Graph coloring, communication complexity, and the stubborn problem

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Abstract

We discuss three equivalent forms of the same problem arising in communication complexity, constraint satisfaction problems, and graph coloring. Some partial results are discussed.

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1 The equivalent forms

A classical result of Graham and Pollak [5] asserts that the edge set of the complete graph on n vertices cannot be partitioned into less than $n-1$ complete bipartite graphs. A natural question is then to ask for some properties of graphs G_ℓ which are edge-disjoint unions of ℓ complete bipartite graphs. An attempt in this direction was proposed by Alon, Saks and Seymour, asking if the chromatic number of G_ℓ is at most $\ell+1$. This wild generalization of Graham and Pollak's theorem was however disproved by Huang and Sudakov [6] who provided graphs with chromatic number $\Omega(\ell^{6/5})$. The $O(\ell^{\log \ell})$ upper-bound being routine to prove, this leaves as open question the *polynomial Alon-Saks-Seymour conjecture* asking if an $O(\ell^c)$ coloring exists for some fixed c.

A communication complexity problem introduced by Yannakakis[8] involves a graph G of size n and the usual suspects Alice and Bob. Alice plays on the stable sets of G and Bob plays on the cliques. Their goal is to exchange the minimum amount of information to decide if Alice's stable set S intersect Bob's clique K. In the nondeterministic version, one asks for the minimum size of a certificate one should give to Alice and Bob to decide whether S intersects K. If indeed S intersects K, the certificate consists in the vertex $x = S \cap K$, hence one just has to describe x, which costs $\log n$. The problem becomes much harder if one want to certify that $S \cap K = \emptyset$ and this is the core of this problem. A natural question is to ask for a $O(\log n)$ upper bound. Yannakakis observed that this would be equivalent to the following *polynomial clique-stable separation conjecture*: There exists a c such that for any graph G on n vertices, there exists $O(n^c)$ vertex bipartitions of G such that for every disjoint stable set S and clique K, one of the bipartitions separates S from K.

A variant of Feder and Vardi celebrated dichotomy conjecture for Constraint Satisfaction Problems, the List Matrix Partition (LMP) problem, see [3], asks whether all $(0,1,*)$ CSP instances are NP–complete or polytime solvable. The LMP was investigated for small matrices, and was completely solved in dimension 4, save for a unique case, known as the

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stubborn problem [1]: Given a complete graph G which edges are labelled by 1,2, or 3, the question is to partition the vertices into three classes V_1, V_2, V_3 so that V_i does not span an edge labelled i. An easy branching majority algorithm computes $O(n^{\log n})$ 2-list-assignments of the vertices such that every solution of the stubborn problem is covered by at least one of these 2-list-assignments. The stubborn problem hence reduces into $O(n^{\log n})$ 2-SAT instances, yielding a pseudo polynomial algorithm. A polynomial algorithm was recently discovered by Cygan et al.[2], but whether the original branching algorithm could be turned into a polynomial algorithm is still open. Precisely one can ask the *polynomial stubborn 2-list cover conjecture* asking if the set of solutions of any instance of the stubborn problem can be covered by $O(n^c)$ instances consisting of lists of size 2.

We show that these three problems are indeed equivalent. One direction was already proved by Alon and Haviv.

We further discuss the polynomial clique-stable separation conjecture by restricting ourselves to some classes of graphs. An open problem of Lovász, see for instance [7], asks for an extending formulation of the stable set polytope of perfect graphs. This does not exist for all graphs, as Fiorini [4] et al. have recently proved, but special classes enjoy this property, like for instance comparability graphs. Furthermore, the existence of extended formulations for perfect graphs would give a polynomial clique-stable set separation for this class, which is still open. It is unlikely that the polynomial clique-stable separation holds for the class of all graphs, hence we propose a milder version of it: for every graph H, the class of H-free graphs (in the induced sense) has the polynomial clique-stable set separation. This question is wide open for the cycle of length 5, this would imply the result for perfect graphs. However, we can show that if H is a split graph, i.e. the union of a clique and a stable set, then there exists a c depending on H such that the clique-stable set separation can be achieved with n^c cuts in the class of H-free graphs. Observe that H can be wild since the edges between the clique and the stable set are not specified, hence no structural description of H-free graphs can be used here. The key tools for finding these n^c cuts is VC-dimension.

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