logic on graphs

Monadic second-order logic (MSO) is an extension of first-order logic (FO) by quantification over sets. On the one-sorted adjacency model of graphs it specifically reads as follows:

▶ **Definition 2.4** (MSO_1 logic of graphs)**.** The language of MSO_1 contains the expressions built from the following elements:

- variables $x, y, \ldots$ for vertices, and $X, Y, \ldots$ for sets of vertices,

- the predicates $x \in X$ and $edge(x, y)$ with the standard meaning,
- equality for variables, the connectives $\wedge, \vee, \neg, \rightarrow$, and the quantifiers $\forall, \exists$ over vertex and vertex-set variables.

Note that we do not allow quantification over edges or sets of edges (as edges are not elements) in MSO_1. If we consider the two-sorted incidence graph model (in which the edges formed another sort of elements), then we get:

▶ **Definition 2.5** (MSO_2 logic of graphs)**.** The language of MSO_2 contains the expressions built from elements of MSO_1 plus the following:

- variables $e, f, \ldots$ for edges, $E, F, \ldots$ for sets of edges, the respective quantifiers, and
- the predicates $e \in F$ and $inc(x, e)$ with the standard meaning.

Already MSO_1 logic is quite powerful as it can express various common hard graph properties; e.g., 3-colourability. The expressive power of MSO_2 is even strictly larger [8] since, for instance, Hamiltonicity has an MSO_2 definition (while not MSO_1). On the other hand, MSO_2 and MSO_1 coincide on the class of trees, or on many other restricted graph classes. Hence we will speak only about MSO_1 on trees, from now on. The large expressive power of MSO logics is the reason for their popularity in algorithmic metatheorems.

The problem to decide, for a sentence ψ in logic $\mathcal{L}$, whether an input structure G satisfies $G \models \psi$, is also commonly called the $\mathcal{L}$ *model checking problem* (of ψ). Hence, for instance, the c-colourability problem for each fixed c is an instance of MSO_1 model checking; where $\psi \equiv \exists X_1, \ldots, X_c. \left[\left(\forall x. \bigvee_{i=1}^{c} x \in X_i \right) \wedge \left(\forall x, y. \bigwedge_{i=1}^{c} (x \notin X_i \vee y \notin X_i \vee \neg\, edge(x, y)) \right) \right]$.

3 Trees of Bounded Height and MSO

The primary purpose of this section is to prove Theorem 3.2; that for any m-coloured tree T of constant height h there exists an efficiently computable subtree $T_0 \subseteq T$ such that, for any MSO_1 sentence ϕ of fixed quantifier rank r, it is $T \models \phi \iff T_0 \models \phi$, and the size of T_0 is bounded by an elementary function of r and m (the dependence on h being non-elementary, though). Particularly, since checking of an MSO_1 property ϕ can be easily solved in time $\mathcal{O}^*\left(2^{c|\phi|}\right)$ on a graph with c vertices (in this case T_0) by recursive exhaustive expansion of all quantifiers of ϕ, this gives a kernelization-based elementary FPT algorithm for MSO_1 model checking of rooted m-coloured trees of constant height h (Corollary 3.3).

We need a bit more formal notation. The *height* h of a rooted tree T is the farthest distance from its root, and a node is at the *level* ℓ if its distance from the root is $h - \ell$. For a node v of a rooted tree T, we call a *limb of* v a subtree of T rooted at some child node of v. Our rooted trees are unordered, and they "grow top-down", i.e. we depict the root on the top. For this section we also switch from considering m-coloured trees to more convenient t-labelled ones, the difference being that one vertex may have several labels at once (and so $m \sim 2^t$). MSO_1 logic is naturally extended to labelled graphs by adding unary predicates $L(x)$ for every label L. We say that two such rooted labelled trees are *l-isomorphic* if there is an isomorphism between them preserving the root and all labels.

3.1 The Reduction Lemma

Concretely, we preprocess a given tree T into a bounded kernel $T_0 \subseteq T$ by recursively deleting from T all limbs which are "repeating (being l-isomorphic) too many times". This is formalized in Lemma 3.1. To describe the exact reduction of T to T_0, we need to define the following recursive "threshold" values, for $i = 0, 1, 2, \ldots$:

$$R_i(q,s,k) = q \cdot N_i(q,s,k)^s, \qquad \text{where} \tag{1}$$

$$N_0(q,s,k) = 2^k + 1 \geq 2 \quad \text{and}$$
$$N_{i+1}(q,s,k) = 2^k \cdot \big(R_i(q,s,k)+1\big)^{N_i(q,s,k)} \leq 2^k \cdot \big(2q \cdot N_i(q,s,k)^s\big)^{N_i(q,s,k)} \tag{2}$$

For clarity, we informally in advance outline the intended meaning of these values R_i and N_i. We say a labelled rooted tree of height i is (q,s,k)-*reduced* if, at any level j, $0 < j \leq i$, each node of T has at most (1) $R_{j-1}(q,s,k)$ pairwise l-isomorphic limbs (which are of height $\leq j-1$). The value (2) $N_j(q,s,k)$ is then an upper bound on the number of all possible non-l-isomorphic rooted k-labelled trees T of height $\leq j$ that are (q,s,k)-reduced. Note that $N_0(q,s,k)$ accounts for all distinct k-labelled single-node trees and the empty tree.

Assume now any MSO_1 sentence (closed formula) ϕ with q element variables and s set variables, and height i. Then, provided $a, b \geq R_i(q,s,k)$ where $k = t + 3q + s$, we show that the sentence ϕ could not distinguish between a disjoint copies and b disjoint copies of any (q,s,k)-reduced rooted t-labelled tree of height i. Altogether formally:

▶ **Lemma 3.1.** *Let T be a rooted t-labelled tree of height h, and let ϕ be an* MSO_1 *sentence with q element quantifiers and s set quantifiers. Suppose that $u \in V(T)$ is a node at level $i+1$ where $i < h$.*

a) If, among all the limbs of u in T, there are more than $R_i(q,s,t+3q+s)$ pairwise l-isomorphic ones, then let $T' \subseteq T$ be obtained by deleting one of the latter limbs from T. Then, $T \models \phi \iff T' \models \phi$.

b) Consequently, there exists a rooted t-labelled tree $T_0 \subseteq T$ such that T_0 is $(q,s,t+3q+s)$-reduced, and $T \models \phi \iff T_0 \models \phi$.

In the case of FO logic, a statement analogous to Lemma 3.1 is obtained using folklore arguments of finite model theory (even full recursive expansion of all q vertex quantifiers in ϕ could "hit" only bounded number of limbs of u and the rest would not matter). However, in the case of MSO logic there are additional nontrivial complications which require new ideas (in addition to standard tools) in the proof. Briefly saying, one has to recursively consider the internal structure of the limbs of u, and show that even an expansion of a vertex-set quantifier in ϕ does not effectively distinguish too many of them (and hence some of them remain irrelevant for the decision whether $T \models \phi$).

Proof of Lemma 3.1. Note first that part b) readily follows by a recursive bottom-up application of a) to the whole tree. Hence we focus on a), and sketch our proof as follows:

(I) We are going to use a so called "quantifier elimination" approach.[1] That means, assuming $T \models \phi \not\iff T' \models \phi$, we look at the "distinguishing choice" of the first quantifier in ϕ, and encode it in the labeling of T (e.g., when $\phi \equiv \exists x.\psi$, we give new exclusive labels to the value of x and to its parent/children in T and T'). By an inductive assumption, we then argue that the shorter formula ψ cannot distinguish between these newly labeled T and T', which is a contradiction.

(II) The traditional quantifier elimination approach—namely of set quantifiers in ϕ, however, might not be directly applicable to even very many pairwise l-isomorphic limbs in T if their size is unbounded. Roughly explaining, the problem is that a single valuation of a set variable on these repeated limbs may potentially pairwise distinguish

[1] This approach has been inspired by recent [7], though here it is applied in a wider setting of MSO logic.

$$
\begin{array}{ccc}
\psi \in \text{“}\mathcal{L}_1 \text{ over } \mathscr{K}\text{”} & & \psi^I \in \text{“}\mathcal{L}_2 \text{ over } \mathscr{M}\text{”} \\
H \in \mathscr{K} & \xrightarrow{\quad I \quad} & G \in \mathscr{M} \\[2em]
G^I \cong H & & G \\
(\text{s.t. } G^I \models \psi) & \xleftarrow{\quad I \quad} & (\text{s.t. } G \models \psi^I)
\end{array}
$$

Figure 3 A basic informal scheme of an interpretation of $\mathrm{Th}_{\mathcal{L}_1}(\mathscr{K})$ into $\mathrm{Th}_{\mathcal{L}_2}(\mathscr{M})$.

all of them. Hence additional combinatorial arguments are necessary to bound the size of the limbs in consideration.

(III) Having successfully resolved technical (II), the rest of the proof is a careful composition of inductive arguments using the formula (1) $R_i(q,s,k) = q \cdot N_i(q,s,k)^s$.

Details can be found in the full paper [12]. ◀

3.2 Algorithmic applications

With some calculus, we summarize the obtained result from an algorithmic point of view. Let $exp^{(i)}(x)$ be the i-fold exponential function defined inductively as follows: $exp^{(0)}(x) = x$ and $exp^{(i+1)}(x) = 2^{exp^{(i)}(x)}$. Note that $exp^{(h)}(x)$ is an elementary function of x for each particular height h. For a rooted t-labelled tree T of height $\leq h$, we call the uniquely-determined maximal (q,s,k)-reduced tree $T_0 \subseteq T$ from Lemma 3.1 b), where $k = t + 3q + s$, a *(q,s,k)-reduction* of the tree T. Then we routinely get:

▶ **Theorem 3.2.** *Let $t, h \geq 1$ be integers, and let ϕ be an* MSO_1 *sentence with q element quantifiers and s set quantifiers. For each rooted t-labelled tree T of height h, the tree $T_0 \subseteq T$ which is a $(q,s,t+3q+s)$-reduction of T and $T_0 \models \phi \iff T \models \phi$, can be computed in linear time (non-parameterized) from T. Moreover, its size is bounded by*

$$|V(T_0)| \leq exp^{(h)}\left[(2^{h+5} - 12) \cdot (t+q+s)(q+s)\right].$$

▶ **Corollary 3.3.** *Let T be a rooted t-labelled tree of constant height $h \geq 1$, and let ϕ be an* MSO_1 *sentence with r quantifiers. Then $T \models \phi$ can be decided by an FPT algorithm in time*

$$\mathcal{O}\left(exp^{(h+1)}\left[2^{h+5} \cdot r(t+r)\right] + |V(T)|\right) = \mathcal{O}\left(exp^{(h+1)}(|\phi|^2) + |V(T)|\right).$$

The arguments of Corollary 3.3 can be further extended to suitable classes of general graphs via the traditional tool of *interpretability* of logic theories [20]. This powerful tool, however, has rather long formal description, and since we are going to use it only ad hoc in some proofs anyway, we provide here only a brief conceptual sketch. Imagine two classes of relational structures $\mathscr{K}, \mathscr{M}$ and two logical languages $\mathcal{L}_1, \mathcal{L}_2$. We say there is an *interpretation I* of the $\mathcal{L}_1$ theory of $\mathscr{K}$ into the $\mathcal{L}_2$ theory of $\mathscr{M}$ if (see Figure 3)

- there exist $\mathcal{L}_2$ formulas which can "define" the domain and the relations of each structure $H \in \mathscr{K}$ inside a suitable structure $G \in \mathscr{M}$, formally $H \simeq G^I$,
- and each formula $\psi \in \mathcal{L}_1$ over $\mathscr{K}$ can be accordingly translated into $\psi^I \in \mathcal{L}_2$ over $\mathscr{M}$ such that "truth is preserved", i.e., $H \models \psi$ iff $G \models \psi^I$ for all such related H, G.

A simple example is an interpretation of the complement of a graph G into G itself via defining the edge relation as $\neg\, edge(x,y)$. A bit more complex example is shown by interpreting a line graph $L(G)$ of a graph G inside G; the domain (vertex set) of $L(G)$ being interpreted in $E(G)$, and the adjacency relation of $L(G)$ defined by the formula $\alpha(e,f) \equiv$

$e \neq f \wedge \exists x.\ inc(x,e) \wedge inc(x,f)$. This example interprets the MSO_1 theory of line graphs in the MSO_2 theory of graphs.

We now return back to the promised extensions. Since the MSO_2 theory of graphs of tree-depth $\leq d$ has an interpretation in coloured trees of depth $\leq d+1$ (a graph G is actually interpreted in W such that $G \subseteq cl(W)$, with labels determining which "back edges" of W belong to G), we get the following generalization of Lampis' [17] from Corollary 3.3: MSO_2 model checking can be done in FPT time which depends elementarily on the checked formula, not only for graphs of bounded vertex cover, but also for those of bounded tree-depth.

▶ **Theorem 3.4.** *Let $\mathscr{D}_d$ denote the class of all graphs of tree-depth $\leq d$, and ϕ be an* MSO_2 *sentence with r quantifiers. Then the problem of deciding $G \models \phi$ for $G \in \mathscr{D}_d$ has an FPT algorithm with runtime $\mathcal{O}\big(\, exp^{(d+2)}\big(2^{3d+7} \cdot r^2\big) + |V(G)|\big)$.*

We also remark on an important aspect of FPT algorithms using width parameters—how to *obtain the associated decomposition* of the input (here of $G \in \mathscr{D}_d$). In the particular case of tree-depth, the answer is rather easy since one can use the linear FPT algorithm for tree-decomposition [2] to compute it (while, say, for clique-width this is an open problem).

Concerning MSO_1 model checking, one can go further. Graphs of *neighbourhood diversity* m (introduced in [17]) are precisely those having a model in which every vertex receives one of m colours, and the existence of an edge between u, v depends solely on the colours of u, v. Clearly, these graphs coincide with those having a tree model of m colours and depth 1, and so we can give an FPT algorithm for MSO_1 model checking on them from Corollary 3.3, which is an alternative derivation for another result of Lampis [17]. We can similarly derive an estimation of the main result of [13] (here just one exponential fold worse).

A common generalization of these particular applications of Corollary 3.3 has been found, together with the new notion of shrub-depth, in this subsequent work:

▶ **Theorem 3.5** (Ganian et al. [14]). *Assume $d \geq 1$ is a fixed integer. Let $\mathscr{G}$ be any graph class of shrub-depth d (Definition 2.3). Then the problem of deciding $G \models \phi$ for the input $G \in \mathscr{G}$ and* MSO_1 *sentence ϕ, can be solved by an FPT algorithm, the runtime of which has an elementary dependence on the parameter ϕ. This assumes G is given on the input alongside with its tree model of depth d.*

4 Expressive power of FO and MSO

Theorem 3.2 has another interesting corollary in the logic domain. Since the size of the reduction T_0 of T is bounded independently of T, the outcome of $T \models \phi$ actually depends on a finite number of fixed-size cases, and one can use even FO logic to express (one would say by brute force) which of these cases is the correct $(q, s, t + 3q + s)$-reduction of T. The outlined arguments lead to the following conclusions.

▶ **Theorem 4.1** (Theorem 3.2). *Let $t, h \geq 1$ be integers, and let ϕ be an* MSO_1 *sentence with q element quantifiers and s set quantifiers. There exists a finite set of rooted t-labelled trees $\mathscr{U}_{h,t,\phi}$ satisfying the following: For any rooted t-labelled tree T of height $\leq h$, it holds $T \models \phi$ if and only if the $(q, s, t + 3q + s)$-reduction of T is l-isomorphic to a member of $\mathscr{U}_{h,t,\phi}$.*

With Theorem 4.1 we get quite close to the very recent achievement of Elberfeld, Grohe, and Tantau [9] who prove that FO and MSO_2 have equal expressive power on the graphs of bounded tree-depth (and that this condition is also necessary on hereditary graph classes). The following weaker statement is actually an easy consequence of our findings, too:

▶ **Corollary 4.2** (Elberfeld, Grohe, and Tantau [9]). *Let h, t be integers, and ϕ an* MSO_1 *sentence. Then there is an FO sentence $\psi_{h,t,\phi}$ such that, for any rooted t-labelled tree T of height $\leq h$, it is $T \models \phi \iff T \models \psi_{h,t,\phi}$.*

It is now a natural question whether and how could our alternative approach to coincidence between FO and MSO on graphs be extended in the same direction.

Indeed, given an MSO_2 sentence ϕ over $\mathscr{D}_d$ (the graphs of tree-depth $\leq d$), we can interpret this in an MSO_1 sentence ϕ^I_d over rooted $(d+1)$-labelled trees of height $\leq d+1$. Then, by Corollary 4.2, we immediately get an FO sentence σ_d equivalent to ϕ^I_d. The problem is, however, that σ_d is a formula over rooted $(d+1)$-labelled trees, and we would like to get an interpretation of σ_d back in the FO theory of the class $\mathscr{D}_d$, which does not seem to be an easy task directly. Still, part of the arguments of [9] can be combined with the approach of Corollary 4.2 to provide an alternative relatively short proof of coincidence between FO and MSO_1 on classes of bounded tree-depth (thus bypassing the Feferman–Vaught–type theorem in [9]).

The reason for specifically mentioning Elberfeld, Grohe, and Tantau's [9] here is actually their main posted question—what are the sufficient and necessary conditions for a hereditary graph class to guarantee the same expressive power of FO and MSO_1? Using Theorem 4.1 and improved ideas based on a proof of Corollary 4.2, we provide a nontrivial sufficient condition which we also conjecture to be necessary.

▶ **Theorem 4.3.** *Let d be an integer and $\mathscr{S}$ be any graph class of shrub-depth d (Definition 2.3). Then for every* MSO_1 *sentence ϕ there is an* FO *sentence $\psi_{d,\phi}$ such that, for any $G \in \mathscr{S}$, it is $G \models \phi \iff G \models \psi_{d,\phi}$. Consequently,* FO *and* MSO_1 *have the same expressive power on $\mathscr{S}$.*

▶ Conjecture 4.4. Consider a hereditary (i.e., closed under induced subgraphs) graph class $\mathscr{S}$. If the expressive powers of FO and MSO_1 are equal on $\mathscr{S}$, then the shrub-depth of $\mathscr{S}$ is bounded (by a suitable constant).

The key to proving Theorem 4.3 is the notion of twin sets. Recall that two vertices $x, y \in V(G)$ are called *twins* if their neighbour sets in $G - x - y$ coincide. Though the edge xy is not specified in this definition, it easily follows that whenever we have a set of pairwise twins in G, then those induce a clique or an independent set.

▶ **Definition 4.5** (Twin sets). Assume $X = \{x_1, \ldots, x_k\}$ and $Y = \{y_1, \ldots, y_k\}$ are disjoint indexed sets (k-tuples) of vertices of a graph G. We say that X, Y are *twin sets* in G if

- the subgraphs of G induced on X and on Y are identical, i.e., $x_i x_j \in E(G)$ iff $y_i y_j \in E(G)$ for all pairs $i, j \in \{1, \ldots, k\}$, and
- for $i = 1, \ldots, k$, the set of neighbours of x_i in $V(G) \setminus (X \cup Y)$ equals that of y_i.

Note that, for simplicity, we consider the twin-sets relation only on disjoint sets, and that this relation is generally not transitive. Although we do not need more for this paper, we suggest that the notion deserves further extended study elsewhere.

A tree model (Definition 2.2) of a graph G can be, informally, viewed as a complete recursive decomposition (of bounded depth) of G into groups of pairwise disjoint pairwise twin sets. Roughly, an application of Lemma 3.1 a) then says that if (at any level) the number of pairwise twin sets in a group is "too large", then one of these sets can be deleted from G without affecting validity of a fixed MSO_1 property on G. Our main task is to describe "reducibility" of a large group of twin sets in G using FO (the sets having bounded size, though), which is more complicated than in the tree-depth case due to lack of some "nice connectivity properties" of a tree-depth decomposition.

Proof outline (Theorem 4.3). We assume a graph $G \in \mathscr{S}$ with a tree model T of constant depth d, and an MSO_1 sentence ϕ. We informally continue as follows.

(I) For every fixed d, one can easily interpret ϕ in an MSO_1 formula ϕ^I_d over T, such that $G \models \phi \iff T \models \phi^I_d$.

(II) By Definition 2.2, pairwise l-isomorphic sibling limbs in T correspond to a group of pairwise twin sets in G. Deleting one of these sets from G is equivalent to deleting the corresponding limb from T. Hence by (I) and Theorem 4.1, there is a finite set $\mathscr{U}_\phi$ of graphs (independent of G) such that $G \models \phi$ iff G "reduces" to a member of $\mathscr{U}_\phi$.

(III) The meaning of "reduction" is analogous to Section 3.1, to a (q,s,k)-reduced subtree of the tree model T. The minor technical differences are; (1) we can describe the reduction using twin sets, without an explicit reference to whole T, and (2) we actually aim at a $(q,s,k)'$-*reduction* which means the reduction threshold values are $R'_j(q,s,k) = \max\{R_j(q,s,k), 2\}$. (We need to guarantee that at least two twin sets of each group remain after the reduction, even in degenerate cases.)

(IV) We provide an FO definition of the fact that G reduces to $H \in \mathscr{U}_\phi$, modulo some technical details. This FO formula ϱ_H depends mainly on d and H (actually on a suitable tree model of H). The desired sentence $\psi_{d,\phi}$ in Theorem 4.3 is then constructed as the (*finite*) disjunction $\psi_{d,\phi} \equiv \bigvee_{H\in\mathscr{U}_\phi} \varrho_H$.

Now we give the crucial technical detail and the related claims which make step (IV) working. Assume T is a tree model of a graph G, and B is a limb of a node v in T, such that W is the set of leaves of B. We say that a tree model T' is obtained from T by *splitting B along $X \subseteq W$* if a disjoint copy B' of B with the same parent v is added into T, and then B is restricted to a rooted Steiner tree of $W \setminus X$ while B' is restricted to a rooted Steiner tree of X' (the corresponding copy of X). A tree model T is *splittable* if some limb in T can be split along some subset X of its leaves, making a tree model T' which represents the same graph G as T does. A tree model is *unsplittable* if it is not splittable. Notice that any tree model can be turned into an unsplittable one; simply since the splitting process must end eventually.

▶ **Lemma 4.6.** *Let H be a graph, and $R \subseteq H$ be an induced subgraph having a tree model T (of m colours and depth d, but this is not relevant). Let T contain two disjoint l-isomorphic limbs B, B' of a node v, and a limb C of a node u. The position of C against B, B' can be arbitrary (it may be $u = v$ or even $C = B$ or $C = B'$), as long as C is disjoint from one of B, B'. Let $W, W' \subseteq V(R)$ denote the sets of leaves of B, B', respectively, and $X \subseteq V(R)$ denote the set of leaves of C. Assume $Y, Z \subseteq V(H) \setminus V(R)$ are such that W, W', Y are pairwise twin sets in H, and that X, Z are also twin sets in H. If $Y \neq Y \cap Z \neq \emptyset$, then the tree model T of R is* splittable.

▶ **Lemma 4.7.** *Let $m, d \geq 1$ and q, s be integers. Assume $G \in \mathcal{TM}_m(d)$ is a graph, and $R \subseteq G$ is an induced subgraph having an* unsplittable *tree model T (of m colours and depth d). Let $\widehat{x_R} = (x_v : v \in V(R))$ be a vector of free variables valued in the respective vertices of R in G. Then there exists an* FO *formula ϱ_T, depending on d, m, q, s, and T, such that the following holds: $G \models \varrho_T(\widehat{x_R})$ if, and only if, $R \subseteq G$ and there exists a tree model $T' \supseteq T$ of G of m colours and depth d, such that the $(q, s, m + 3q + s)'$-reduction of T' is T.*

The importance of Lemma 4.6 in the proof of Lemma 4.7 is, informally, in that one can focus just on including every vertex of $G - R$ into some set which is twin (possibly after recursive reduction) to suitable limbs of T, while such sets will then never overlap. See Figure 4. With Lemma 4.7 at hand, it is then straightforward (though technical and not

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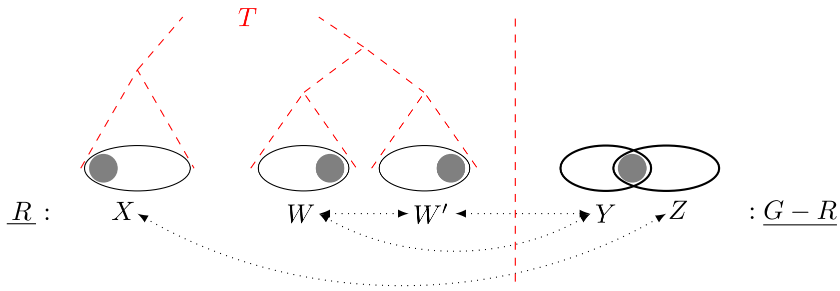

Figure 4 A situation which cannot happen, in a graph G with an unsplittable tree model T of an induced subgraph $R \subseteq G$, and with the sets W, W', Y and X, Z as in Lemma 4.6.

short) to finish the proof of Theorem 4.3 along the aforementioned outline. Details can be found in the full paper [12]. ◀

5 Conclusions

We briefly recapitulate the two-fold contribution of our primary result; that the MSO model checking problem on the universe of coloured trees of bounded height can be reduced to a kernel of size bounded by an elementary function of the formula. Firstly, it allows us to easily obtain nontrivial extensions of Lampis' and Ganian's result and to fill the gap set by Courcelle's theorem and the negative result of Frick and Grohe.

Secondly, it provides an alternative simple and intuitive way of understanding of *why* on some classes of graphs FO and MSO logics coincide. In this respect, the most important property of our kernel is that, after seeing more than a certain number of copies of a certain substructure in the input graph, the validity of an MSO formula in question does not change any further. While such a behavior is natural for FO properties, it is somehow surprising to see it for much wider MSO. This "loss of expressiveness" of MSO (getting down to the FO level) is inherited by graph classes of bounded tree-depth and shrub-depth.

Finally, we briefly discuss why we believe Conjecture 4.4 holds true. It is known [9] that each subgraph closed class of graphs such that FO $=$ MSO$_2$ has to have bounded tree-depth. Both classes of bounded tree-depth and classes of bounded shrub-depth are interpretable in trees of bounded depth, the main difference is how "dense" they are. By allowing "too many" edges in graphs of bounded shrub-depth, we basically lost the ability to address edges of the interpreted graph in the underlying tree and hence also the ability to quantify over these edges and sets of edges (notice that this also means that our class of graphs is no longer closed under taking subgraphs, but is still hereditary). Since this is exactly the difference between MSO$_1$ and MSO$_2$, classes of graphs of bounded shrub-depth are natural candidates for exactly those hereditary classes where FO $=$ MSO$_1$.

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