Husserl and Hilbert on Completeness and Husserl's Term Rewrite-based Theory of Multiplicity (Invited Talk)

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Abstract

Hilbert and Husserl presented axiomatic arithmetic theories in different ways and proposed two different notions of "completeness" for arithmetic, at the turning of the 20th Century (1900–1901). The former led to the completion axiom, the latter completion of rewriting. We look into the latter in comparison with the former. The key notion to understand the latter is the notion of *definite multiplicity* or *manifold* (Mannigfaltigkeit). We show that his notion of multiplicity is understood by means of term rewrite theory in a very coherent manner, and that his notion of "definite" multiplicity is understood as the relational web (or tissue) structure, the core part of which is a "convergent" term rewrite proof structure. We examine how Husserl introduced his term rewrite theory in 1901 in the context of a controversy with Hilbert on the notion of completeness, and in the context of solving the justification problem of the use of imaginaries in mathematics, which was an important issue in the foundations of mathematics in the period.

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1 Introduction

Two characteristic notions of completeness of arithmetic appeared at the same place, Göttingen in Germany, in 1900–1901, at the same Faculty of Philosophy, one introduced by Hilbert of the Mathematics Section, and the other introduced by Husserl of the Philosophy Section. The notion of completeness by Hilbert is well known: completeness in his sense ensures existence of a categorical model of an axiomatic system. On the other hand, Husserl's notion of completeness is not well known: completeness in his sense ensures a mathematical multiplicity or manifold (Mannigfaltigkeit) to be "definite". His notion of definite multiplicity has not been clarified very well until today although many efforts for clarifications have been made by a large number of former works. The purpose of this paper is to show that Husserl's definite multiplicity, hence completeness, can be well understood by the helps of term rewrite theory.

We explain how Husserl's notion of completeness is different from Hilbert and how he reached his idea by interpreting Hilbert's axiomatization of arithmetic in a slightly different way. We, in particular, show that Husserl introduced the various basic notions of term rewrite proofs. His motivation of the study on completeness was originated from "the justification problem of the use of imaginaries in mathematics", which is concerned with the conservation problem in the modern logical sense. He reached his solution in 1901 and presented it at the Mathematical Society Meeting invited and organized by Hilbert at Göttingen in the same year. The lectures is now called his Double Lecture (November and December). as he gave two talks there. He gave a solution using the notions of syntactic completeness and consistency (syntactic completeness for the original system and consistency for enlarged system with imaginaries) as the first rough outline. However he went further to clarify as to what was the completeness condition, then he explained it by means of, essentially, term rewrite theory. Husserl introduced his notion of definiteness of multiplicity or manifold to explain his notion of completeness. We show that a multiplicity of an axiomatic system, in his sense, is the whole web (or tissue) structure of rewrite equational-proof formation steps (or reduction moves) and term-formation steps (or moves), and that the notion of definiteness of a multiplicity corresponds to the notion of convergence in the modern term rewrite theory. A form of (Knuth-Bendix) completion procedure was also proposed by Husserl in 1901, in order to make a non-definite multiplicity definite. We shall explain these with some textual evidences (with the use of Husserl's several manuscripts in 1901); more detailed references and quotations as well as more philosophical discussions of the subject will appear in the subsequent papers. Husserl in fact used the word "term" and considered the notion of multiplicity as the term-rewrite computational content of an axiomatic system. But he also used the words, "concept" (of a term), and "object of of a concept" (of term), keeping his philosophical framework of significative intention-objectivity in the Logical Investigations. Moreover, the term-rewrite reductions to normal forms was considered as the significative fulfillment framework of the Sixth Investigation, to some extent. We shall discuss the further philosophical discussions related to these subjects in the subsequent philosophical paper.[1]

Roughly speaking, by completeness of arithmetic Hilbert meant the maximal extension of a model to reach a categorical (unique) model, which is the continuum, while Husserl meant the minimal term model based-reductive proofs web (tissue) structure, which is, roughly speaking, his notion of definite multiplicity. Husserl argued, against Hilbert, that this way to understand completeness is needed to solve the problem of justifying the use of imaginaries in mathematics.[2] Husserl distinguished the two notions of completeness clearly from the point of view of the justification problem of imaginaries, as if the axiomatic system were complete in the Hilbert sense (which he called "essentially complete" compared with his "outer-essentially complete"), the purpose of him to investigate in the problem would be hidden since no possibility of extending the original complete axiomatic system to an enlarged system with imaginearies would remain[*Hua XII*, p.445] . Hence, in order to understand the background issues and the motivation how Husserl reached a theory of rewriting proofs, we explain the problem of justifying the use of imaginaries in mathematics briefly. The problem of justification of the imaginaries had been one of the issues in philosophy of mathematics since Leibniz in the 17th Century, but in particular in the period of the end of the 19th Century. Husserl reached in 1901 to a general theory of convergent term rewriting and completion in the context of attacking this question.

The question of justifying the use of imaginaries in mathematics had been raised by various scholars both in mathematics and in philosophy at the turning of the century, and

[1] In this paper, we do not compare our work with former works related to the Double Lecture, except for a limited number of references and comments. We shall discuss further comparisions in the subsequent paper.

[2] It is also noted that Hilbert first considered an axiomatic foundations of geometry and reduced his foundational issues on consistency of geometry to that of (real number) arithmetic, while Husserl did in the opposite way; he believed his term rewrite theory on arithmetic should work with geometry.

the research question was in the air.[3] A typical way of the questioning was: "How can we justify the use of imaginary numbers in the course of proving a proposition or calculating an equation referring only to concepts on real numbers? Can such a proof or calculation through imaginaries be always transformed to a proof or calculation referring only to concepts on real numbers?"

Another example would be: "How can we justify an analytic proof techniques for proving of a theorem on elementary number theory?"

Husserl considered various cases of a relatively real system and its enlarged system with relatively imaginaries with respect to the real, and attacked the problem of imaginaries in a general arithmetical setting in 1901, to tried to find a general condition (from the term-rewrite theoretic view) to settle the problem.[4] Hilbert in his peak period of logic-foundational studies (in the 1920's) also considered a general framework of enlarged system with imaginaries, although Hilbert stuck to a fixed "contentual" finitist mathematical system for the original "real" system.[5]

The problem of justification of imaginaries in mathematics in the Husserl-Hilbert style is understood (at least roughly). as the problem to show (a sort of) "conservative relation" of axiomatic systems. Consider two formal axiomatic systems $S_1 \subset S_2$ where

S_1: original [real] system,
S_2: enlarged system with some imaginaries

[Conservation Property] For any real proposition A of S_1, if A is provable in S_2 then A is always provable in S_1, namely,

$$S_2 \vdash A \Longrightarrow S_1 \vdash A,$$

then S_2 is called a conservative extension of S_1 (and S_1 is called a conservative subsystem of S_2)[6]

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[3] See [Hartimo 2007] [Hartimo 2010] for the historical background and context at the end of the Century on the problem of justification of the use of imaginaries around the turning of the century.

[4] They include any variant of the usual formal axiomatic natural number theory, as well as any formal theory of integers, of rationals, of real numbers, of complex numbers, etc.[Schuhmann and Schuhmann 2001, p.105–p.106], [*Hua XII*, p.442–p.443] as well as (propositional) logical calculus [*Hua XII*, p.487–p.488]

[5] Cf. [Hilbert 1926], also [Detlefsen 1947] and [Kreisel 1958], for Hilbert's consistency proof program. It is possible that Hilbert's (revised) presentation of the consistency program in the mid 1920's, using the conservation problem framework, was a result of the influence of Husserl's 1901 talk to Hilbert, which was pointed out in [Okada 1987].

[6] The conservative delation problems, hence the justification problems, had been discussed in history of the development of mathematics (although the property cannot always be expected as Gödel showed in his Incompleteness Theorem (1931). Note that both Leibniz and Gödel emphasizes the usefulness of introducing imaginaries by pointing out the effect of speeding-up and servayability of proving the real propositions. Leibniz, pointed out usefulness of placing Lemma (hence the use of cut-rule in the sense of Gentzen, while the investigation into rewriting of a proof with cut-rules (hence with lemmas) into a cut-free direct proof was one of the main research paradigms employed by the Hilbert School in the 1030's in order to solve the conservation-justification problem. In the case of Leibniz, he needed the use of infinitesimal (real) numbers, such as dx, and infinite (real) numbers, $1/dx$ in his introduction of differential and integral calculus in the 17th Century. Of course, he himself faced the question how to justify the use of such numbers. Now, there are two well known ways to justify the use; one is to introduce the "contextual" rewriting with of "limit"; for example, $df(x)/dx$ is defined contextually with the ϵ-δ "description", which was introduced by Cauchy, only in the beginning of the 19th Century, then more logically by Bolzano and Weierstrass (Husserl worked as an assistant to Weierstrass before he moved to philosophy.). The other way is to introduce nonstandard analysis; Abraham Robinson, in the 1950-60's, was first who adapted Tarskian model theory/formal semantics theory to Leibniz representation dx as the nonstandard real numbers in his introduction of non-standard analysis, where the notion of elementary extension of a model is essential. Leibniz himself also suggested the first way, but also expressed that a certain algebraic or abstract rewriting works in practice (cf. [Okada 1987])

Husserl prepared various manuscripts in the winter semester of 1901, and presented his "solution" to the problem, which he believed to work for various arithmetic systems at two successive talks, which is now called the Double Lecture, at the Göttingen Mathematical Society Meeting organized by Hilbert. At the first talk he claimed, among others, the following.

(Claim α) If the following are satisfied, then the use of imaginaries is justified.

(1) the original narrower system is (syntactically) complete, and
(2) the enlarged system is consistent.

Namely, the above two conditions imply the conservation property i.e., the enlarged system is a conservative extension over the original, hence the use of imaginaries introduced in an enlarged system is justified under these conditions.[7]

He explained his notion of completeness more precisely by the use of the notion of multiplicity or manifold (Mannigfaltigkeit). The notion of completeness is characterized by "definiteness" of a multiplicity. The word "multiplicity" of an axiomatic system and the word "domain of an axiomatic system" were exchangeably used by him.[8] The originality of this paper is to clarify (or propose to read) the notion of definite multiplicity of Husserl in terms of term rewrite theory. In fact, Husserl went beyond just the notion of syntactic completeness by going into the notion of (definite) multiplicity. Since his notion of enlargement of definite multiplicity guarantees consistency[9], he expresses his claim as follows.

(Claim β) An axiomatic theory is complete "If the [an] axiomatic mathematical theory determines its mathematical domain (multiplicity) "definitely," without leaving any ambiguity in the structure of the "domain", and then the use of imaginaries is justified".

Husserl called a multiplicity which is determined definitely a definite multiplicity.

Before we go to the next Section, for philosophical readers we make here a remark; Husserl reached his notion of definite multiplicity with term rewriting theory as the result of his various different former studies of him. The following different questions and studies merged at the same time in 1901 winter when he reached the solution.

1. justification of the use of imaginaries in mathematics,
2. mathematical multiplicity/manifold,
3. general conditions of extending/overloading mathematical operators/functions (consistent overloading use of function symbols), or now called of overloading (of function symbols) in Computer Science,
4. general theory of decision problem for an axiomatic equational system.
5. phenomenological notion of significative fulfillment as a fulfillment of arithmetic terms .
6. studies in categorial intuition (or syntax-oriented intuitive evidence, including intuition on formal arithmetic).

[7] This was pointed out in [Okada 1987] and [Majer 1997].
[8] This fact might have misled the commentators to understand the notion as a set theoretical model of the axiomatic system for the long time history of study over 60 years.
[9] as we shall see later, he considered extending an original system under the condition that normal (irreducible) constructor terms do not collapse

Unfortunately in this short paper we cannot explain and discuss the whole picture of how Husserl united 1-6 above during his development in his philosophy of logic and mathematics. We would like to pick up some minimum backgrounds and issues from these lists, and discuss a birth of an ideas of definite multiplicity and rewriting theory by Husserl. We leave more detailed discussions on the controversy between Hilbert and husserl, from the philosophical and mathematical points of view in the subsequent papers.

2 Hilbert's axiomatic system of arithmetic and Husserl's interpretation of Hilbert's system toward term rewrite theory

Hilbert published in 1900 his article on his formulation of arithmetic axioms in 1900 (in the same line as his "Foundations of geometry (1899)". Husserl modified Hilbert's axiomatization of arithmetic in 1901, which shows us some important differences between the attitudes of the two figures on the notion of completeness.[10] Husserl presented a formal axiomatic system in a form very similar to Hilbert's, with a slight modification. Here, this similarity and slight modification are both important, in our opinion, to understand Husserl's notion of completeness, and of his term rewrite based-notion of multiplicity.

Hilbert's presentation of axioms for arithmetic in question is composed of four groups of axioms, as well known:

I. Axioms of linking (junction)
II. Axioms of calculation
III. Axioms of ordering
IV. Axioms of continuity (composed of the Archimedean axiom and the axiom of completeness (closure) saying that the model of the axiomatic system is categorical, in the sense that the maximal closure of the models is unique..)

Now we go through to check Husserl's modified understanding of Hilbert's axiomatic system of arithmetic.

Axioms of Continuity The Group IV (the continuity) is composed of the Archimedean axiom (IV-1) and the axiom of completeness (IV-2). The axiom of completeness says the that the only maximally extended model (unique up to isomorphism) of the models of axiomatic system of (I)-(IV-1) is the model of the whole system (I)-(IV).[11]

Although this completeness axiom is placed to intend to give the unique determination of semantical model-structure by mean of a syntactic axioms,, the expression of the completeness

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[10] Husserl's analysis on Hilbert's axioms of arithmetic appeared only in Schumann-Schumman's edition [Schuhmann and Schuhmann 2001] of the Double Lecture manuscript, not in the original edition[*Hua XII*]. It is plausible that this part (Husserl's critical modification of Hilbert's axiomatic system) was written by Husserl only after his first lecture of the Double-Lecture after the discussion with Hilbert, where Hilbert was among the audience. Hilbert had already published "on the number concept" in 1900 [Hilbert 1900] in which he presented formal axiomatic system for arithmetic with his notion of completion.

[11] Hilbert expresses categoricity of the model of its own axiomatic system. Hilbert expressed the axiom of completeness as follows in [Hilbert 1900] [Ewald 1996], He wrote:

> It is not possible to add to the system of numbers another system of things so that the axioms I, II, III and IV-1 [namely, all the axioms except this completeness axiom itself] are also all satisfied in the combined system; in short, the numbers form a system of things which is incapable of being extended while continuing to satisfy all the axioms.

Here, a "system of things" means a "model" in the contemporary logical sense.

axiom itself refers to the semantic notion. Hence, it cannot be understood in the framework of the contemporary syntax-semantics distinction. Husserl proposed to stay on the syntactic side to express axioms. This is the starting point of Husserl's way of considering his notion of completeness.

Hence, Husserl, of course, abandoned the axioms Group IV, i.e., axioms of continuity. He explained before Hilbert in the audience that the Hilbertian completeness axiom excludes possibility of extending axiomatic system with imaginaries because the completeness axiom of Hilbert requires maximality (non-extendability) of the model,, (hence excludes the question of imaginaries itself under this axiomatic setting.[12]

Axioms of Linking The linking axioms (Group I) in both Hilbert's and Husserl's formation state that the primitive operation symbols (function-symbols) carry out a linking among the terms on the term formations (generations). Namely, when "+" ("×", respectively) is used with terms, say s and t, a new term $s + t$ ($s \times t$) is formed: + links the two terms s and t to the new composed terms $s + t$, and $s \times t$. Although Husserl's presentation of the axiomatic system of arithmetic (he presented, as an example, system of rational number) was surprisingly similar to Hilbert's there is an important difference between them on this Linking Axioms. For Hilbert the new linkage $s+t$ (or *stimest*) gives a "determinant" number. On the other hand, Husserl was concerned with possibility of indeterminacy with concrete presence of calculation axioms,. In fact, he was concerned with non-confluent rewrite calculations and non-terminating rewrite calculation. For him, determinacy of terms needs to be characterized by means of the term rewrite structure imposed by the Calculation Axioms, which is the basis of his notion of multiplicity.[13]

In the case of Husserl, for a term, say t (in Husserl's terminology, operational complexity), "t" is called "provably existent" when t is reduced to be a constructor-based normal term, say n, with the help of the axioms of calculation. Husserl's completeness and definiteness of multiplicity means, as we shall see in next Section, that this linking edges for the term-formations are most compactly, hence minimally determined, in accordance with the minimal term-model, while Hilbert's completeness or closure axiom means that the linkages are fixed in the maximally expanded way in the sense of the categorical model.[14]

Axioms of Ordering This part is the same as that of Hilbert.[15]

Axioms of Calculation Now, Group II, in which an outstanding difference can be found, as Husserl needs to claim that an axiomatic system forms the proof structure of definite multiplicity, which requires at least a ground convergent term rewrite proof structure (as we see in the next Section more closely). for which he understand that under the setting of the calculation axioms any closed term should have rewriting deductive steps i.e., to a unique

[12] This explanation was put just before the main part of his completeness proof of arithmetic in the Double Lecture manuscript.

[13] We use the English words "junction" and "linking" interchangeably for the translation of "Verknüpfung." On the other hand, when Husserl describes the term-rewrite based-reduction structure of a multiplicity he uses the word "term" (Glied).

[14] The axioms of linking in Hilbert include not only the term formation definition but also the characterization of idempotent-functions and converse-functions, as he consider fields. Husserl employed this Hilbert line to define constructors, $0, 1$ following Hilbert, for his notion of constructive multiplicity. see below..

[15] Group III, the ordering axioms of the ordered field, is exactly the same for both Husserl and Hilbert although Husserl writes down precisely only a few examples of Hilbert's full axioms.

normal term, (a unique term representing a rational number as he uses the system of rational numbers as an example at the Double Lecture). It is particularly interesting to see that four pages earlier in the (Schuhmann-Schuhmann 2001) edition of the same manuscript, Husserl tried to formulate the axiomatic systems and to start with an Hilbertian axiom of calculation (Group II); there he put first the commutativity axiom for "+" (i.e., $a + b = b + a$),[16] which is one of the six axioms of Hilbert's Group II (Hilbert's six axioms are the commutativity, associativity and distributivity for the two primitive function symbols $+$ and $\times$.), in the later part of the Double Lecture manuscript, Husserl did not mention this commutativity rule to discuss Group II axioms. It means, in the author's opinion, that Husserl intended to change the form of Calculation Axioms (Group II) to make the rewrite rules oriented, by changing the algebraic rules to the rewrite rules. He presented a reductive rewrite evaluations in the 6th Logical Investigation in 1900 (Section 60) , (in his phenomenological terminology, "significative fulfillment"). (In the Double Lecture and other manuscripts in 1901 which we mentioned rarely used the phenomenological vocabulary but still he used the words "fulfillment" and "adequation".)

He did not present concrete list of rewrite rules as the calculation axioms, but his way was to present a general term rewrite theory in the sense that what kind of condition the rewrite rules should satisfy in order to the axiomatic system convergent, hence complete.[17]

Although it is not very clear what are the exact form of new calculation rules of Husserl, in any case, it is very clear that Husserl stepped out of Hilbert's setting of calculation rules here and realized the need for completely different axiomatizations of equational calculation rules to govern calculation of each operation (function) represented by a function symbol, which shows that each one-step move from one joint to another on the joint-web (tissue) structure of multiplicity, which can be performed by means of equational deduction (namely, a particular application of an algebraic general axiom(s) of calculation needs to correspond to the underlying one step ground-term rewriting).

We might need too point out here that the explicit primitive recursive calculation axioms were presented only more than 20 years later by Skolem. In particular he also presented (in an informal way) the mathematical induction scheme in his Primitive Recursive Arithmetic (PRA).

3 Husserl's definite multiplicity as the convergent term rewriting proofs web

Now we have reached the stage to discuss Husserl's notion of definite multiplicity in the Double Lecture and other important and matured manuscripts in 1901 and to explain how the notion is directly related to term rewrite theory.

We first explain the notion of multiplicity (manifold) of an axiomatic system in Husserl's sense. Husserl also uses the word "domain of an axiomatic system" to express a multiplicity. Husserl's multiplicity has been interpreted by many philosophers and logicians for over 40 years that this domain-multiplicity means a set theoretical domain, namely a model in the sense of model theory.[18] We show now that this is not the case by our reading and that by a multiplicity he means the whole network (web or tissue) of rewrite equational proofs and term formations.[19]

[16] [Schuhmann and Schuhmann 2001, p.113]
[17] In fact, Husserl allowed to introduce any number of function symbols in Group I (Linkage Axiom), hence it is natural to presume that he consider calculation axioms to calculate those function symbols.
[18] Some other views may be found in [Hartimo 2010]
[19] In fact, enlarging an original axiomatic system by adding imaginary propositions as new axioms on the

(A) Multiplicity Husserl's multiplicity of an axiomatic system is understood as the following whole web (or tissue) or graph structure (relation-web or relation-linkings):

(a) the nodes represented by terms of the system, and the edges represented by following relations between terms;
(b) term formation steps following the linking axioms,
(c) rewrite (i.e., oriented equational) proof steps following the calculation axioms.
(d) provably decidable atomic relations of normal (irreducible) terms in the rewrite proof sense.

A multiplicity is said to be definite if the web or graph is tightly determined. In particular he requires convergence of the rewrite edges.

(B) Definite Multiplicity A multiplicity is called definite (in the strong sense) when the underlying term rewrite system is convergent. (confluent and terminating).

There is no word "convergence" nor other words which are used in term rewrite theory, but by definiteness he meant the direction-oriented equational deductions confluent and terminating.

He often confirms that an arithmetical multiplicity is ground convergent (namely, convergent on the ground (closed) terms level, and left open the convergence on the variables level, and he believes that the calculation axioms could set so that the convergence on the ground level holds. In this paper we distinguish the definite only on the ground convergence case from the general case.

(B') Definite Multiplicity in the weak sense A multiplicity is called definite in the weak sense when the underlying term rewrite system is ground convergent. (confluent and terminating on the closed terms level).

Husserl also defines constructiveness of the definite multiplicity, where the constructor terms are pre-given, namely formed by linking axioms based on given constructors.

(C) Constructive Multiplicity When the term formation steps are based with constructors, and the calculation axioms preserve the constructor terms as (at least a part of) the normal (irreducible terms), the whole web (or tissue) is "constructive" multiplicity[20].

He focuses the notion of constructive multiplicity especially in the Double Lecture. He presumes that the constructor terms are irreducible in the sense that they are normal terms. The termination property comes with this setting. He calls the pre-given distinguished set of intended normal terms as the number-series. He follows Frege and Hilbert regarding this naming. He, however, also calls the number-series as the "standard" or "measure" . He gives his dynamic term-rewriting view that the measure plays the role of measuring any term in the multiplicity-web (as its value) under the definiteness condition. He imposes a completion procedure on a non-confluent constructive multiplicity by adding new direction-oriented calculation axioms upon necessity so that the resulting multiplicity becomes "definite"

one hand and by enlarging a domain of the original axiomatic system by adding imaginary elements are not equivalent when one assumes the "domain" (or "multiplicity") in the sense of a set theoretical domain and this difficulty is discussed by Husserl himself. Our reading of the domain (multiplicity) as the whole proofs and terms web makes sense and works well by understanding with term rewrite theory in our opinion.

[20] He also uses the word "mathematical multiplicity".

constructive multiplicity; we shall discuss this (Knuth-Bendix type) completion procedure of Husserl later. Husserl claims that

(Claim γ) Rewrite-provability in the sense of direct rewrite proof [without the use of symmetric axiom] is logically equivalent to logical equational provability of equational proof system [with the use of symmetric axiom] in the case that the multiplicity is definite.

This lemma is of course an essential lemma well known in nowadays term rewrite theory. But, to be honest, we should point out that Husserl does not (or could not) treat the termination property directly and explicitly. He rather claims that for a constructive multiplicity case the pre-given set of constructor terms plays as the irreducible terms, and any calculation axioms needs to preserve the irreducibility of the constructor terms (although he allows possibility of reduction paths not falling into the pre-given constructor terms. Therefore, he needs to consider a completion procedure.

Husserl's solution to the problem of justification of the use of imaginaries is expressed as follows.

(Claim δ [Husserl's Solution with the definiteness-completeness condition] If the multiplicity of the original system is definite and the multiplicity of the enlarged system preserves the normal forms of the original, the problem (conservation problem) is positively solved.

Here, we resume to classify the types of multiplicity for an axiomatic formal (deductive) system,according to Husserl:

(i) a (not necessarily definite) multiplicity
(ii) a definite multiplicity
(iii) a constructive (or mathematical) (not necessarily definite) multiplicity
(iv) a constructive definite multiplicity

About Completion from non-definite into definite multiplicity As mentioned above, he also introduces a completion procedure to make the calculation axiom convergent (hence the multiplicity definite) by adding calculation rules for a disjoint (critical) pair. He seems that he was too optimistic about the termination property with respect to this completion procedure as he mainly considers the case of constructive multiplicities where the intended normal terms are pre-given and that the new rules between critical pair can be directed into a constructor normal term side. He call a "disjunctive" moves for the pair of terms which cause non-confluence (hence goes to different irreducible terms.[21] Hence, he claims as follows.

(Claim η) By adding rules for disjoint moves from a position non-convergent constructive multiplicity becomes definite (in a finite steps).

Hierarchical theory of multiplicities, or a part of Mathesis Universalis Husserl often requires definiteness to the multiplicity of enlarged system too, in addition to the preservation of the original normal terms. This is because he considers hierarchical extensions freely by enlarging systems step by step. The accumulated whole is called theory of multiplicities or theory of theories.[22]

[21] See [Dershowitz and Jouannaud 1990] for the basic definition of critical pair.
[22] His concrete example of hierarchical theory of multiplicities includes positive number system up to the complex number system as well as logic and geometrical systems), as he presented them concretely in the Double Lecture and related manuscripts.

This was the first (and last) concrete presentation of Husserl's idea of Mathesis Universalis, which he emphasized to aim at establishment in his logical investigations , at the end of his Vol. 1 of the Logical Investigations (Prolegomena), where he also mentioned the problem of justification of the use of imaginaries as an important but had not yet solved the problem.

Husserl describes a multiplicity as the whole *relations-web (or relations-tissue)* of an axiomatic system.[23] "Should a system of axioms define its objects by a *web of relations* (or the form of such a web)" further only by means of materialization, [which means substitutions], "every object must be unambiguously [uniquely] determined by its interrelations [i.e.,by the relation-web]". Then, he continues:

> *Any object is formally the simple position in the relation-web*, i.e., in the relation-form where the objects can be situated, and the form of relation must be so well established that it must be so well formally differentiated in an ultimate manner. If it leaves here indeterminations, it would then again possible to go further in the formal characterization of the *relation-web (tissue)*.[24]

This is a remarkable comment of Husserl on the term rewrite proofs; he says that any (mathematical object of) term in the equational proof system is considered a "simple position" in the whole rewrite-relation-web structure, namely multiplicity, where the objects are situated.[25]

Our "Figure" illustrates an image of a constructive multiplicity web for a simple primitive recursive (hence non-overlapping) calculation axioms just for "+" and "×" where there is the "standard' (or sometimes called "Kernel") ', which is the series of normal terms of numbers[26].

It is definite on the ground level as well as the variable rewrite level, the logical level requires non-equational rules for the ground level though (see below). the multiplicity web of the natural number system is open to expand to various enlarged systems(of integers, of rationals, of computable reals, etc.) with preserving conservation on the ground level.[27]

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[23] E.g., [*Hua XII*, p.474, 475] (Husserl uses the word "joints (or terms) [Glieder]" instead of "web" in the double-lecture to express it.

[24] [*Hua XII*, p.475]. The author had no chance to carefully look at the English edition of this volume translated by Prof. Dallas Willard during the preparation of this paper. The author plans to consult and quote Prof. Willard's English edition for the preparation of his forthcoming paper on this subject. The author just points out here that the English edition uses the word "network of relations" in stead of "web of relations", which is also very suitable and coherent with our reading.

[25] This is partly a result of the influence from Hilbert's formalistic holism standpoint on the Foundations of Geometry (1999), from which Husserl learned that the geometrical primitives, such and point or line, are not defined separately but should be meaningful in the relation to the whole axiomatic system as the whole. See [Okada 2004]

[26] Husserl describes the term rewrite structure with standard as follows, for example.

> The numbers are the *standards of operation* [Operetionsetalons] in an defined operation-domain; these are the joints of a complete whole totality neither augmentable, nor diminishable, of unique and pairwisely different [untereinander nicht äquivalenten] operation-character, which are the lowest specific differences in this sphere of operation and which have the property that any real operation of its domain must have its provably [nachweislich] equivalent in an characterization of this whole. [*Hua XII*, p.475].

[27] On the Figure there are inequality edges between normal constructor terms. This corresponds to a proof of $\neg s = t$ in HusseOnrl's sense, which is in the convergent term rewrite sense; he emphasizes that "$\neg s = t$ is provable" is not in the logical provability sense, but in the sense of convergent rewriting. Namely, for two terms s and t $\neg s = t$ is provable in a (constructive) definite multiplicity when the normal terms of them are different (otherwise equal). Since the completeness in the sense of Husserl is based on the minimal term model, this means $\neg s = t$. We recall that he mainly the considers constructive multiplicity

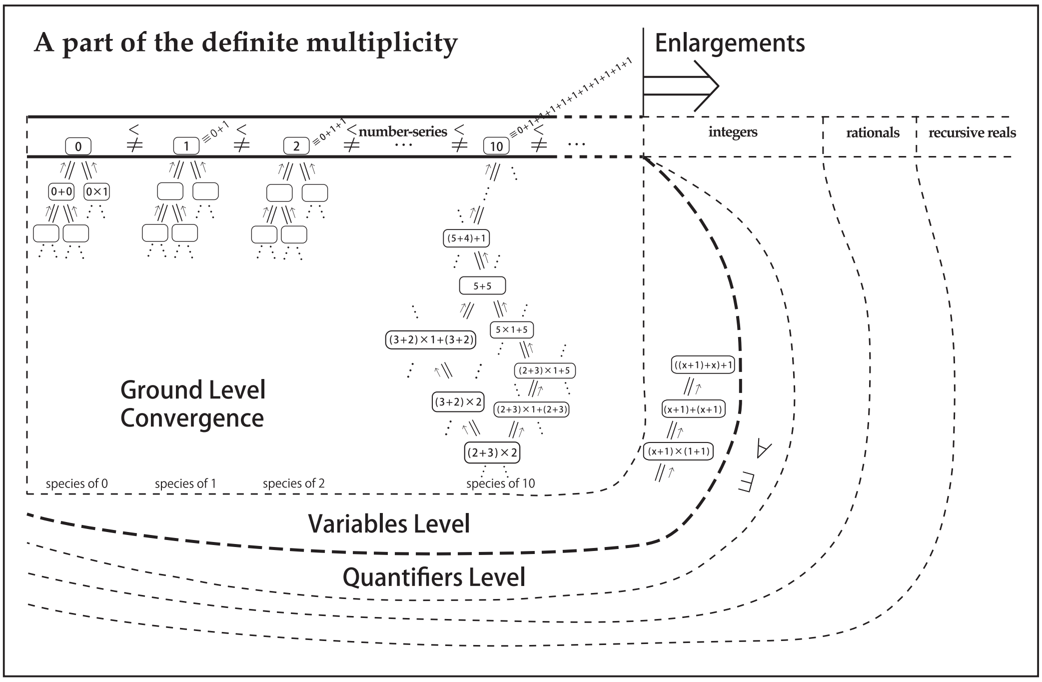

Figure 1 The above gives the reader a rough picture of Husserlian constructive definite multiplicity of the axiomatic system including axioms:

- $\mathsf{x} + 0 = \mathsf{x}$,
- $\mathsf{x} + (\mathsf{y} + 1) = (\mathsf{x} + \mathsf{y}) + 1$,
- $\mathsf{x} \times 1 = \mathsf{x}$,
- $\mathsf{x} \times (\mathsf{y} + 1) = (\mathsf{x} \times \mathsf{y}) + \mathsf{x}$

Husserl believed that his completeness/definiteness works not only for the theories of natural numbers, of integers, of rationals, but also for the theories of real numbers and of complex numbers in the uniform way (although as we now know that it does not work with real number theory as the the decision of the primitive numerical equality-relation is not decidable anymore. Hence, it needs to be limited to the rewrite-computable or recursive reals (or in the case of an abstract real closed field system) if one wishes to defend his computational view of universal arithmetic.)

More About Completion Procedure

Husserl explains on the non-definite, namely non-convergent case that some suitable equational axioms should be added so that the multiplicity becomes convergent, in his sense that the

case as the ideal definite multiplicities. The different views about the linking axioms between Hilbert and Husserl becomes more clear when we understand. In the Hilbert sense, any term (composed by the linking axioms) exists, and the linking axioms are understood as existential axioms. This is because, $t = t$ implies $\exists x(t = x)$. On the other hand, for Husserl, term t composed by linking axiom only has an intentional meaning, or representation of concept of t. The existence of t is shown only when it reduced to a constructer term (namely, there is at least a rewriting path from t to a constructer term in the web structure of the constructive multiplicity web. See Section 2 above.

axiomatic system complete. He says that he addition should be done so that the "result" is the same, in other words, the equational provability is equivalent to the rewrite proof system, which means completion in the sense of Knuth-Bendix.

He explains the completion in a most detailed and reasonable way with respect to a constructive multiplicity, where the constructor-based normal forms are "pre-given" with the axiomatic setting (of Linking Axioms).

> [I]f any relation between them [two terms in a multiplicity] were ambiguous or undetermined [which means that the two terms go to different irreducible terms], I could then add the axioms which would introduce the determination; any undetermined relation should be, on the basis of the axioms, transformable into a determined relation. [*Hua XII*, p.497]

He tries to clarify this further and says:

> If there are two ways of determination which gives the same result [we read this that if there are two ways to reach different irreducible terms s and t from a term u], which shows both s and t are provably the same as u], we could then fix them arbitrarily [namely, one could add a rewrite rule from one irreducible term to the other.], then it should have an [additional] axiom which unites them. [*Hua XII*, p.498]

This is understood as a so called "Knuth-Bendix completion procedure" . The procedure was introduced in the 1960s from the computer scientific context by Knuth and Bendix [Knuth and Bendix 1970] and which was presented as a general procedure by Huet-Oppen in the 1980s Cf. [Dershowitz and Jouannaud 1990] for the general historical information and basic notions on the term rewrite theory in theoretical computational science, although Husserl's setting was the case of terminating (constructive) rewrite systems, [28]

Husserl's completion procedure is to add new equational axioms with a direction to make the underlying non-confluent term-rewriting system confluent, while the provability power of the axiomatic system unchanged; namely the additions of new axioms are redundant in the sense of logical provability, but necessary in the sense of computation.[29]

About the variables level

As mentioned in the previous Section, the equivalence between the rewrite-provability and the equational provability becomes delicate with the non-ground term rewrite case because one usually needs to add some additional non-rewrite deductive principle or inference rule in addition to the purely equational calculation axioms in order to deduce an algebraic equational proposition with variables . For example, $x + y = y + x$ is "true" in the sense of a (standard) model of arithmetic, but one usually needs the mathematical induction

[28] There were similar procedure defined in mathematics, especially in algebra-related fields independently, but the procedure appeared in mathematics had been very much dependent on a rather specific field, and even Knuth-Bendix' presentation was the word problem oriented. So, Huet-Oppen[Huet and Oppen 1980] seems a very first to propose it as a procedure on a term rewrite-oriented equational system in general. But, we would lime to point out the idea of the procedure appeared in Husserl (1901) for his theory of term rewriting even yet conceptually, in the context of philosophy of logic and mathematics.l

[29] When he explains the (completion) procedure in a most detail way Husserl presumes that the underlying axiomatic system has the constructors, hence a multiplicity has the constructor terms as irreducible normal forms. But a term in the multiplicity might property guarantees the uniqueness property of the irreducible terms, which is of a specific importance for Husserl as (the concept of) each irreducible term serves as a definite "specific difference" by the constructor based normal-irreducible terms

rule to prove it. Note that both $x = y$ and $y = x$ are irreducible terms in a simple (non-overlap) convergent rewrite system such as the primitive recursive rule system in the example above. the convergence does not correspond to the truth of the standard model on the standard/measure. Husserl were very much aware of the delicate issue of the variable level of definite multiplicity. Husserl confirms ground convergence of arithmetical systems mentioned in the previous Section. And he also tells convergence on the variable rewrite level of the system (without commutativity). But he also asked himself the precise reason why it is. In fact he asks himself in footnote 39 (at the very last footnote in the Double Lecture manuscript, of the Sschumann-Schumann edition) saying "Why?" which was added as the footnote for a passage where he explains the ground convergence of arithmetical system (of rationals). [Schuhmann and Schuhmann 2001].

We add a small remark about the further development of equational proof system of arithmetic. This line of systematic study of proof system for arithmetic began only more than 20 years later than Husserl by Skolem's Primitive Recursive Arithmetic[30]. But, Skolem's (rather informal) principle of mathematical induction required non-equational logical inference (implication or conditional) in addition to the purely equational language. (One could express it in terms of the natural-deduction style inference rule, Induction Rule, as below, with the non-local but global inference rule. See the Induction Rule in the footnote below.) It was a philosopher, Wittgenstein, who first reformulated the mathematical induction rule in an equational way, which is now called the Uniqueness Rule, without using logic (implication).[31] Wittgenstein read and studied Skolem carefully and proposed his equational Uniqueness Rule without logic, as an alternative representation of mathematical Induction in the 1920's. The hypothetical appearance of a proposition at the induction step is reduced to equational Uniqueness (Inference) Rule. It is by this Rule form of Induction that the algebraic rules, for example, $x + y = y + x$ with variables x and y, is equationally provable without logic, Wittgenstein's philosophy student, Goodstein, who was a constructivist mathematician, took the Wittgenstein's Uniqueness Rule as the basis of his Recursive Number Theory. He gave the equivalence proof between the Induction Rule and the Uniqueness Rule under the presence of logic (with implication-conditional) although Wittgenstein took it for granted in his philosophical discussions.[32] Goodstein also had a version of purely equational representation (Goodstein Induction) of the Induction Rule (see the footnote below). However, it requires additional axioms of positive minus $x \dot{-} x = 0$ (for natural numbers) and the absolute value function, which are , however,not direction oriented rewrite axioms (although of course equational). The spirit of this Goodstein's equational axiomatization of arithmetic was oriented by the line of Wittgenstein's formulation (i.e., reduction of logical implication into equational calculus). It was Lambek and some others much later who developed equational type systems with rewrite rules of Mal'cev operator (which plays the role of logical implication by rewrite rules). With See [Okada 1999] for the rewrite theoretic discussions of Goodstein and Lambek Uniqueness Rule, where it is exposed that one way of direction-oriented Lambek

[30] Also, in 1931, Gödel's incompleteness appeared and told that there are true universal proposition on the variables level which is not provable in any axiomatic consistent arithmetic. This means that even if it has a definite multiplicity in the strong sense, hence convergent on the variable level, there is always a "true" equality which cannot be reached by going down through in the definite multiplicity web.

[31] Note that Husserl and Wittgenstein are known as two of the most philosophers in the Western world in the 20th Century. It is interesting to see that both studied and researched deeply equational theories.

[32] Skolem's Primitive recursive Arithmetic was taken as the basis of their Finitist System by Hilbert-Bernays after the appearance of Gödel's Incompleteness Theorem. Then, Gödel extended it to higher types in his interpretation (Dialectica Interpretation) of Gentzen's consistency proof.

Uniqueness Rule makes the whole rewrite system confluent and the opposite way of orientation makes it terminating, on the variables levels of type theories. This fact tells that the type theories with uniqueness still have a semi-definite multiplicity (i.e., the computational content) in the sense of Husserl even on the variable-higher type levels, although they are never convergent, namely never truly definite, as the definiteness with the uniqueness rule conflicts with the incompleteness theorem of Gödel. [Okada 2004] [Marion and Okada 2012] for the philosophical discussion on the equational proofs on the variables level of Wittgenstein and Goodstein.[33]

Conclusion

We presented that the Husserl proposed, in 1901, his own view on the notion of completeness by modifying Hilbert's axiomatic system of arithmetic (1900). He gave a sufficient condition for solving the problem of justifying the use of imaginaries in mathematics, which can be understood as the conservation condition in the modern logical sense, The condition was given with two stages. On the first stage he gave the condition by the use of the notions of syntactic completeness and consistency. Then, on the second (higher) stage of his research and presentation, he explained the notion of completeness more precisely. He claimed that an axiomatic system is complete (in his sense) if and only if the multiplicity (manifold) of the system is "definite". The notion of multiplicity and that of definiteness of multiplicity are the key notions to understand the whole picture of Husserl's theory and his solution to the problem. In this paper, we clarified what is these key notions. We claimed that these notions are coherently understood by means of general term rewrite theory. In particular, a multiplicity is understood as relational-web (or tissue) where term rewrite proof-steps (moves) both of the closed terms level and general terms level are the basic part of the multiplicity-web. The definiteness corresponds to convergence of this part. He considered a constructors-based definite multiplicity an ideal definite multiplecity. This tells that Husserl's view of axiomatic (arithmeticcal) systems was very much oriented by the computational view and his view was very much advanced in terms of computation theory, which were developed much later in modern logic and theoretical computer science. Husserl also introduced completion procedure of the underlying rewrite system. He gave general conditions to enlarge a rewrite system with preserving conservation. We think that Husserl's notion of multiplicity and completeness successfully extract the rewrite based-computational content from a given

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1. The induction rule :

$$(Induction)\quad \frac{f(x,0) = g(x,0) \quad \begin{array}{c}[f(x,y) = g(x,y)] \\ \vdots \\ f(x,Sy) = g(x,Sy)\end{array}}{f(x,y) = g(x,y)}$$

2. Uniqueness rule:

$$(Uniquness)\quad \frac{f(x,0) = g(x,0) \quad f(x,Sy) = h(x,y,f(x,y)) \quad g(x,Sy) = h(x,y,g(x,y))}{f(x,y) = g(x,y)}$$

3. Goodstein Induction:

$$(Goodstein\ Ind.)\quad \frac{f(x,0) = 0 \quad (1 \dot{-} fxy)\cdot fx(Sy) = 0}{f(x,y) = 0}$$

arithmetic axiom system. We characterized Hilbert's notion of completeness (of 1900) as the maximally expanded categorical model, while Husserl's notion of completeness as the minimal term model although Husserl's notion of multiplicity is not just a model but more like a type theoretic-proof theoretic structure (even limited to the first order terms), where proof formation steps and term formation steps are the basic parts of the multiplicity web. We also discussed potentials and limitation of Husserl's line of the research paradigm from the equational arithmetical point of view, in the domains of philosophy of mathematics and of theory of term rewriting.

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