One-variable first-order linear temporal logics with counting

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Abstract

First-order temporal logics are notorious for their bad computational behaviour. It is known that even the two-variable monadic fragment is highly undecidable over various timelines. However, following the introduction of the monodic formulas (where temporal operators can be applied only to subformulas with at most one free variable), there has been a renewed interest in understanding extensions of the one-variable fragment and identifying those that are decidable. Here we analyse the one-variable fragment of temporal logic extended with counting (to two), interpreted in models with constant, decreasing, and expanding first-order domains. We show that over most classes of linear orders these logics are (sometimes highly) undecidable, even without constant and function symbols, and with the sole temporal operator 'eventually'. A more general result says that the bimodal logic of commuting linear and pseudo-equivalence relations is undecidable. The proofs are by reductions of various counter machine problems.

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1 Introduction and results

Though first-order temporal logics are natural and expressive languages for querying temporal databases [3, 4] and reasoning about knowledge that changes in time [15], their practical use has been discouraged by their high computational complexity. It is well-known that even the two-variable monadic fragment is undecidable over various timelines, and Π^1_1-hard over the natural numbers [27, 28, 19, 6, 7]. In contrast to classical first-order logic where the decision problems of its fragments were studied in great detail [2], there have been only a few attempts on finding well-behaved decidable fragments of first-order temporal logics, mostly by restricting the available quantifier patterns [4, 14, 15].

In this paper we contribute to this research line by studying the one-variable 'future' fragment of linear temporal logic with counting (to two), interpreted in models over various timelines, and having constant, decreasing, or expanding first-order domains. Our language $\mathrm{FOLTL}^{\neq}$ is strong enough to express uniqueness of a property of domain elements ($\exists^{=1}x$), and the 'elsewhere' quantifier ($\forall^{\neq}x$). However, $\mathrm{FOLTL}^{\neq}$ has no equality, no constant or function symbols, and its only temporal operators are 'eventually' and 'always in the future'. $\mathrm{FOLTL}^{\neq}$ is weaker than the two-variable monadic *monodic* fragment with equality, where temporal operators can be applied only to subformulas with at most one free variable. (This fragment with the 'next time' operator is known to be Π^1_1-hard over the natural numbers [31, 5].) $\mathrm{FOLTL}^{\neq}$ is connected to bimodal product logics [8, 7], and to the temporalisation of the expressive description logic $\mathcal{CQ}$ with one global universal role [30]. Here are some examples of $\mathrm{FOLTL}^{\neq}$-formulas:

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- [4] "*An order can only be submitted once:*" $\forall x\,\Box_F\big(\mathsf{Subm}(x)\to\Box_F\neg\mathsf{Subm}(x)\big)$.
- The Barcan formula: $\exists x\,\Diamond_F\mathsf{P}(x)\leftrightarrow\Diamond_F\exists x\,\mathsf{P}(x)$.
- "*Every day has its unique dog:*" $\Box_F\exists^{=1}x\,\big(\mathsf{Dog}(x)\land\Box_F\neg\mathsf{Dog}(x)\big)$.
- "*It's only me who is always unlucky:*" $\forall^{\neq}x\,\Diamond_F\mathsf{Lucky}(x)$.

While the addition of 'elsewhere' quantifiers to the two-variable fragment of classical first-order logic does not increase the coNEXPTIME complexity of its validity problem [12, 21], we show that adding the same feature to the (decidable) one-variable fragment of first-order temporal logic results in (sometimes highly) undecidable logics over most timelines, not only in models with constant domains, but even those with decreasing and expanding first-order domains. Our main results on the FOLTL$^{\neq}$-validity problem are summarised in the following table:

	$\langle\omega,<\rangle$	all finite linear orders	all linear orders	all modally discrete linear orders
constant domains	Π^1_1-hard Theorem 1	Π^0_1-hard Theorem 2	undecidable, r.e. Theorems 5, 10	Π^1_1-hard Theorem 8
decreasing domains	Π^0_1-hard Theorem 2	Π^0_1-hard Theorem 2	undecidable, r.e. Theorems 7, 10	Π^1_1-hard Theorem 8
expanding domains	undecidable r.e.? Theorem 4	decidable Ackermann-hard Theorems 11, 3	decidable? r.e. Theorem 10	decidable? r.e.?

The structure of the paper is as follows. In Section 2 we provide the definitions of the studied logics. In Section 3 we introduce counter machines, and show several ways of simulating their behaviour in FOLTL$^{\neq}$. In Sections 4–6, with the help of the techniques developed in Section 3, we reduce various counter machine problems to FOLTL$^{\neq}$-satisfiability over different timelines and first-order domain settings. In Section 7 we discuss the connection between FOLTL$^{\neq}$ and propositional bimodal logics, and give a general undecidability result (Theorem 12). Finally, in Section 8 we formulate some open problems. Complete proofs will appear in the full paper.

2 Definitions and basic properties

We define FOLTL$^{\neq}$-*formulas* by the following grammar:

$$\phi ::\quad \mathsf{P}(x)\mid\neg\phi\mid\phi\land\psi\mid\exists x\,\phi\mid\exists^{\geq 2}x\,\phi\mid\Diamond_F\phi$$

where P ranges over an infinite set $\mathcal{P}$ of *monadic* predicate symbols. We use the usual abbreviations $\lor$, $\to$, $\leftrightarrow$, $\forall x$, $\Box_F$, and also $\Box_F^+\phi := \phi\land\Box_F\phi$.

A FOLTL-*model* is a tuple $\mathfrak{M}=\big\langle\langle T,<\rangle,D_t,I\big\rangle_{t\in T}$, where $\langle T,<\rangle$ is a linear order[1], representing the *timeline*, D_t is a non-empty set, the *domain at moment* t, for each $t\in T$, and I is a function associating with every $t\in T$ a first-order structure $I(t)=\langle D_t,\mathsf{P}^{I(t)}\rangle_{\mathsf{P}\in\mathcal{P}}$. We say that $\mathfrak{M}$ is *based on* the linear order $\langle T,<\rangle$. $\mathfrak{M}$ is a *constant* (resp. *decreasing*, *expanding*) *domain model*, if $D_t=D_{t'}$, (resp. $D_t\supseteq D_{t'}$, $D_t\subseteq D_{t'}$) whenever $t,t'\in T$ and

[1] By a linear order we mean a *strict* one. This is for simplifying the presentation only. For reflexive orders, see the 'interval trick' in Section 5.

$t < t'$. A constant domain model is clearly both a decreasing and expanding domain model as well, and can be represented as a triple $\langle\langle T, <\rangle, D, I\rangle$.

The *truth-relation* $(\mathfrak{M}, t) \models^a \phi$ (or simply $t \models^a \phi$ if $\mathfrak{M}$ is understood) is defined, for all $t \in T$ and $a \in D_t$, by induction on ϕ as follows:

- $t \models^a \mathsf{P}(x)$ iff $a \in \mathsf{P}^{I(t)}$, $t \models^a \neg\phi$ iff $t \not\models^a \phi$, $t \models^a \phi \wedge \psi$ iff $t \models^a \phi$ and $t \models^a \psi$,
- $t \models^a \exists x\, \phi$ iff there exists $b \in D_t$ with $t \models^b \phi$,
- $t \models^a \exists^{\geq 2} x\, \phi$ iff there exist $b, b' \in D_t$ with $b \neq b'$, $t \models^b \phi$ and $t \models^{b'} \phi$,
- $t \models^a \Diamond_F \phi$ iff there is $t' \in T$ such that $t' > t$, $a \in D_{t'}$ and $t' \models^a \phi$.

We say that ϕ is *true* (*satisfiable*) *in* $\mathfrak{M}$, if $t \models^a \phi$ holds for all (some) $t \in T$ and $a \in D_t$. Given a class $\mathcal{C}$ of linear orders, we say that ϕ is FOLTL$^{\neq}$-*valid in constant* (*decreasing*, *expanding*) *domain models over* $\mathcal{C}$, if ϕ is true in every constant (decreasing, expanding) domain FOLTL-model based on some linear order from $\mathcal{C}$. It is not hard to see that for every class $\mathcal{C}$ of linear orders, FOLTL$^{\neq}$-validity in decreasing (expanding) domain models over $\mathcal{C}$ is reducible to FOLTL$^{\neq}$-validity in constant domain models over $\mathcal{C}$.

We introduce the following abbreviations:

$$\exists^{=1} x\, \phi :: \quad \exists x\, \phi \wedge \neg\exists^{\geq 2} x\, \phi, \qquad \exists^{\neq} x\, \phi :: \quad (\neg\phi \wedge \exists x\, \phi) \vee \exists^{\geq 2} x\, \phi, \qquad \forall^{\neq} x\, \phi :: \quad \neg\exists^{\neq} x\, \neg\phi.$$

It is straightforward to see that they have the intended semantics:

- $t \models^a \exists^{=1} x\, \phi$ iff there exists a unique $b \in D_t$ with $t \models^b \phi$,
- $t \models^a \forall^{\neq} x\, \phi$ iff $t \models^b \phi$, for every $b \in D_t$ with $b \neq a$.

Further, we could have chosen $\exists^{\neq} x$ as our primary connective instead of $\exists x$ and $\exists^{\geq 2} x$, as

$$\exists x\, \phi \leftrightarrow \phi \vee \exists^{\neq} x\, \phi \qquad \text{and} \qquad \exists^{\geq 2} x\, \phi \leftrightarrow \exists x\, (\phi \wedge \exists^{\neq} x\, \phi). \tag{1}$$

3 Encoding counter machines in FOLTL-models

A *Minsky* or *counter machine* M is described by a finite set Q of states, a set $H \subseteq Q$ of terminal states, a finite set $C = \{c_0, \ldots, c_{N-1}\}$ of counters, a finite nonempty set $I_q \subseteq Op_C \times Q$ of instructions, for each $q \in Q - H$, where each operation in Op_C is one of the following forms, for some $i < N$:

- c_i^{++} —*increment counter* c_i *by one*,
- c_i^{--} —*decrement counter* c_i *by one*,
- $c_i^{??}$ —*test whether counter* c_i *is empty*.

A *configuration* of M is a tuple $\langle q, \mathbf{c}\rangle$ with $q \in Q$ representing the current state, and an N-tuple $\mathbf{c} = \langle c_0, \ldots, c_{N-1}\rangle$ of natural numbers representing the current contents of the counters. We say that there is a (*reliable*) *step* between configurations $\sigma = \langle q, \mathbf{c}\rangle$ and $\sigma' = \langle q', \mathbf{c}'\rangle$ (written $\sigma \to \sigma'$) iff there is $i < N$ such that

- either $c'_i = c_i + 1$, $c'_j = c_j$ for $j \neq i$, $j < N$, and $\langle c_i^{++}, q'\rangle \in I_q$,
- or $c'_i = c_i - 1$, $c'_j = c_j$ for $j \neq i$, $j < N$, and $\langle c_i^{--}, q'\rangle \in I_q$,
- or $c'_i = c_i = 0$, $c'_j = c_j$ for $j < N$, and $\langle c_i^{??}, q'\rangle \in I_q$.

We write $\sigma \to_{lossy} \sigma'$ if there are configurations $\sigma^1 = \langle q, \mathbf{c}^1\rangle$ and $\sigma^2 = \langle q', \mathbf{c}^2\rangle$ such that $\sigma^1 \to \sigma^2$, $c_i \geq c_i^1$ and $c_i^2 \geq c'_i$ for every $i < N$. A sequence $\langle \sigma_n : n < B\rangle$ of configurations, with $0 < B \leq \omega$, is called a *run* (resp. *lossy run*), if $\sigma_{n-1} \to \sigma_n$ (resp. $\sigma_{n-1} \to_{lossy} \sigma_n$) holds for every $0 < n < B$.

FOLTL$^{\neq}$ does not have the 'next time' temporal operator. So in order to simulate the behaviour of counter machines in FOLTL-models, first we generate an infinite 'diagonal', using two monadic predicate symbols, N (for 'next') and S (for 'state'). The limited counting capabilities of $\mathrm{FOLTL}^{\neq}$ will be used in forcing the *uniqueness* of the diagonal. To this end, let $\mathsf{diag}_\infty^{dec}$ be the conjunction of the following formulas:

$$\mathsf{S}(x) \wedge \forall x\, \Box_F^+\big(\mathsf{S}(x) \to \exists^{\neq} x\, \mathsf{N}(x)\big),$$

$$\forall x\, \Box_F^+\big[\mathsf{N}(x) \to \big(\forall^{\neq} x\, \neg\mathsf{N}(x) \wedge \Diamond_F \mathsf{S}(x) \wedge \Box_F\Box_F\neg\mathsf{S}(x)\big)\big], \tag{2}$$

$$\forall x\, \Box_F^+\big[\mathsf{S}(x) \to \big(\forall^{\neq} x\, \neg\mathsf{S}(x) \wedge \Box_F\neg\mathsf{S}(x)\big)\big]. \tag{3}$$

The formula $\mathsf{diag}_\infty^{dec}$ forces a unique infinite diagonal not only in constant but also in decreasing domain models. A straightforward induction proves the following:

▶ **Lemma 1.** Suppose that $t_0 \models^{a_0} \mathsf{diag}_\infty^{dec}$ in some decreasing domain FOLTL-model $\big\langle \langle T,<\rangle, D_t, I\big\rangle_{t\in T}$. Then there are sequences $\langle t_n \in T : n < \omega\rangle$ and $\langle a_n \in D_{t_n} : n < \omega\rangle$ such that the following hold, for all $n<\omega$ and $a \in D_{t_n}$: if $n>0$ then t_n is the immediate $<$-successor of t_{n-1}, $t_n \models^a \mathsf{S}(x)$ iff $a = a_n$, and $t_n \models^a \mathsf{N}(x)$ iff $a = a_{n+1}$.

Observe that in Lemma 1, if $\langle T,<\rangle$ is $\langle\omega,<\rangle$ and $t_0 = 0$, then $t_n = n$ for all $n<\omega$.

Constant domain models. We begin by showing how to encode runs that start with all-0 counters by going *forward* along the created diagonal. For each counter $i < N$, we take two fresh predicate symbols C_i^+ and C_i^- that will be used to mark those domain points where M increments and decrements counter c_i at each moment of time. The actual content of counter c_i is represented by those domain points where $\mathsf{C}_i^+(x) \wedge \neg\mathsf{C}_i^-(x)$ holds. The following formula ensures that each point of our constant domain is used only once, and only previously incremented points can be decremented:

$$\bigwedge_{i<N} \forall x\, \Box_F^+\big[\big(\mathsf{C}_i^+(x) \to \Box_F \mathsf{C}_i^+(x)\big) \wedge \big(\mathsf{C}_i^-(x) \to \Box_F \mathsf{C}_i^-(x)\big) \wedge \big(\mathsf{C}_i^-(x) \to \mathsf{C}_i^+(x)\big)\big]. \tag{4}$$

For each $i<N$, the following formulas simulate the possible changes in the counters:

$$\mathsf{Fix}_i :: \ \forall x\, \big(\Box_F \mathsf{C}_i^+(x) \to \mathsf{C}_i^+(x)\big) \wedge \forall x\, \big(\Box_F \mathsf{C}_i^-(x) \to \mathsf{C}_i^-(x)\big),$$

$$\mathsf{Inc}_i :: \ \exists^{=1} x\, \big(\neg\mathsf{C}_i^+(x) \wedge \Box_F \mathsf{C}_i^+(x)\big) \wedge \forall x\, \big(\Box_F \mathsf{C}_i^-(x) \to \mathsf{C}_i^-(x)\big),$$

$$\mathsf{Dec}_i :: \ \exists^{=1} x\, \big(\mathsf{C}_i^+(x) \wedge \neg\mathsf{C}_i^-(x) \wedge \Box_F \mathsf{C}_i^-(x)\big) \wedge \forall x\, \big(\Box_F \mathsf{C}_i^+(x) \to \mathsf{C}_i^+(x)\big).$$

Using these formulas, we can encode the steps of M. For each $\iota \in Op_C$, we define the formula do_ι by taking

$$\mathsf{do}_\iota :: \begin{cases} \mathsf{Inc}_i \wedge \displaystyle\bigwedge_{i\neq j<N} \mathsf{Fix}_j, & \text{if } \iota = c_i^{++}, \\ \mathsf{Dec}_i \wedge \displaystyle\bigwedge_{i\neq j<N} \mathsf{Fix}_j, & \text{if } \iota = c_i^{--}, \\ \forall x\, \big(\mathsf{C}_i^+(x) \to \mathsf{C}_i^-(x)\big) \wedge \displaystyle\bigwedge_{j<N} \mathsf{Fix}_j, & \text{if } \iota = c_i^{??}. \end{cases}$$

Now we can encode runs that start with all-0 counters. For each $q \in Q$, we take a fresh predicate symbol S_q, and define φ_M to be the conjunction of (4) and the following formulas:

$$\bigwedge_{i<N} \forall x\, \big(\neg\mathsf{C}_i^+(x) \wedge \neg\mathsf{C}_i^-(x)\big),$$

$$\forall x\, \Box_F^+\Big[\mathsf{S}(x) \leftrightarrow \bigvee_{q\in Q}\Big(\mathsf{S}_q(x) \wedge \bigwedge_{q\neq q'\in Q} \neg\mathsf{S}_{q'}(x)\Big)\Big], \tag{5}$$

$$\forall x\, \Box_F^+ \bigwedge_{q\in Q-H}\Big[\mathsf{S}_q(x) \to \bigvee_{\langle\iota,q'\rangle\in I_q}\Big(\mathsf{do}_\iota \wedge \forall x\, \big[\mathsf{N}(x) \to \Box_F\big(\mathsf{S}(x) \to \mathsf{S}_{q'}(x)\big)\big]\Big)\Big].$$

The following lemma says that in constant domain models, going forward along the diagonal points generated in Lemma 1, we can force (finite or infinite) runs of M:

▶ **Lemma 2.** Suppose $t_0 \models^{a_0} \mathsf{diag}^{dec}_\infty \wedge \varphi_M$ in some constant domain FOLTL-model $\big\langle\langle T,<\rangle, D, I\big\rangle$. For all $n<\omega$ and $i<N$, let

$$q_n := q, \text{ if } t_n \models^{a_n} \mathsf{S}_q(x), \qquad c_i(n) := |\{a \in D : t_n \models^a \mathsf{C}_i^+(x) \wedge \neg\mathsf{C}_i^-(x)\}|.$$

Then $\big\langle\langle q_n, \mathbf{c}(n)\rangle : n < B\big\rangle$ is a well-defined run of M starting with all-0 counters, whenever $0 < B \leq \omega$ is such that $t_n \models^{a_n} \bigwedge_{h\in H} \neg\mathsf{S}_h(x)$, for every $n < B$.

Decreasing domain models. We can also encode runs that start with all-0 counters by going *backward* along the diagonal. Moreover, this way we have more control over the points representing the content of the counters, and so we can simulate runs not only in constant but also in decreasing domain models.

We take a fresh predicate symbol start, intended to mark the start of runs and being constant along each first-order domain. In decreasing domain models we can say this by

$$\forall x\, \Box_F^+\big(\mathsf{start}(x) \to \forall x\, \mathsf{start}(x)\big). \tag{6}$$

For each counter $i<N$, we take a fresh predicate symbol C_i. The actual content of counter c_i will be represented by those points where $\mathsf{C}_i(x)$ holds. We want to force these points only to be among the domain points a_n generated in Lemma 1. We can achieve this by the following formulas, simulating the possible changes in the counters:

$$\mathsf{AllC}_i(x) :: \ \Diamond_F \mathsf{N}(x) \wedge \Box_F\big(\mathsf{N}(x) \vee \Diamond_F\mathsf{N}(x) \to \mathsf{C}_i(x)\big),$$

$$\mathsf{Fix}_i^{bw} :: \ \forall x\, \big(\mathsf{C}_i(x) \leftrightarrow \mathsf{AllC}_i(x)\big), \tag{7}$$

$$\mathsf{Inc}_i^{bw} :: \ \forall x\, \big[\mathsf{C}_i(x) \leftrightarrow \big(\mathsf{N}(x) \vee \mathsf{AllC}_i(x)\big)\big], \tag{8}$$

$$\mathsf{Dec}_i^{bw} :: \ \forall x\, \big(\mathsf{C}_i(x) \to \mathsf{AllC}_i(x)\big) \wedge \exists^{=1}x\, \big(\neg\mathsf{C}_i(x) \wedge \mathsf{AllC}_i(x)\big). \tag{9}$$

Next, we encode the steps of M, going backward along the diagonal. For every $\iota \in Op_C$, we define the formula do_ι^{bw} by taking

$$\mathsf{do}_\iota^{bw} :: \begin{cases} \mathsf{Inc}_i^{bw} \wedge \bigwedge_{i\neq j<N} \mathsf{Fix}_j^{bw}, & \text{if } \iota = c_i^{++}, \\ \mathsf{Dec}_i^{bw} \wedge \bigwedge_{i\neq j<N} \mathsf{Fix}_j^{bw}, & \text{if } \iota = c_i^{--}, \\ \forall x\, \neg\mathsf{C}_i(x) \wedge \bigwedge_{j<N} \mathsf{Fix}_j^{bw}, & \text{if } \iota = c_i^{??}. \end{cases}$$

Finally, it remains to encode runs that start with all-0 counters, by going backward along the diagonal. Given a counter machine M, we define $\varphi_M^{bw\text{-}dec}$ to be the conjunction of (5), (6), and the following formulas:

$$\forall x\, \Box_F^+\big(\mathsf{S}(x) \wedge \mathsf{start}(x) \to \bigwedge_{i<N} \forall x\, \neg\mathsf{C}_i(x)\big),$$

$$\forall x\, \Box_F^+ \bigwedge_{q\in Q-H} \Big[\Big(\mathsf{S}(x) \wedge \neg\mathsf{start}(x) \wedge \exists x\, \big(\mathsf{N}(x) \wedge \Diamond_F \mathsf{S}_q(x)\big)\Big) \to \bigvee_{\langle\iota,q'\rangle\in I_q} \big(\mathsf{do}_\iota^{bw} \wedge \mathsf{S}_{q'}(x)\big)\Big].$$

The next lemma says that in decreasing domain models, starting at a 'start' point and going backward along the diagonal generated in Lemma 1, we can force *finite* runs of M:

▶ **Lemma 3.** Suppose $t_0 \models^{a_0} \mathsf{diag}^{dec}_\infty \wedge \varphi_M^{bw\text{-}dec}$ in some decreasing domain FOLTL-model $\big\langle\langle T,<\rangle, D_t, I\big\rangle_{t\in T}$. For all $n<\omega$ and $i<N$, let

$$q_n := q, \text{ if } t_n \models^{a_n} \mathsf{S}_q(x), \qquad c_i(n) := |\{a \in D_{t_n} : t_n \models^a \mathsf{C}_i(x)\}|, \qquad \sigma_n := \langle q_n, \mathbf{c}(n)\rangle. \tag{10}$$

Then $\langle \sigma_{n_2}, \sigma_{n_2-1}, \dots \sigma_{n_1} \rangle$ is a well-defined run of M starting with all-0 counters, whenever $n_1 \le n_2 < \omega$ is such that $t_{n_2} \models^{a_{n_2}} \mathsf{start}(x)$ and $t_n \models^{a_n} \neg\mathsf{start}(x) \wedge \bigwedge_{h\in H} \neg\mathsf{S}_h(x)$, for every n with $n_1 \le n < n_2$.

Expanding domain models. We can still say something about counter machine runs by going backward along the diagonal in expanding domain models. However, in this case some of the content of the counters might get lost as the runs progress, so we can force only *lossy* runs. To this end, let $\mathsf{diag}_\infty^{exp}$ and $\varphi_M^{bw\text{-}exp}$ be obtained from $\mathsf{diag}_\infty^{dec}$ and $\varphi_M^{bw\text{-}dec}$, respectively, by simultaneously replacing all occurrences of the prefix $\forall x\, \Box_F^+$ with $\Box_F^+ \forall x$. Then let φ_M^{lossy} be obtained from $\varphi_M^{bw\text{-}exp}$ by replacing all occurrences of the formulas (7)–(9) by their 'lossy versions':

$$\mathsf{Fix}_i^{bw\text{-}lossy} :: \ \forall x\, \big(\mathsf{C}_i(x) \to \mathsf{AllC}_i(x)\big),$$
$$\mathsf{Inc}_i^{bw\text{-}lossy} :: \ \forall x\, \big[\mathsf{C}_i(x) \to \big(\mathsf{N}(x) \vee \mathsf{AllC}_i(x)\big)\big],$$
$$\mathsf{Dec}_i^{bw\text{-}lossy} :: \ \forall x\, \big(\mathsf{C}_i(x) \to \mathsf{AllC}_i(x)\big) \wedge \exists x\, \big(\neg\mathsf{C}_i(x) \wedge \mathsf{AllC}_i(x)\big).$$

Then we have the expanding domain version of Lemma 1 for $\mathsf{diag}_\infty^{exp}$, and the following 'lossy analogue' of Lemma 3:

▶ **Lemma 4.** Suppose $t_0 \models^{a_0} \mathsf{diag}_\infty^{exp} \wedge \varphi_M^{lossy}$ in an expanding domain FOLTL-model $\big\langle \langle T, < \rangle, D_t, I \big\rangle_{t\in T}$. Then $\langle \sigma_{n_2}, \sigma_{n_2-1}, \dots \sigma_{n_1} \rangle$, as defined in (10), is a well-defined lossy run of M starting with all-0 counters, whenever $n_1 \le n_2 < \omega$ is such that $t_{n_2} \models^{a_{n_2}} \mathsf{start}(x)$ and $t_n \models^{a_n} \neg\mathsf{start}(x) \wedge \bigwedge_{h\in H} \neg\mathsf{S}_h(x)$, for every n with $n_1 \le n < n_2$.

Observe that in this case the counting capabilities of $\mathrm{FOLTL}^{\neq}$ are only used in forcing the uniqueness of the diagonal in the expanding domain version of Lemma 1.

4 $\mathrm{FOLTL}^{\neq}$ over $\langle \omega, < \rangle$ and finite linear orders

▶ **Theorem 1.** *$\mathrm{FOLTL}^{\neq}$-validity is Π_1^1-hard in constant domain models over $\langle \omega, < \rangle$.*

We prove this theorem by reducing the following Σ_1^1-complete [1] problem to $\mathrm{FOLTL}^{\neq}$-satisfiability in constant domain models over $\langle \omega, < \rangle$:

CM recurrence:

Given a counter machine M and two states q_0, q_r, is there a run starting from $\langle q_0, \mathbf{0} \rangle$ and visiting q_r infinitely often?

The following claim is a straightforward consequence of Lemma 2:

▶ **Claim 1.1.** Suppose $\mathsf{diag}_\infty^{dec} \wedge \varphi_M \wedge \mathsf{S}_{q_0}(x) \wedge \forall x\, \Box_F^+ \bigwedge_{h\in H} \neg\mathsf{S}_h(x)$ is satisfiable in some constant domain FOLTL-model. Then M has an infinite run starting from $\langle q_0, \mathbf{0} \rangle$.

Now it clearly follows from Claim 1.1 that if

$$\mathsf{diag}_\infty^{dec} \wedge \varphi_M \wedge \mathsf{S}_{q_0}(x) \wedge \forall x\, \Box_F^+ \bigwedge_{h\in H} \neg\mathsf{S}_h(x) \wedge \Box_F \Diamond_F \exists x\, \mathsf{S}_{q_r}(x) \tag{11}$$

is satisfiable in some constant domain FOLTL-model based on $\langle \omega, < \rangle$, then M has a run starting from $\langle q_0, \mathbf{0} \rangle$ and visiting q_r infinitely often. (Observe that this is not necessarily true for models based on arbitrary timelines.)

On the other hand, if M has a run $\big\langle \langle q_n, \mathbf{c}(n) \rangle : n < \omega \big\rangle$ that visits q_r infinitely often and $\mathbf{c}(0) = \mathbf{0}$, then we define a constant domain FOLTL-model $\mathfrak{M}_\infty^{fw} = \big\langle \langle \omega, < \rangle, \omega, I \big\rangle$ as follows. For all $n < \omega$ and $q \in Q$, let

$$\mathsf{S}^{I(n)} := \{n\}, \quad \mathsf{N}^{I(n)} := \{n+1\} \ \text{ and } \ \mathsf{S}_q^{I(n)} := \begin{cases} \{n\}, & \text{if } q = q_n, \\ \emptyset, & \text{otherwise.} \end{cases} \tag{12}$$

Further, for all $i < N$ and $n < \omega$, we define inductively the sets $\mathsf{C}_i^{+I(n)}$ and $\mathsf{C}_i^{-I(n)}$. We let $\mathsf{C}_i^{+I(0)} = \mathsf{C}_i^{-I(0)} := \emptyset$, and then

$$\mathsf{C}_i^{+I(n+1)} := \begin{cases} \mathsf{C}_i^{+I(n)} \cup \{n\}, & \text{if } c_i(n+1) = c_i(n) + 1, \\ \mathsf{C}_i^{+I(n)}, & \text{otherwise.} \end{cases} \tag{13}$$

$$\mathsf{C}_i^{-I(n+1)} := \begin{cases} \mathsf{C}_i^{-I(n)} \cup \{min(\mathsf{C}_i^{+I(n)})\}, & \text{if } c_i(n+1) = c_i(n) - 1, \\ \mathsf{C}_i^{-I(n)}, & \text{otherwise.} \end{cases} \tag{14}$$

Then it is easy to check that $(\mathfrak{M}_\infty^{fw}, 0) \models^0$ (11), proving Theorem 1.

▶ **Theorem 2.** FOLTL$^{\neq}$*-validity is* Π_1^0*-hard*

1. *in decreasing domain models over* $\langle \omega, < \rangle$*,*
2. *in decreasing domain models over the class of all finite linear orders,*
3. *in models with finite decreasing domains over* $\langle \omega, < \rangle$*.*

We prove the theorem using a reduction of the following Σ_1^0-complete [20] problem:

CM reachability:

Given a counter machine M and two states q_0, q_r, is there a run from $\langle q_0, \mathbf{0} \rangle$ to some configuration $\langle q_r, \mathbf{c} \rangle$?

In order to prove **1**, define the formula reach^{dec} by taking

$$\mathsf{reach}^{dec} :: \quad \mathsf{S}_{q_r}(x) \wedge \forall x\, \Box_F^+ \big[\exists x\, \Diamond_F \mathsf{start}(x) \to \big(\neg \mathsf{start}(x) \wedge \forall x \bigwedge_{h \in H} \neg \mathsf{S}_h(x)\big)\big] \wedge$$
$$\forall x\, \Box_F^+ \big(\mathsf{S}(x) \wedge \mathsf{start}(x) \to \mathsf{S}_{q_0}(x)\big).$$

The following claim is a consequence of Lemma 3:

▶ **Claim 2.1.** Suppose $\mathsf{diag}_\infty^{dec} \wedge \varphi_M^{bw\text{-}dec} \wedge \mathsf{reach}^{dec} \wedge \exists x\, \big(\mathsf{start}(x) \vee \Diamond_F \mathsf{start}(x)\big)$ is satisfiable in some decreasing domain FOLTL-model based on $\langle \omega, < \rangle$. Then there is a run of M starting with $\langle q_0, \mathbf{0} \rangle$ and reaching q_r.

On the other hand, if M has a run $\big\langle \langle q_n, \mathbf{c}(n) \rangle : n \le K \big\rangle$ with $\mathbf{c}(0) = \mathbf{0}$ and $q_K = q_r$, then we define a constant domain FOLTL-model $\big\langle \langle \omega, < \rangle, \omega, I \big\rangle$ as follows. For all $n < \omega$, $q \in Q$,

$$\mathsf{S}^{I(n)} := \{n\}, \quad \mathsf{N}^{I(n)} := \{n+1\} \ \text{ and } \ \mathsf{S}_q^{I(n)} := \begin{cases} \{n\}, & \text{if } n \le K \text{ and } q = q_{K-n}, \\ \{n\}, & \text{if } n > K \text{ and } q = q_h, \\ \emptyset, & \text{otherwise,} \end{cases} \tag{15}$$

for some fixed $h \in H$. Further, for all $i < N$ and $n < \omega$, we define inductively the sets $\mathsf{C}_i^{I(n)}$. We let $\mathsf{C}_i^{I(n)} := \emptyset$ whenever $n \ge K$, and then for every $n < K$, we let

$$\mathsf{C}_i^{I(K-n-1)} := \begin{cases} \mathsf{C}_i^{I(K-n)} \cup \{K-n\}, & \text{if } c_i(n+1) = c_i(n) + 1, \\ \mathsf{C}_i^{I(K-n)} - \{min(\mathsf{C}_i^{I(K-n)})\}, & \text{if } c_i(n+1) = c_i(n) - 1, \\ \mathsf{C}_i^{I(K-n)}, & \text{otherwise.} \end{cases} \tag{16}$$

Finally, let $\mathsf{start}^{I(K)} := \omega$, and $\mathsf{start}^{I(n)} := \emptyset$ for all $n \neq K$, $n < \omega$. Then it is not hard to check that $\mathsf{diag}_\infty^{dec} \wedge \varphi_M^{bw\text{-}dec} \wedge \mathsf{reach}^{dec} \wedge \exists x\, \big(\mathsf{start}(x) \vee \Diamond_F \mathsf{start}(x)\big)$ is satisfiable in this model.

However, $\mathsf{diag}_\infty^{dec}$ is clearly not satisfiable in models based on finite timelines, or in models with finite domains. Let $\mathsf{diag}_{fin}^{dec}$ be obtained from $\mathsf{diag}_\infty^{dec}$ by replacing the conjunct (2) with

$$\forall x\, \Box_F^+ \big(\mathsf{N}(x) \to \forall^{\neq} x\, \neg \mathsf{N}(x)\big) \wedge \forall x\, \Box_F^+ \big[\mathsf{N}(x) \wedge \neg \mathsf{start}(x) \to \big(\Diamond_F \mathsf{S}(x) \wedge \Box_F \Box_F \neg \mathsf{S}(x)\big)\big].$$

Now it is easy to see that $\mathsf{diag}^{dec}_{fin} \wedge \varphi^{bw\text{-}dec}_{M} \wedge \mathsf{reach}^{dec}$ is satisfiable in a decreasing domain FOLTL-model where either its timeline or all its domains are finite iff M has a run starting with $\langle q_0, \mathbf{0}\rangle$ and reaching q_r, completing the proof of Theorem 2.

▶ **Theorem 3.** FOLTL$^{\neq}$*-validity is Ackermann-hard in expanding domain models over the class of all finite linear orders.*

The decidability (and the finite expanding domain property) of this logic follows from its reducibility to certain propositional bimodal logics, see Theorem 11 in Section 7. Here we prove the lower bound in Theorem 3 by a reduction the following problem:

LCM reachability:

Given a counter machine M, a configuration $\sigma_0 = \langle q_0, \mathbf{0}\rangle$, and a state q_r, is there a lossy run from σ_0 to some configuration $\langle q_r, \mathbf{c}\rangle$?

It is shown in [24] that this problem, without the restriction that σ_0 has all-0 counters, is Ackermann-hard. It is not hard to see that this restriction does not matter: For every M and σ_0 one can define a machine M^{σ_0} that first performs incrementation steps filling the counters up to their 'σ_0-level', and then performs M's actions. Then M has a lossy run from σ_0 reaching q_r iff M^{σ_0} has a lossy run starting with all-0 counters and reaching q_r.

Let $\mathsf{diag}^{exp}_{fin}$ and reach^{exp} obtained from $\mathsf{diag}^{dec}_{fin}$ and reach^{dec}, respectively, by replacing all occurrences of the prefix $\forall x\,\Box^+_F$ with $\Box^+_F\forall x$. The next claim is a consequence of Lemma 4:

▶ **Claim 3.1.** If $\mathsf{diag}^{exp}_{fin} \wedge \varphi^{lossy}_{M} \wedge \mathsf{reach}^{exp}$ is satisfiable in an expanding domain FOLTL-model based on a finite linear order, then M has a lossy run starting with $\langle q_0, \mathbf{0}\rangle$ and reaching q_r.

On the other hand, suppose M has a lossy run $\big\langle \langle q_n, \mathbf{c}(n)\rangle : n \le K\big\rangle$ with $\mathbf{c}(0) = \mathbf{0}$ and $q_K = q_r$. Then we can define a constant domain FOLTL-model $\big\langle \langle T, <\rangle, D, I\big\rangle$ that satisfies $\mathsf{diag}^{exp}_{fin} \wedge \varphi^{lossy}_{M} \wedge \mathsf{reach}^{exp}$ by taking $T = D = \{0, \dots, K\}$, the restriction of (15), $\mathsf{start}^{I(K)} := D$, $\mathsf{start}^{I(n)} := \emptyset$ for all $n < K$, $\mathsf{C}_i^{I(K)} := \emptyset$ for all $i < N$, and the following in place of (16), for all $i < N$ and $n < K$:

$$\mathsf{C}_i^{I(K-n-1)} := \begin{cases} \mathsf{C}_i^{I(K-n)} \cup \{K-n\}, & \text{if } c_i(n+1) = c_i(n)+1,\\ \text{any subset of } \mathsf{C}_i^{I(K-n)} \text{ of size } c_i(n+1), & \text{if } c_i(n+1) \le c_i(n). \end{cases}$$

This completes the proof of Theorem 3.

▶ **Theorem 4.** FOLTL$^{\neq}$*-validity is undecidable in expanding domain models over $\langle \omega, <\rangle$.*

We prove the theorem by a reduction of the following Π^0_1-complete problem [16, 18, 23] to the FOLTL$^{\neq}$-satisfiability problem in question:

LCM ω-reachability:

Given a counter machine M, a configuration $\sigma_0 = \langle q_0, \mathbf{0}\rangle$ and a state q_r, is it the case that for every $n < \omega$ M has a lossy run starting with σ_0 and visiting q_r at least n times?

(The idea of our reduction is similar to the one used in [16] for a formalism more expressive than FOLTL$^{\neq}$.) Take a fresh predicate symbol R, and define rec^{fw} as the conjunction of the following formulas:

$$\Box_F\Diamond_F\mathsf{start}(x) \wedge \Box_F\forall x\,\big(\mathsf{start}(x) \to \forall x\,\mathsf{start}(x)\big),$$

$$\Box_F\forall x\,\Big(\mathsf{start}(x) \to \exists x\,\big[\mathsf{R}(x) \wedge \Diamond_F\mathsf{S}(x) \wedge \Box_F\big(\Diamond_F\mathsf{S}(x) \to \neg\mathsf{start}(x)\big)\big]\Big),$$

$$\Box_F\forall x\,\big[\mathsf{R}(x) \to \Box_F\big(\mathsf{S}(x) \to \mathsf{S}_{q_r}(x)\big)\big],$$

$$\Box_F \forall x \Big[\mathsf{S}_{q_r}(x) \to \exists x \Big(\mathsf{R}(x) \wedge \Diamond_F\big(\mathsf{start}(x) \wedge \Diamond_F \mathsf{S}(x)\big) \wedge \Box_F\big[\mathsf{start}(x) \wedge \Diamond_F \mathsf{S}(x) \to \Box_F\big(\Diamond_F \mathsf{S}(x) \to \neg\mathsf{start}(x)\big)\big]\Big)\Big],$$

$$\Box_F^+ \forall x\, \big(\mathsf{R}(x) \to \Box_F \neg \mathsf{R}(x)\big).$$

▶ **Claim 4.1.** If $\mathsf{diag}_\infty^{exp} \wedge \mathsf{rec}^{fw}$ is satisfiable in some expanding domain FOLTL-model based on $\langle \omega, < \rangle$, then there is an infinite sequence $\langle k_n < \omega : n < \omega \rangle$ such that, for all $n < \omega$, $k_n \models \forall x\, \mathsf{start}(x)$, and if $n > 0$ then $|\{k : k_{n-1} < k \leq k_n$ and $k \models^{a_k} \mathsf{S}_{q_r}(x)\}| \geq n$.

(Observe that Claim 4.1 is not necessarily true for models based on arbitrary timelines.) Now the following claim is a consequence of Claim 4.1 and Lemma 4:

▶ **Claim 4.2.** Suppose that the formula

$$\mathsf{diag}_\infty^{exp} \wedge \varphi_M^{lossy} \wedge \mathsf{rec}^{fw} \wedge \Box_F \forall x\, \big(\mathsf{S}(x) \wedge \mathsf{start}(x) \to \mathsf{S}_{q_0}(x)\big) \wedge \Box_F \forall x \bigwedge_{h \in H} \neg \mathsf{S}_h(x) \tag{17}$$

is satisfiable in an expanding domain FOLTL-model based on $\langle \omega, < \rangle$. Then, for every $n < \omega$, M has a lossy run starting with $\langle q_0, \mathbf{0} \rangle$ and visiting q_r at least n times.

On the other hand, if for every $n < \omega$, M has a lossy run ρ_n of M starting with $\langle q_0, \mathbf{0} \rangle$ and visiting q_r at least n times, then (17) is satisfiable in the constant domain FOLTL-model sketched in Fig. 1, completing the proof of Theorem 4.

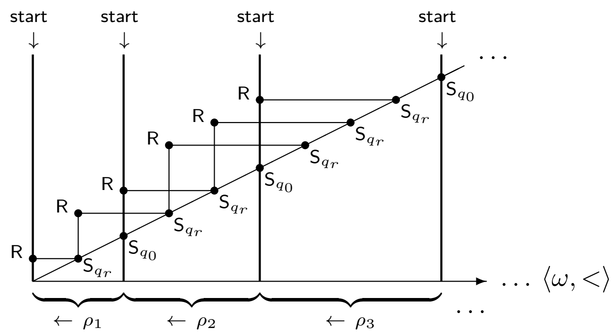

Figure 1 Sketch of a constant domain FOLTL-model based on $\langle \omega, < \rangle$ satisfying the formula (17).

5 FOLTL$^{\neq}$ over arbitrary linear orders

As FOLTL$^{\neq}$-validity in constant (expanding, decreasing) domain models over the class of all linear orders is recursively enumerable (see Theorem 10), we cannot expect 'CM reachability' or 'CM recurrence' to be reduced to its satisfiability problem. However, in the constant domain case at least, we can still reduce the following undecidable [20] problem:

CM non-termination:

Given a counter machine M and a state q_0, is there an infinite run starting from $\langle q_0, \mathbf{0} \rangle$?

Then Claim 1.1 and a FOLTL-model defined as in (12)–(14) give us the following:

▶ **Theorem 5.** *FOLTL$^{\neq}$-validity is undecidable in constant domain models over the class of all linear orders.*

Observe that a formula of the form $\Diamond_F \mathsf{S}(x) \wedge \Box_F \Box_F \neg \mathsf{S}(x)$ is clearly not satisfiable in any FOLTL-model based on a dense linear order, and so our formulas generating diagonals are not satisfiable in such a model either. However, below we show that the formulas used in Sections 3 and 4 can be modified to prove the following generalisation of Theorem 5:

▶ **Theorem 6.** FOLTL$^{\neq}$*-validity is undecidable in constant domain models over any class of linear orders containing a linear order that has an infinite ascending chain.*

We use a version of the 'interval trick' suggested in [22, 26, 9]. Take a fresh predicate symbol Tick and define a new temporal operator $\blacklozenge_F$ and its dual $\blacksquare_F$ by setting, for any FOLTL$^{\neq}$-formula $\psi(x)$,

$$\blacklozenge_F\psi(x) :: \ \big[\mathsf{Tick}(x) \to \Diamond_F\big(\neg\mathsf{Tick}(x) \land (\psi(x) \lor \Diamond_F\psi(x))\big)\big] \land \big[\neg\mathsf{Tick}(x) \to \Diamond_F\big(\mathsf{Tick}(x) \land (\psi(x) \lor \Diamond_F\psi(x))\big)\big].$$

In order to properly simulate 'next time', we need the following property of $\mathsf{Tick}(x)$:

$$\big(\exists x\,\mathsf{Tick}(x) \leftrightarrow \forall x\,\mathsf{Tick}(x)\big) \land \Box_F\big(\exists x\,\mathsf{Tick}(x) \leftrightarrow \forall x\,\mathsf{Tick}(x)\big). \qquad (18)$$

Suppose that $r \models (18)$ in some constant domain FOLTL-model $\big\langle\langle T,<\rangle, D, I\big\rangle$. We define a new relation $\prec$ on $T_r = \{t \in T : r < t\}$ by taking, for all $t, t' \in T_r$,

$$t \prec t' \quad \text{iff} \quad \exists z\,\big[t < z \le t' \text{ and, for all } a \in D,\ \big(t \models^a \mathsf{Tick}(x) \ \leftrightarrow\ z \models^a \neg\mathsf{Tick}(x)\big)\big].$$

It is straightforward to check that for all $t \in T_r$ and $a \in D$, $t \models^a \blacklozenge_F\psi$ iff there is $t' \in T_r$ with $t \prec t'$ and $t' \models^a \psi$. Also, $\prec$ is transitive and asymmetric, but $\langle T_r, \prec\rangle$ is not necessarily a linear order. Instead of trichotomy, we only have that either $t \prec t'$ or $t' \prec t$ or $t \sim t'$ hold, where $\sim$ is the following equivalence relation on T_r: $t \sim t'$ iff for all z with $min(t,t') \le z \le max(t,t')$ and all $a \in D$, $\big(z \models^a \mathsf{Tick}(x) \ \leftrightarrow\ min(t,t') \models^a \mathsf{Tick}(x)\big)$. We would like our predicates to be constant in any $\sim$-class. To achieve this, for a predicate symbol P, let $\mathsf{interval_P}$ denote conjunction of (18) and the following formulas:

$$\forall x\,\Box_F\big(\mathsf{P}(x) \to \blacksquare_F\neg\mathsf{P}(x)\big),$$
$$\forall x\,\Box_F\big(\Diamond_F\mathsf{P}(x) \land \blacksquare_F\neg\mathsf{P}(x) \to \mathsf{P}(x)\big),$$
$$\forall x\,\Box_F\big(\mathsf{P}(x) \land \neg\blacklozenge_F\top(x) \to \Box_F\mathsf{P}(x)\big),$$
$$\forall x\,\Box_F\big(\mathsf{P}(x) \land \blacklozenge_F\top(x) \to \blacklozenge_F\mathsf{P}_{next}(x)\big),$$
$$\forall x\,\Box_F\big(\mathsf{P}(x) \to \Box_F(\blacklozenge_F\mathsf{P}_{next}(x) \to \mathsf{P}(x))\big),$$

where P_{next} is a fresh predicate symbol, and $\top(x)$ is a shorthand for $\mathsf{P}(x) \lor \neg\mathsf{P}(x)$.

▶ **Claim 6.1.** Suppose that $r \models \mathsf{interval_P}$, and take $t, t' \in T_r$ with $t < t'$ and $t \sim t'$. Then, for all $a \in D$, $t \models^a \mathsf{P}(x)$ iff $t' \models^a \mathsf{P}(x)$.

Now we have the following generalisation of Claim 1.1:

▶ **Claim 6.2.** Let ϕ_M be obtained from $\mathsf{diag}_\infty^{dec} \land \varphi_M \land \mathsf{S}_{q_0}(x) \land \forall x\,\Box_F^+\bigwedge_{h\in H}\neg\mathsf{S}_h(x)$ by replacing each occurrence of $\Diamond_F$ and $\Box_F$ with $\blacklozenge_F$ and $\blacksquare_F$, and adding the conjuncts $\mathsf{interval_P}$ for each occurring predicate symbol P. If $t_0 \models^{a_0} \phi_M$ in some constant domain FOLTL-model based on a linear order having an infinite ascending chain starting at t_0, then M has an infinite run starting from $\langle q_0, \mathbf{0}\rangle$.

For the other direction, suppose that M has an infinite run starting from $\langle q_0, \mathbf{0}\rangle$. Let $\langle T,<\rangle$ be a linear order in our class containing an infinite ascending chain $t_0 < \cdots < t_n < \ldots$. Take the constant domain FOLTL-model $\mathfrak{M}_\infty^{fw}$ defined in (12)–(14). We define a constant domain model $\mathfrak{N}_\infty^{fw} = \big\langle\langle T,<\rangle, \omega, J\big\rangle$ by taking, for all $t \in T$ and $\mathsf{P} \in \{\mathsf{N}, \mathsf{S}, \mathsf{C}_i^+, \mathsf{C}_i^-, \mathsf{S}_q\}_{i<N,\, q\in Q}$,

$$\mathsf{Tick}^{J(t)} = \begin{cases} \omega, & \text{if } t_{n+1} < t \le t_n,\ n \text{ is even,} \\ \emptyset, & \text{otherwise,} \end{cases} \qquad \mathsf{P}^{J(t)} = \begin{cases} \mathsf{P}^{I(n)}, & \text{if } t_{n+1} < t \le t_n, \\ \emptyset, & \text{otherwise,} \end{cases}$$

$$\mathsf{P}_{next}^{J(t)} = \begin{cases} \mathsf{P}^{I(n)}, & \text{if either } n > 0 \text{ and } t_n < t \le t_{n-1}, \text{ or } n = 0 \text{ and } t > t_0, \\ \emptyset, & \text{otherwise.} \end{cases}$$

Then $(\mathfrak{N}_\infty^{fw}, t_0) \models^0 \phi_M$, completing the proof of Theorem 6.

6 $\mathrm{FOLTL}^{\neq}$ over timelines with infinite descending chains

We say that a linear order $\langle T,<\rangle$ *has a rooted infinite descending chain* if there exists $\langle t_n \in T : n \leq \omega\rangle$ with $t_\omega < \cdots < t_n < \cdots < t_0$. In this section we show that, in FOLTL-models based on such linear orders, we can also simulate counter machine runs along a diagonal that is *generated* backward. Let $\mathsf{diag}_\infty^{bw\text{-}dec}$ be the conjunction of (2)–(3) and the following formulas:

$$\Diamond_F\big(\mathsf{S}(x) \wedge \mathsf{start}(x) \wedge \forall x\, \Box_F \neg \mathsf{S}(x)\big),$$
$$\forall x\, \Diamond_F \mathsf{N}(x),$$
$$\forall x\, \Box_F\big(\mathsf{N}(x) \to \exists x\, \mathsf{S}(x)\big),$$

and recall the formula $\varphi_M^{bw\text{-}dec}$ from Section 3. Then we have the following analogues of Lemmas 1 and 3:

▶ **Lemma 5.** Suppose that $r \models^{\alpha_0} \mathsf{diag}_\infty^{bw\text{-}dec}$ in some decreasing domain FOLTL-model $\big\langle \langle T,<\rangle, D_t, I\big\rangle_{t\in T}$. Then there are sequences $\langle \tau_n \in T : n < \omega\rangle$ and $\langle \alpha_n \in D_{\tau_n} : n < \omega\rangle$ such that $\tau_0 \models^{\alpha_0} \mathsf{start}(x)$, $t \models^{a} \neg\mathsf{S}(x)$ for all $t > \tau_0$ and $a \in D_t$, and the following hold, for all $n < \omega$ and $a \in D_{\tau_n}$: $r < \tau_n$, if $n > 0$ then τ_{n-1} is the immediate $<$-successor of τ_n, $\tau_n \models^a \mathsf{S}(x)$ iff $a = \alpha_n$, and $t_n \models^a \mathsf{N}(x)$ iff $n > 0$ and $a = \alpha_{n-1}$.

▶ **Lemma 6.** Suppose $r \models^{\alpha_0} \mathsf{diag}_\infty^{bw\text{-}dec} \wedge \varphi_M^{bw\text{-}dec}$ in a decreasing domain FOLTL-model $\big\langle \langle T,<\rangle, D_t, I\big\rangle_{t\in T}$. For all $i < N$ and all $n < \omega$, let

$$q_n := q, \text{ if } \tau_n \models^{\alpha_n} \mathsf{S}_q(x), \qquad c_i(n) := |\{a \in D_{\tau_n} : \tau_n \models^a \mathsf{C}_i(x)\}|.$$

Then $\big\langle \langle q_n, \mathbf{c}(n)\rangle : n < B\big\rangle$ is a well-defined run of M starting with all-0 counters, whenever $B \leq \omega$ is such that $\tau_n \models^{\alpha_n} \neg\mathsf{start}(x) \wedge \bigwedge_{h\in H} \neg\mathsf{S}_h(x)$, for every $n < B$.

Using these lemmas, first we prove the following generalisation of Theorem 5:

▶ **Theorem 7.** $\mathrm{FOLTL}^{\neq}$*-validity is undecidable in decreasing domain models over the class of all linear orders.*

We reduce the 'CM non-termination' problem to $\mathrm{FOLTL}^{\neq}$-satisfiability in the above class of models. On the one hand, Lemma 6 implies the 'backward' analogue of Claim 1.1:

▶ **Claim 7.1.** Suppose χ_M is satisfiable in a decreasing domain FOLTL-model, where

$$\chi_M :: \quad \mathsf{diag}_\infty^{bw\text{-}dec} \wedge \varphi_M^{bw\text{-}dec} \wedge \forall x\, \Box_F\big(\mathsf{S}(x) \wedge \mathsf{start}(x) \to \mathsf{S}_{q_0}(x)\big) \wedge \forall x\, \Box_F^+ \bigwedge_{h\in H} \neg\mathsf{S}_h(x)$$
$$\wedge\, \forall x\, \Box_F\big(\mathsf{start}(x) \to \Box_F \neg\mathsf{start}(x)\big). \quad (19)$$

Then M has an infinite run starting from $\langle q_0, \mathbf{0}\rangle$.

On the other hand, if M has an infinite run $\big\langle \langle q_n, \mathbf{c}(n)\rangle : n < \omega\big\rangle$ with $\mathbf{c}(0) = \mathbf{0}$, then we define a constant domain FOLTL-model $\mathfrak{N}_\infty^{bw} = \big\langle \langle \omega+1, >\rangle, \omega, I\big\rangle$ as follows. Let $\mathsf{P}^{I(\omega)} = \emptyset$, for all $\mathsf{P} \in \{\mathsf{N}, \mathsf{S}, \mathsf{C}_i, \mathsf{S}_q\}_{i<N,\, q\in Q}$, and for all $n < \omega$ and $q \in Q$, let

$$\mathsf{S}^{I(n)} := \{n\}, \quad \mathsf{N}^{I(n)} := \begin{cases} \{n-1\}, & \text{if } n > 0,\\ \emptyset, & \text{if } n = 0,\end{cases} \quad \text{and } \mathsf{S}_q^{I(n)} := \begin{cases} \{n\}, & \text{if } q_n = q,\\ \emptyset, & \text{otherwise.}\end{cases} \quad (20)$$

Further, for all $i < N$, $n < \omega$, we define inductively the sets $\mathsf{C}_i^{I(n)}$. We let $\mathsf{C}_i^{I(0)} := \emptyset$, and

$$\mathsf{C}_i^{I(n+1)} := \begin{cases} \mathsf{C}_i^{I(n)} \cup \{n\}, & \text{if } c_i(n+1) = c_i(n)+1,\\ \mathsf{C}_i^{I(n)} - \{min(\mathsf{C}_i^{I(n)})\}, & \text{if } c_i(n+1) = c_i(n)-1,\\ \mathsf{C}_i^{I(n)}, & \text{otherwise.}\end{cases} \quad (21)$$

Finally, let $\mathsf{start}^{I(0)} = \omega$, and $\mathsf{start}^{I(n)} = \emptyset$ for all $0 < n \leq \omega$. Then it is easy to check that $(\mathfrak{N}_\infty^{bw}, \omega) \models^0$ (19), completing the proof of Theorem 7.

The proof of our next theorem uses the same counter machine problem, 'CM recurrence', as the proof of Theorem 1. We call a linear order *modally discrete* if there is no infinite sequence $\langle t_n \in T : n \leq \omega \rangle$ with $t_0 < t_1 < \cdots < t_n < \cdots < t_\omega$. Note that as the only temporal operators of FOLTL$^{\neq}$ are $\Diamond_F$ and $\Box_F$, 'full' discreteness (that is, having no two points with infinitely many points in between) is not FOLTL$^{\neq}$-expressible, but modal discreteness is (see e.g. [11]). Special cases of modally discrete linear orders are *Noetherian orders* (no ascending chains of points) and arbitrary models of the '$\Diamond_F\Box_F$-theory' of $\langle \omega, < \rangle$.

▶ **Theorem 8.** *FOLTL$^{\neq}$-validity is Π_1^1-hard in decreasing domain models over any class of modally discrete linear orders containing a linear order that has a rooted infinite descending chain.*

We define the formula rec^{bw} as the conjunction of the following formulas:

$$\forall x\, \Box_F \big(\mathsf{S}(x) \to \exists x\, \mathsf{R}(x)\big),$$
$$\forall x\, \Box_F \big(\mathsf{R}(x) \to \Box_F \neg \mathsf{S}(x)\big),$$
$$\forall x\, \big(\Diamond_F \mathsf{R}(x) \to \Box_F (\mathsf{S}(x) \to \mathsf{S}_{q_r}(x))\big),$$
$$\forall^{\neq} x\, \Box_F \big(\mathsf{S}(x) \to \exists x\, \mathsf{N}(x)\big).$$

In the following claim we use the diagonal generated backward in Lemma 5:

▶ **Claim 8.1.** Suppose that $r \models^{\alpha_0} \mathsf{diag}_\infty^{bw\text{-}dec} \wedge \mathsf{rec}^{bw}$ in some decreasing domain FOLTL-model based on a modally discrete linear order. Then there are infinitely many n such that $\tau_n \models^{\alpha_n} \mathsf{S}_{q_r}(x)$.

So by Claims 7.1 and 8.1, if (19) $\wedge\, \mathsf{rec}^{bw}$ is satisfiable in some decreasing domain FOLTL-model based on a modally discrete linear order, then M has a run starting from $\langle q_0, \mathbf{0} \rangle$ and visiting q_r infinitely often.

On the other hand, suppose that M has run $\big\langle \langle q_n, \mathbf{c}(n) \rangle : n < \omega \big\rangle$ such that $\mathbf{c}(0) = 0$ and $q_{k_n} = q_r$ for an infinite sequence $\langle k_n : n < \omega \rangle$. Let $\langle T, < \rangle$ be a modally discrete linear order in our class that has a rooted infinite descending chain $\tau_\omega < \cdots < \tau_n < \cdots < \tau_0$. We may assume that τ_n is the immediate $<$-successor of τ_{n+1}, for all $n < \omega$. Take the constant domain FOLTL-model $\mathfrak{N}_\infty^{bw}$ defined in (20)–(21). We define a constant domain model $\mathfrak{N} = \big\langle \langle T, < \rangle, \omega, J \big\rangle$ by taking, for all $t \in T$, $\mathsf{P} \in \{\mathsf{N}, \mathsf{S}, \mathsf{C}_i, \mathsf{start}, \mathsf{S}_q\}_{i<N,\, q \in Q}$,

$$\mathsf{P}^{J(t)} = \begin{cases} \mathsf{P}^{I(n)}, & \text{if } t = \tau_n,\ n < \omega, \\ \emptyset, & \text{otherwise,} \end{cases} \qquad \mathsf{R}^{J(t)} = \begin{cases} \{k_n\}, & \text{if } t = \tau_n,\ n < \omega, \\ \emptyset, & \text{otherwise.} \end{cases}$$

Then $(\mathfrak{N}, \tau_\omega) \models^0 (19) \wedge \mathsf{rec}^{bw}$, completing the proof of Theorem 8.

7 FOLTL$^{\neq}$ and propositional bimodal logics

There is a well-known connection between finite variable fragments of first-order temporal logics and propositional multimodal logics where the first-order quantifiers are simulated by **S5**-modalities [7]. Here we describe this connection for the version of FOLTL$^{\neq}$ that has $\exists^{\neq} x$ as its sole quantifier (see (1)).

Bimodal formulas are defined by the following grammar:

$$\varphi :: \quad \mathsf{P} \mid \neg\varphi \mid \varphi \wedge \psi \mid \Diamond_0 \varphi \mid \Diamond_1 \varphi$$

where (with a slight abuse of notation) P ranges over an infinite set $\mathcal{P}$ of propositional variables. Then clearly there is a bijection * from FOLTL$^{\neq}$-formulas to bimodal formulas, mapping each $\mathsf{P}(x)$ to P, $\Diamond_F\phi$ to $\Diamond_0\phi^\star$, $\exists^{\neq}x\,\phi$ to $\Diamond_1\phi^\star$, and commuting with the Booleans.

Bimodal formulas are evaluated in *models* $\mathfrak{M} = \langle W, R_0, R_1, \nu\rangle$, where R_0, R_1 are binary relations over a nonempty set W, and ν is function from $\mathcal{P}$ to the subsets of W. We say that such an $\mathfrak{M}$ is a model *over* the bi-relational structure $\langle W, R_0, R_1\rangle$. For any model $\mathfrak{M}$, $w \in W$, and formula φ, we define the *truth-relation* $\mathfrak{M}, w \models \varphi$ (or just $w \models \varphi$ is $\mathfrak{M}$ is understood) by induction on φ:

- $w \models \mathsf{P}$ iff $w \in \nu(\mathsf{P})$, $w \models \neg\varphi$ iff $w \not\models \varphi$, $w \models \varphi \wedge \psi$ iff $w \models \varphi$ and $w \models \psi$,
- $w \models \Diamond_i\varphi$ iff there is $v \in W$ such that wR_iv and $v \models \varphi$, for $i = 0, 1$.

We say that φ is *true in a model* $\mathfrak{M}$, whenever $\mathfrak{M}, w \models \varphi$ holds for all $w \in W$. If for some set L of bimodal formulas, φ is true in $\mathfrak{M}$ for every φ in L, then we say that $\mathfrak{M}$ is a *model of L*. Given a class $\mathcal{C}$ of models, the *logic of* $\mathcal{C}$, denoted by $\mathsf{Log}\,\mathcal{C}$, is the set of all bimodal formulas that are true in each model from $\mathcal{C}$.

Every FOLTL-model $\mathfrak{M} = \big\langle \langle T, <\rangle, D_t, I\big\rangle_{t\in T}$ can be transformed to a modal model $\mathfrak{M}^\star = \langle W, R_0, R_1, \nu\rangle$, where $W = \{\langle t, a\rangle : t \in T,\, a \in D_t\}$, $\langle t, a\rangle R_0 \langle t', a'\rangle$ iff $t < t'$ and $a = a'$, $\langle t, a\rangle R_1 \langle t', a'\rangle$ iff $a \neq a'$ and $t = t'$, $\nu(\mathsf{P}) = \{\langle t, a\rangle : t \models^a \mathsf{P}(x)\}$. Such an R_0 is always transitive and weakly connected[2], and R_1 is a pseudo-equivalence relation[3]. So $\mathfrak{M}^\star$ is a model of the *fusion* (or *independent join*) $\mathbf{K4.3} \oplus \mathbf{Diff}$ of the unimodal logics $\mathbf{K4.3}$ (the logic of all models over transitive and weakly connected relations) and $\mathbf{Diff}$ (the logic of all models over pseudo-equivalence relations [25]). Using the methods of [17], it can be shown that in fact $\mathbf{K4.3} \oplus \mathbf{Diff} = \mathsf{Log}\{\mathfrak{M}^\star : \mathfrak{M}$ is a FOLTL-model$\}$, and so for any FOLTL$^{\neq}$-formula ϕ, ϕ is FOLTL$^{\neq}$-valid in arbitrary domain FOLTL-models over the class of all linear orders iff $\phi^\star$ belongs to the bimodal logic $\mathbf{K4.3} \oplus \mathbf{Diff}$. Therefore, the following theorem follows from the results of Wolter [29] and Spaan [26] on fusions:

▶ **Theorem 9.** FOLTL$^{\neq}$*-validity is decidable and* PSPACE*-hard in arbitrary domain models over the class of all linear orders.*

If $\mathfrak{M}$ is a (decreasing, expanding) constant domain FOLTL-model, then $\mathfrak{M}^\star$ is what is called in the literature [8, 7, 10, 17] a (*decreasing*, *expanding*) *product model.* So it is not hard to see that $\mathsf{Log}\{\mathfrak{M}^\star : \mathfrak{M}$ is a constant domain FOLTL-model$\}$ coincides with the *product logic* $\mathbf{K4.3}\times\mathbf{Diff}$, and so by Theorem 5 this bimodal logic is undecidable. Also, by similar results on modal product logics (see [8] or [7, Thm.3.17]), we obtain the following general theorem:

▶ **Theorem 10.** *If $\mathcal{C}$ is a class of linear orders that is definable by a recursive set of first-order sentences (in the language with a binary predicate and equality), then* FOLTL$^{\neq}$*-validity is recursively enumerable in constant, decreasing, or expanding domain models over $\mathcal{C}$.*

Further, Theorem 1 in [10] implies the following:

▶ **Theorem 11.** FOLTL$^{\neq}$*-validity is decidable and has the finite domain property in expanding domain models over the class of all finite linear orders.*

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[2] A relation R is called *weakly connected* if $\forall xyz\big(xRy \wedge xRz \to (y = z \vee yRz \vee zRy)\big)$.

[3] A relation R is called a *pseudo-equivalence* if it is symmetric and $\forall xyz\big(xRy \wedge yRz \to (xRz \vee x = z)\big)$.

Observe that if $\mathfrak{M}$ is a constant domain FOLTL-model, then the two relations R_0 and R_1 of $\mathfrak{M}^\star$ commute. So $\mathfrak{M}^\star$ is a model of the bimodal logic

$$[\mathbf{K4.3}, \mathbf{Diff}] := \mathsf{Log}\{\langle W, R_0, R_1, \nu\rangle : R_0 \text{ is transitive and weakly connected}, \\ R_1 \text{ is a pseudo-equivalence}, R_0 \text{ and } R_1 \text{ commute}\}.$$

However, $[\mathbf{K4.3}, \mathbf{Diff}]$ is far from being equal to $\mathbf{K4.3} \times \mathbf{Diff}$. In fact, there are infinitely many logics in between, see [13]. So the following theorem generalises Theorem 5:

▶ **Theorem 12.** *No bimodal logic between* $[\mathbf{K4.3}, \mathbf{Diff}]$ *and* $\mathbf{K4.3} \times \mathbf{Diff}$ *is decidable.*

8 Conclusion and open problems

We have shown that $\mathrm{FOLTL}^{\neq}$ is very complex over various classes of linear orders, whenever the models have constant, decreasing, or expanding domains. Several questions about expanding domain cases are left unanswered:

1) Is $\mathrm{FOLTL}^{\neq}$ decidable in expanding domain models over the class of all linear orders?
2) Is $\mathrm{FOLTL}^{\neq}$-validity recursively enumerable in expanding domain models over $\langle\omega, <\rangle$?
3) Is $\mathrm{FOLTL}^{\neq}$-validity decidable or recursively enumerable in expanding domain models over the class of all modally discrete linear orders?

By generalising our techniques to the propositional bimodal setting, we have shown that the bimodal logic of commuting linear and pseudo-equivalence relations is undecidable. Related open questions are the following:

4) Is one half of commutativity between the **K4.3** and **Diff** modalities enough to obtain undecidability?
5) Is the bimodal logic $[\mathbf{K4}, \mathbf{Diff}]$ of commuting transitive and pseudo-equivalence relations decidable? Is the product logic $\mathbf{K4} \times \mathbf{Diff}$ decidable?
6) The bimodal reformulation of 1): Is the expanding product logic $\mathbf{K4.3} \times^{exp} \mathbf{Diff}$ decidable?

In our proofs we used reductions of counter machine problems. Other lower bound results about bimodal logics with grid-like models use reductions of tiling or Turing machine problems [7, 9, 22]. On the one hand, it is not hard to re-prove the same results using counter machine reductions. On the other, it seems tiling and Turing machine techniques require more control over the $\omega \times \omega$-grid than the limited expressivity of $\mathrm{FOLTL}^{\neq}$ provides. In order to understand the boundary of each technique, it would be interesting to find tiling or Turing machine reductions for the results of the present paper.

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