# Polynomial Kernels for $\lambda$-extendible Properties Parameterized Above the Poljak-Turzík Bound

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## Abstract

Poljak and Turzík (*Discrete Mathematics* 1986) introduced the notion of $\lambda$-extendible properties of graphs as a generalization of the property of being bipartite. They showed that for any $0 < \lambda < 1$ and $\lambda$-extendible property $\Pi$, any connected graph $G$ on $n$ vertices and $m$ edges contains a spanning subgraph $H \in \Pi$ with at least $\lambda m + \frac{1-\lambda}{2}(n-1)$ edges. The property of being bipartite is $\lambda$-extendible for $\lambda = 1/2$, and so the Poljak-Turzík bound generalizes the well-known Edwards-Erdős bound for Max-Cut. Other examples of $\lambda$-extendible properties include: being an acyclic oriented graph, a balanced signed graph, or a $q$-colorable graph for some $q \in \mathbb{N}$.

Mnich et al. (*FSTTCS* 2012) defined the closely related notion of *strong* $\lambda$-extendibility. They showed that the problem of finding a subgraph satisfying a given strongly $\lambda$-extendible property $\Pi$ is fixed-parameter tractable (FPT) when parameterized above the Poljak-Turzík bound—*does there exist a spanning subgraph $H$ of a connected graph $G$ such that $H \in \Pi$ and $H$ has at least $\lambda m + \frac{1-\lambda}{2}(n-1) + k$ edges?*—subject to the condition that the problem is FPT on a certain simple class of graphs called *almost-forests of cliques*. This generalized an earlier result of Crowston et al. (*ICALP* 2012) for Max-Cut, to all strongly $\lambda$-extendible properties which satisfy the additional criterion.

In this paper we settle the kernelization complexity of nearly all problems parameterized above Poljak-Turzík bounds, in the affirmative. We show that these problems admit quadratic kernels (cubic when $\lambda = 1/2$), *without using* the assumption that the problem is FPT on almost-forests of cliques. Thus our results not only remove the technical condition of being FPT on almost-forests of cliques from previous results, but also unify and extend previously known kernelization results in this direction. Our results add to the select list of *generic* kernelization results known in the literature.

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## 1 Introduction

In parameterized complexity each problem instance $I$ comes with a parameter $k$, and a parameterized problem is said to be *fixed parameter tractable* (FPT) if for each instance $(I, k)$ the problem can be solved in time $f(k)|I|^{\mathcal{O}(1)}$ where $f$ is some computable function. The parameterized problem is said to admit a *polynomial kernel* if there is a polynomial time algorithm, called a *kernelization* algorithm, that reduces the input instance down to


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an instance with size bounded by a polynomial $p(k)$ in $k$, while preserving the answer. This reduced instance is called a $p(k)$ *kernel* for the problem. The study of kernelization is a major research frontier of Parameterized Complexity; many important recent advances in the area pertain to kernelization. These include general results showing that certain classes of parameterized problems have polynomial kernels [1, 4, 14, 13] and randomized kernelization based on matroid tools [19, 18]. The recent development of a framework for ruling out polynomial kernels under certain complexity-theoretic assumptions [3, 9, 15] has added a new dimension to the field and strengthened its connections to classical complexity. For overviews of kernelization we refer to surveys [2, 16] and to the corresponding chapters in books on Parameterized Complexity [17, 23]. In this paper we give a generic kernelization result for a class of problems parameterized above guaranteed lower bounds.

**Context and Related Work.** Many interesting graph problems are about finding a largest subgraph $H$ of the input graph $G$, where graph $H$ satisfies some specified property and its size is defined as the number of its edges. For many properties this problem is NP-hard, and for some of these we know nontrivial lower bounds for the size of $H$. In these latter cases, the apposite parameterization "by problem size" is: Given graph $G$ and parameter $k \in \mathbb{N}$, does $G$ have a subgraph $H$ which has (i) the specified property and (ii) at least $k$ *more* edges than the best known lower bound? MAX-CUT is a sterling example of such a problem. The problem asks for a largest *bipartite* subgraph $H$ of the input graph $G$; it is NP-complete [25], and the well-known *Edwards-Erdős bound* [11, 12] tells us that any connected loop-less graph on $n$ vertices and $m$ edges has a bipartite subgraph with at least $\frac{m}{2} + \frac{n-1}{4}$ edges. This lower bound is also the best possible, in the sense that it is tight for an infinite family of graphs—for example, for the set of all cliques with an odd number of vertices.

Poljak and Turzík investigated the *reason* why bipartite subgraphs satisfy the Edwards-Erdős bound, and they abstracted out a sufficient condition for *any* graph property to have such a lower bound. They defined the notion of a $\lambda$-extendible property for $0 < \lambda < 1$, and showed that for any $\lambda$-extendible property $\Pi$, any connected graph $G = (V, E)$ contains a spanning subgraph $H = (V, F) \in \Pi$ with at least $\lambda|E| + \frac{1-\lambda}{2}(|V|-1)$ edges [24]. The property of being bipartite is $\lambda$-extendible for $\lambda = 1/2$, and so the Poljak and Turzík result implies the Edwards-Erdős bound. Other examples of $\lambda$-extendible properties—with different values of $\lambda$—include $q$-colorability and acyclicity in oriented graphs.

In their pioneering paper which introduced the notion of "above-guarantee" parameterization, Mahajan and Raman [20] posed the parameterized tractability of MAX-CUT above its tight lower bound (MAX-CUT ATLB)—*Given a connected graph $G$ with $n$ vertices and $m$ edges and a parameter $k \in \mathbb{N}$, does $G$ have a bipartite subgraph with at least $\frac{m}{2} + \frac{n-1}{4} + k$ edges?*—as an open problem. This was recently resolved by Crowston et al. who showed that MAX-CUT ATLB can be solved in $2^{\mathcal{O}(k)} \cdot n^4$ time and has a kernel with $\mathcal{O}(k^5)$ vertices [7]. Following this, Mnich et al. [21] generalized the FPT result of Crowston et al. to *all* graph properties which (i) satisfy a (potentially) stronger notion which they dubbed *strong $\lambda$-extendibility*, and (ii) are FPT on a certain simple class of graphs called *almost-forests of cliques*. That is, they showed that for any strongly $\lambda$-extendible graph property $\Pi$ which satisfies the simplicity criterion, the following problem—called ABOVE POLJAK-TURZÍK ($\Pi$), or APT($\Pi$) for short—is FPT: *Given a connected graph $G$ with $n$ vertices and $m$ edges and a parameter $k \in \mathbb{N}$, does $G$ have a spanning subgraph $H \in \Pi$ with at least $\lambda m + \frac{1-\lambda}{2}(n-1) + k$ edges?* Problems which satisfy these conditions include MAX-CUT, ORIENTED MAX ACYCLIC DIGRAPH, MAX $q$-COLORABLE SUBGRAPH and, more generally, any graph property which is equivalent to having a homomorphism to a fixed vertex-transitive graph [21].

**Our Results and their Implications.** Our main result is that for almost all strongly $\lambda$-extendible properties $\Pi$ of (possibly oriented or edge-labelled) graphs, the Above Poljak-Turzík ($\Pi$) problem has kernels with $\mathcal{O}(k^2)$ or $\mathcal{O}(k^3)$ vertices. Here "almost all" includes the following: (i) *all* strongly $\lambda$-extendible properties for $\lambda \neq \frac{1}{2}$, (ii) *all* strongly $\lambda$-extendible properties which contain all orientations and labels (if applicable) of the graph $K_3$ (triangle), and (iii) all *hereditary* strongly $\lambda$-extendible properties for simple or oriented graphs. In particular, our result implies kernels with $\mathcal{O}(k^2)$ vertices for Max $q$-Colorable Subgraph and other problems defined by homomorphisms to vertex-transitive graphs.

We address both the questions left open by Mnich et al. [21], albeit in different ways. Firstly, we resolve the kernelization question for strongly $\lambda$-extendible properties, except for the special cases of non-hereditary $\frac{1}{2}$-extendible properties which do not contain some orientation or labelling of the triangle, or hereditary $\frac{1}{2}$-extendible properties which do not contain some labelling of the triangle. Note that for non-hereditary properties, we may expect to find kernelization very difficult, as a large subgraph with the property can disappear entirely if we delete even a small part of the graph. For the cases when the membership of the triangle depends on its labelling, we may expect the rules of kernelization to depend greatly on the family of labellings, and so it is difficult to produce a general result.

Secondly, we get rid of the simplicity criterion required by Mnich et al. Showing that a specific problem is FPT on almost-forests of cliques takes—in general—a non-trivial amount of work, as can be seen from the corresponding proofs for Max-Cut [6, Lemma 9], Oriented Max Acyclic Digraph, and having a homomorphism to a vertex transitive graph [22, Lemmas 27, 31]. Mnich et al. had proposed that a way to get around this problem was to find a logic which captures all problems which are FPT on almost-forests of cliques, and had left open the problem of finding the right logic. The proof of our main result shows that *all* strongly $\lambda$-extendible properties—save for the special cases —are FPT on almost-forests of cliques: in fact, that they have polynomial size kernels on this class of graphs. No special logic is required to capture these problems, and this answers their second open problem.

Formally, our main result is as follows:

▶ **Theorem 1.** *Let $0 < \lambda < 1$, and let $\Pi$ be a strongly $\lambda$-extendible property of (possibly oriented and/or labelled) graphs. Then the* Above Poljak-Turzík ($\Pi$) *problem has a kernel on $\mathcal{O}(k^2)$ vertices if conditions 0a or 0b holds, and a kernel on $\mathcal{O}(k^3)$ vertices if only 0c holds:*
**1.** $\lambda \neq \frac{1}{2}$*;*
**2.** *All orientations and labels (if applicable) of the graph $K_3$ belong to $\Pi$;*
**3.** $\Pi$ *is a hereditary property of simple or oriented graphs.*

As a corollary, we get that a number of specific problems have polynomial kernels when parameterized above their respective Poljak-Turzík bounds:

▶ **Corollary 2.** *The* Above Poljak-Turzík ($\Pi$) *parameterization of* Max $q$-Colorable Subgraph*, $q > 2$, has a kernel on $\mathcal{O}(k^2)$ vertices, and the* Above Poljak-Turzík ($\Pi$) *parameterization of* Oriented Max Acyclic Digraph *has a kernel on $\mathcal{O}(k^3)$ vertices.*

**An outline of the proof.** We now give an intuitive outline of our proof of Theorem 1. Our proof starts from a key result of Mnich et al.

▶ Proposition 1 ([21]). Let $\Pi$ be a strongly $\lambda$-extendible property and let $(G, k)$ be an instance of APT($\Pi$). Then in polynomial time, we can either decide that $(G, k)$ is a Yes-instance or find a set $S \subseteq V(G)$ such that $|S| < \frac{6k}{1-\lambda}$ and $G - S$ is a forest of cliques.

Proposition 1 is a classical WIN/WIN result, and either outputs that the given instance is a YES instance or outputs a set $S \subseteq V(G); |S| < \frac{6k}{1-\lambda}$. In the former case we return a trivial YES instance. In the latter case we know that $G - S$ is a forest of cliques and $|S| < \frac{6k}{1-\lambda}$; thus $G - S$ has a very special structure. For $\lambda \neq \frac{1}{2}$, or when all orientations or labels of the graph $K_3$ have the property, we show combinatorially that if the combined sizes of the cliques are too big then either we can get some "extra edges", or we can apply a reduction rule. We then show that the reduced instance has size polynomial in $k$. For $\lambda = \frac{1}{2}$, we need the extra technical condition that the property be hereditary, and defined only for simple or oriented graphs. In this case we can show that either the problem either contains (all orientation of) $K_3$, or is exactly Max-Cut, or that we can bound the number and sizes of the cliques. In any of these cases the problem admits a polynomial kernel.

A block of a graph $G$ is a maximal 2-connected subgraph of $G$. Note that a block $B$ of $G$ may consist of a single vertex and no edges, if that vertex is isolated in $G$.

Let $G, S$ be as described above, and let $Q$ be the set of cut vertices of $G - S$. For any block $B$ of $G-S$, let $B_{\mathsf{int}} = V(B) \setminus Q$ be the *interior* of $B$. Let $\mathcal{B}$ be the set of blocks of $G-S$. A *block neighbor* of a block $B$ is a block $B'$ such that $|V(B) \cap V(B')| = 1$. Given a sequence of blocks $B_0, B_1, \dots, B_l, B_{l+1}$ in $G - S$, the subgraph induced by $V(B_1) \cup \dots \cup V(B_l)$ is a *block path* if, for every $1 \leq i \leq l$, $V(B_i)$ contains exactly two vertices from $Q$ and $B_i$ has exactly two block neighbors $B_{i-1}$ and $B_{i+1}$. A block $B$ in $G - S$ is a *leaf block* if $V(B)$ contains exactly one vertex from $Q$. A block in $G - S$ is an *isolated block* if it has no block neighbors.

Let $\mathcal{B}_0$ and $\mathcal{B}_1$ be the set of isolated blocks and leaf blocks, respectively, contained in $\mathcal{B}$. Let $\mathcal{B}_2$ be the set of blocks $B \in \mathcal{B}$ such that $B = B_i$, $1 \leq i \leq l$, for some block path $B_0, \dots, B_{l+1}$. Finally, let $\mathcal{B}_{\geq 3} = \mathcal{B} \setminus (\mathcal{B}_0 \cup \mathcal{B}_1 \cup \mathcal{B}_2)$.

In order to bound the number of vertices in $G-S$ it is enough to bound (i) the number of blocks, and (ii) the size of each block. When $\lambda \neq \frac{1}{2}$ or the property includes all orientation and labellings of $K_3$, we show (Lemma 20) that all blocks with two or more vertices have positive excess. Using this fact, we can bound the number of vertices in blocks of $\mathcal{B}_1$ or $\mathcal{B}_2$ directly, and it remains only to bound $|\mathcal{B}_0|$. In the remaining case, we have to bound each of $|\mathcal{B}_0|, |\mathcal{B}_1|, |\mathcal{B}_2|$ and the size of each block separately. We bound these numbers over a number of lemmas.

Due to space constraints, many proofs are omitted. See [8] for a full version.

## 2 Definitions

We use "graph" to denote simple graphs without self-loops, directions, or labels, and use standard graph terminology used by Diestel [10] for the terms which we do not explicitly define. Each edge in an *oriented* graph has one of two directions $\{<, >\}$, while each edge in a *labelled* graph has an associated label $\ell \in L$ chosen from a finite set $L$. A *graph property* is a subclass of the class of all (possibly labelled and/or oriented) graphs. For a labelled and/or oriented graph $G$, we use $U(G)$ to denote the underlying simple graph; for any graph property of simple graphs, we say that $G$ has the property if $U(G)$ does: for instance, $G$ is connected if $U(G)$ is. For a (possibly labelled and/or oriented) graph $G = (V, E)$ and weight function $w : E(G) \to \mathbb{R}^+$, we use $w(F)$ to denote the sum of the weights of all the edges in $F \subseteq E$. We use $K_j$ to denote the complete simple graph on $j$ vertices for $j \in \mathbb{N}$, and $K$ to denote an arbitrary complete simple graph. For a graph property $\Pi$, we say that $K_j \in \Pi$ if $G \in \Pi$ for every (oriented, labelled) graph $G$ such that $U(G) = K_j$. A connected (possibly labelled and/or oriented) graph is a *tree of cliques* if the vertex set of each block of the graph forms a clique. We use $\mathcal{C}(G)$ to denote the set of connected components of graph $G$. A *forest of cliques* is a graph whose connected components are trees of cliques. We use $Q(G)$ to denote the set of cut vertices of graph $G$; thus $G$ is 2-connected if and only if $Q(G) = \emptyset$.

Mnich et al. [21] defined the following variant of Poljak and Turzík's notion of $\lambda$-extendibility [24].

▶ **Definition 3.** Let $\mathcal{G}$ be a class of (possibly labelled and/or oriented) graphs and let $0 < \lambda < 1$. A graph property $\Pi$ is *strongly* $\lambda$*-extendible* on $\mathcal{G}$ if it satisfies the following properties:

- Inclusiveness $\{G \in \mathcal{G} : U(G) \in \{K_1, K_2\}\} \subseteq \Pi$;
- Block additivity $G \in \mathcal{G}$ belongs to $\Pi$ if and only if every block of $G$ belongs to $\Pi$;
- Strong $\lambda$-subgraph extension Let $G \in \mathcal{G}$ and let $(U, W)$ be a partition of $V(G)$, such that $G[U] \in \Pi$ and $G[W] \in \Pi$. For any weight function $w : E(G) \to \mathbb{R}^+$ there exists an $F \subseteq E(U, W)$ with $w(F) \geq \lambda w(E(U, W))$, such that $G - (E(U, W) \setminus F) \in \Pi$.

In the rest of the paper we use $\mathcal{G}$ to denote a class of (possibly labelled and/or oriented) graphs, and $\Pi$ to denote an arbitrary—but fixed—strongly $\lambda$-extendible property defined on $\mathcal{G}$ for some $0 < \lambda < 1$. The focus of our work is the following "above-guarantee" parameterized problem:

| Above Poljak-Turzík ($\Pi$) (APT($\Pi$)) | |
|---|---|
| *Input:* | A connected graph $G = (V, E)$ and an integer $k$. |
| *Parameter:* | $k$ |
| *Question:* | Is there a spanning subgraph $H = (V, F) \in \Pi$ of $G$ such that $\lvert F\rvert \geq \lambda \lvert E\rvert + \frac{1-\lambda}{2}(\lvert V\rvert - 1) + k$? |

Let $G \in \mathcal{G}$. A $\Pi$*-subgraph* of $G$ is a subgraph of $G$ which is in $\Pi$. Let $\beta_\Pi(G)$ denote the maximum number of edges in any $\Pi$-subgraph of $G$, and let $\gamma_\Pi(G)$ denote the Poljak-Turzík bound on $G$; that is, $\gamma_\Pi(G) = \lambda|E(G)| + \frac{1-\lambda}{2}(|V(G)| - |\mathcal{C}(G)|)$. The *excess of* $\Pi$ *on* $G$, denoted $ex_\Pi(G)$, is equal to $\beta_\Pi(G) - \gamma_\Pi(G)$. We omit the subscript $\Pi$ when it is clear from the context. We use $ex(K_j)$ to denote the minimum value of $ex(G)$ for any (oriented, labelled) graph $G$ such that $K_j = U(G)$. Thus, for example, if $ex(K_3) = t$ then any graph $G$ with underlying graph $K_3$ has a $\Pi$-subgraph with at least $\gamma(G) + t$ edges, regardless of orientations or labellings on the edges of $G$. We say that a strongly $\lambda$-extendible property *diverges on cliques* if there exists $j \in \mathbb{N}$ such that $ex(K_j) > \frac{1-\lambda}{2}$. We say that a simple connected graph $\widetilde{K}$ is an *almost-clique* if there exists $V' \subseteq V(\widetilde{K})$ with $|V'| \leq 1$ (possibly $V'$ is empty) such that $\widetilde{K} - V'$ is a clique. For an almost-clique $\widetilde{K}$, we use $ex(\widetilde{K})$ to denote the minimum value of $ex(G)$ for any (oriented, labelled) graph $G$ such that $\widetilde{K} = U(G)$, and we say that $\widetilde{K} \in \Pi$ if and only if $G \in \Pi$ for every (oriented, labelled) graph $G$ with underlying graph $\widetilde{K}$.

▶ **Definition 4.** We use $AK^+_\Pi$ to denote the class of all graphs $G \in \mathcal{G}$ such that $U(G)$ is an almost-clique and $ex_\Pi(G) > 0$. For any strongly $\lambda$-extendible property which diverges on cliques, we use $\inf_{AK}$ to denote the value $\inf_{(G \in AK^+)} ex(G)$.

Note that the class $AK^+_\Pi$ contains an infinite number of graphs. Hence, it could be the case that $\inf_{AK} = 0$. In the next section, we will show that for any strongly $\lambda$-extendible property which diverges on cliques, it holds that $\inf_{AK} > 0$.

# 3 Preliminary Results

We begin with some preliminary results. The first two lemmas state how, in two special cases, the excess of a graph $G$ can be bounded in terms of the excesses of its subgraphs.

▶ **Lemma 5.** *Let $G$ be a connected (possibly labelled and/or oriented) graph and $v$ a cut vertex of $G$. Then $\gamma(G) = \sum_{X\in\mathcal{C}(G-\{v\})} \gamma(G[X \cup \{v\}])$ and $\beta(G) = \sum_{X\in\mathcal{C}(G-\{v\})} \beta(G[X \cup \{v\}])$.*

▶ **Lemma 6.** *Let $G \in \mathcal{G}$ be a connected graph and let $V(G) = V_1 \cup V_2$ such that $V_1 \cap V_2 = \emptyset$, $V_1 \neq \emptyset$, $V_2 \neq \emptyset$. Let $c_1$ be the number of components of $G[V_1]$ and $c_2$ be the number of components of $G[V_2]$. If $ex(G[V_1]) \geq k_1$ and $ex(G[V_2]) \geq k_2$, then $ex(G) \geq k_1 + k_2 - \frac{1-\lambda}{2}(c_1 + c_2 - 1)$.*

**Proof.** Observe that the size of a minimum spanning forest for $G[V_i]$ is $|V_i| - c_i$, for $i \in \{1, 2\}$, and the size of a minimum spanning tree for $G$ is $|V_1| + |V_2| - 1$. Therefore $\gamma(G) = \gamma(G[V_1]) + \gamma(G[V_2]) + \lambda|E(V_1, V_2)| + \frac{1-\lambda}{2}(c_1 + c_2 - 1)$. Let $H_i$ be a $\Pi$-subgraph of $G[V_i]$, for $i \in \{1, 2\}$. By the strong $\lambda$-subgraph extension property, there exists a $\Pi$-subgraph $H$ of $G$ such that $H = (V_1 \cup V_2, E(H_1) \cup E(H_2) \cup F)$, where $F \subseteq E(V_1, V_2)$ is such that $|F| \geq \lambda|E(V_1, V_2)|$. Therefore $\beta(G) \geq \beta(G[V_1]) + \beta(G[V_2]) + \lambda|E(V_1, V_2)|$. It follows that $ex(G) = \beta(G) - \gamma(G) \geq \beta(G[V_1]) + \beta(G[V_2]) - \gamma(G[V_1]) - \gamma(G[V_2]) - \frac{1-\lambda}{2}(c_1 + c_2 - 1) = ex(G[V_1]) + ex(G[V_2]) - \frac{1-\lambda}{2}(c_1 + c_2 - 1) \geq k_1 + k_2 - \frac{1-\lambda}{2}(c_1 + c_2 - 1)$. ◀

We now prove some useful facts about strongly $\lambda$-extendible properties which diverge on cliques. In particular, we show that for a property $\Pi$ which diverges on cliques, $ex(K_j)$ increases as $j$ increases; this motivated our choice of the name. We also show that $\inf_{AK}$ is necessarily a constant greater than 0.

▶ **Lemma 7.** *Let $ex(K_j) = a \geq \frac{1-\lambda}{2}$ for some $j \in \mathbb{N}$. Then, for every almost-clique $\widetilde{K}$ with at least $j+1$ vertices, $ex(\widetilde{K}) \geq a - \frac{1-\lambda}{2}$.*

**Proof.** Let $G \in \mathcal{G}$ be a graph such that $U(G) = \widetilde{K}$, where $\widetilde{K}$ is an almost-clique with at least $j+1$ vertices. Consider a partition of $V(G)$ into two sets $V_1$ and $V_2$ such that $U(G[V_1]) = K_j$ and $G[V_2]$ is connected. Then, by Lemma 6, $ex(G) \geq a - \frac{1-\lambda}{2}$. ◀

▶ **Lemma 8.** *Let $\Pi$ be a strongly $\lambda$-extendible property which diverges on cliques, and let $j, a$ be such that $ex(K_j) = \frac{1-\lambda}{2} + a$, $a > 0$. Then $ex(K_{rj}) \geq \frac{1-\lambda}{2} + ra$ for each $r \in \mathbb{N}$. Furthermore, $\lim_{s\to+\infty} ex(K_s) = +\infty$.*

**Proof.** We will prove the first claim by induction. The case $r = 1$ being trivial, suppose that the claim holds for $r$: we will prove it for $r + 1$. Consider a partition of $V(K_{(r+1)j})$ into two sets $U$, with $jr$ vertices, and $W$, with $j$ vertices. It holds by hypothesis that $ex(K_{(r+1)j}[W]) \geq \frac{1-\lambda}{2} + a$ and, by induction assumption, $ex(K_{(r+1)j}[U]) \geq \frac{1-\lambda}{2} + ra$. Therefore, by Lemma 6, $ex(K_{(r+1)j}) \geq \frac{1-\lambda}{2} + (r + 1)a$.

This shows that there exists a subsequence $\{s_r\}_{r\in\mathbb{N}}$ of $\{s\}_{s\in\mathbb{N}}$ such that $ex(K_{s_r})$ is a strictly increasing function of $r$. To conclude the proof, it is enough to apply Lemma 7. ◀

▶ **Lemma 9.** *Let $\Pi$ be a strongly $\lambda$-extendible property which diverges on cliques. Then $\inf_{AK} > 0$.*

**Proof.** Since $\Pi$ diverges on cliques, there exists $j$ such that $ex(K_j) \geq \frac{1-\lambda}{2} + a$, $a > 0$. Then, by Lemma 6, for every graph $G \in AK^+$ with at least $j + 1$ vertices, $ex(G) \geq a$. Therefore, $\inf_{AK} \geq \min(a, \min_{\{G\in AK^+:|V(G)|\leq j\}} ex(G)) > 0$ (note that $\{G \in AK^+ : |V(G)| \leq j\}$ is a finite set, hence the minimum of $ex(G)$ is defined and is positive). ◀

## 4 Polynomial kernel for divergence

In this section we show that APT(Π) has a polynomial kernel, as long as Π diverges on cliques and all cliques with at least two vertices have positive excess. We also give some technical lemmas which will be required in the next section.

Recall the partition $\mathcal{B}_0, \mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_{\geq 3}$ of the blocks of $G - S$. Since $|S| < \frac{6k}{1-\lambda}$, and the number of cut vertices in $G - S$ is bounded by the number of blocks in $G - S$, it is enough to prove upper bounds on $|\mathcal{B}_0|, |\mathcal{B}_1|, |\mathcal{B}_2|, |\mathcal{B}_{\geq 3}|$, and $|B_{\mathsf{int}}|$ for every block $B$ in $G - S$.

▶ **Definition 10.** Let $v$ be a cut vertex of $G$ and let $X \subseteq V(G) \setminus \{v\}$ be such that $G[X]$ is a component of $G - \{v\}$ and the underlying graph of $G[X \cup \{v\}]$ is a 2-connected almost-clique. Then we say that $G[X \cup \{v\}]$ is a *dangling component* with root $v$.

It is possible that a graph $G$ could have many dangling components, all with excess 0. This makes it possible to have large NO-instances. Therefore we require the following reduction rule, which can be run in polynomial time.

▶ Reduction Rule 1. Let $G \in \mathcal{G}$ be a connected graph with at least two 2-connected components and let $G'$ be a dangling component. Then if $ex(G') = 0$, delete $G' - \{v\}$ (where $v$ is root of $G'$) and leave $k$ the same.

▶ **Lemma 11.** *Let Π be a strongly λ-extendible property which diverges on cliques and let $G$ be a connected graph reduced under Reduction Rule 1. Then the number of dangling components is less than $\frac{k}{\inf_{AK}}$, or the instance is a YES-instance.*

▶ **Theorem 12.** *Let Π be a strongly λ-extendible property which diverges on cliques and let $G$ be a connected graph reduced under Reduction Rule 1. If there exists $s \in S$ such that $\sum_{B \in \mathcal{B}} |N_G(B_{\mathsf{int}}) \cap \{s\}|$ is at least $(\frac{16}{1-\lambda} + \frac{2}{\inf_{AK}})k - 2$, then the instance is a YES-instance.*

**Proof.** Let $U = \{s\}$. For every block $B$ of $G - S$ such that $|N_G(B_{\mathsf{int}}) \cap \{s\}| = 1$, pick a vertex in $N(s) \cap B_{\mathsf{int}}$ and add it to $U$. Since the vertices are chosen in the interior of different blocks of $G - S$, $G[U]$ is a star and therefore it is in Π by block additivity. By Lemma 5, $ex(G[U]) = \frac{1-\lambda}{2}d$, where $d$ is the degree of $s$ in $G[U]$. Let $c$ be the number of components of $G - U$, and assume that $U$ is constructed such that $d$ is maximal and $c$ is minimal. By Lemma 6, $ex(G) \geq \frac{1-\lambda}{2}(d - c)$.

We will now show that $c$ is bounded. The number of components of $G - U$ which contain a vertex of $S \setminus \{s\}$ is bounded by $(|S| - 1) < \frac{6k}{1-\lambda} - 1$. In addition, the number of components of $G - S$ which contain at least two blocks from which a vertex has been added to $U$ is at most $\frac{d}{2}$.

Now, let $T$ be a component of $G - S$ such that in the graph $G$, no vertex in $T - U$ has a neighbor in $S \setminus \{s\}$ and $|U \cap V(T)| = 1$. Firstly, suppose that $T$ contains only one block $B$ of $G - S$. Let $\{v\} = U \cap V(T)$. Note that, by the current assumptions, $N(S \setminus \{s\}) \cap V(T) \subseteq \{v\}$. If $v$ is the only neighbor of $s$ in $T$, then it is a cut vertex in $G$, hence $B$ is a dangling component of $G$. If $s$ has another neighbor $v'$ in $T$ and $v$ has no neighbor in $S$ different from $s$, then $s$ is a cut vertex, therefore $G[V(B) \cup \{s\}]$ is a dangling component. Finally, if $v$ has at least two neighbors in $S$ and $s$ has at least another neighbor $v'$ in $T$, let $U'$ be the star obtained by replacing $v$ with $v'$ in $U$, and observe that $T$ is connected to $S \setminus \{s\}$ in $G - U'$, contradicting the minimality of $c$.

Now, suppose that $T$ contains at least two blocks of $G - S$. In this case, every block except $B$ does not contain neighbors of $S$. In particular, this holds for at least one leaf block $B'$ in $T$. Hence, $B'$ is a dangling component.

This ensures that carefully choosing the vertices of $U$ we may assume that any component of $G-U$ still contains a vertex of $S \setminus \{s\}$, or contains at least two blocks from which a vertex of $U$ has been chosen, or contains part of a dangling component. Hence, the number of components of $G-U$ is at most $\frac{6k}{1-\lambda} - 1 + \frac{d}{2} + \frac{k}{\inf_{AK}}$.

Therefore, if $d \geq (\frac{16}{1-\lambda} + \frac{2}{\inf_{AK}})k - 2$, then $ex(G) \geq k$. ◀

The next corollary follows from the fact that any block in $\mathcal{B}_0$ or $\mathcal{B}_1$ which contains no vertices adjacent to $S$ is a dangling component, and the number of these must be bounded since each has a positive excess.

▶ **Corollary 13.** *Let $\Pi$ be a strongly $\lambda$-extendible property which diverges on cliques and let $G$ be a connected graph reduced under Reduction Rule 1. Then $|\mathcal{B}_0| + |\mathcal{B}_1| \leq ((\frac{16}{1-\lambda} + \frac{2}{\inf_{AK}})k - 2)\frac{6k}{1-\lambda} + \frac{k}{\inf_{AK}}$, or the instance is a* Yes*-instance.*

The next corollary follows from the fact that every component in $G-S$ must contain a vertex which is adjacent to $S$ in $G$.

▶ **Corollary 14.** *Let $\Pi$ be a strongly $\lambda$-extendible property which diverges on cliques and let $G$ be a connected graph reduced under Reduction Rule 1. Then either $G-S$ has at most $((\frac{16}{1-\lambda} + \frac{2}{\inf_{AK}})k - 2)\frac{6k}{1-\lambda} + \frac{k}{\inf_{AK}}$ components, or the instance is a* Yes*-instance.*

As $\inf_{AK} > 0$, if we have too many blocks in $G-S$ with positive excess, we have a YES-instance.

▶ **Lemma 15.** *Let $\Pi$ be a strongly $\lambda$-extendible property which diverges on cliques. If $G-S$ contains at least $((\frac{16}{1-\lambda} + \frac{2}{\inf_{AK}})k - 1)\frac{6k}{\inf_{AK}(1-\lambda)} + \frac{k}{(\inf_{AK})^2} + \frac{k-1}{\inf_{AK}}$ blocks with positive excess, then the instance is a* Yes*-instance.*

The next two lemmas are not used in the proof of Theorem 18, but are needed for the next section.

▶ **Lemma 16.** *Let $G \in \mathcal{G}$ be a connected graph and $S \subseteq V(G)$ be such that $G-S$ is a forest of cliques. Then $|\mathcal{B}_{\geq 3}| \leq 3|\mathcal{B}_1|$.*

▶ **Lemma 17.** *Let $\Pi$ be a strongly $\lambda$-extendible property which diverges on cliques, and let $j, a$ be such that $ex(K_j) = \frac{1-\lambda}{2} + a$, $a > 0$. If $|B_{int}| \geq \lceil \frac{4k}{a} + \frac{1-\lambda}{2a} \rceil j$ for any block $B$ of $G-S$, then the instance is a* Yes*-instance.*

▶ **Theorem 18.** *Let $\Pi$ be a strongly $\lambda$-extendible property which diverges on cliques, and suppose $ex(K_i) > 0$ for all $i \geq 2$. Then APT($\Pi$) has a kernel with at most $\mathcal{O}(k^2)$ vertices.*

**Proof.** Let $j \in \mathbb{N}$ be such that $ex(K_j) = \frac{1-\lambda}{2} + a$, where $a > 0$. Let $V(G-S) = V' \cup V'' \cup V''$, where $V'$ is the set of vertices contained in blocks with exactly one vertex, $V''$ is the set of vertices contained in blocks with between 2 and $j-1$ vertices and $V'''$ is the set of vertices contained in blocks with at least $j$ vertices (note that in general $V'' \cap V''' \neq \emptyset$). Observe that the blocks containing $V'$ are isolated blocks, therefore by Corollary 13 there exists a constant $c_1$ depending only on $\Pi$ such that $|V'| \leq c_1 k^2$, or the instance is a YES-instance. By Lemma 15, there exists a constant $c_2$ depending only on $\Pi$ such that $|V''| \leq c_2(j-1)k^2$, or the instance is a YES-instance.

Now, let $\mathcal{B}''$ be the set of blocks of $G-S$ which contain at least $j$ vertices. For every block $B \in \mathcal{B}''$, let $rj + l$ be the number of its vertices, where $0 \leq l < j$. Note that, by Lemma 8 and Lemma 6, $ex(B) \geq ra$. Let $d = \sum_{B \in \mathcal{B}''} r$ and let $G''$ be the union of all components of $G-S$ which contain a block in $\mathcal{B}''$. By Corollary 14, we may assume that $G''$ has at most

$((\frac{16}{1-\lambda}+\frac{2}{\inf_{AK}})k-2)\frac{6k}{1-\lambda}+\frac{k}{\inf_{AK}}$ components. Furthermore, by repeated use of Lemma 5, we get that $ex(G'') \geq da$. Observe that $G-G''$ has at most $|S| \leq \frac{6k}{1-\lambda}$ components: then, by Lemma 6, $ex(G) \geq da - \frac{1-\lambda}{2}(((\frac{16}{1-\lambda}+\frac{2}{\inf_{AK}})k-2)\frac{6k}{1-\lambda}+\frac{k}{\inf_{AK}}+\frac{6k}{1-\lambda}-1)$. Therefore if $d \geq ((\frac{16}{1-\lambda}+\frac{2}{\inf_{AK}})k-1)\frac{6k}{a(1-\lambda)}+\frac{k}{(a\inf_{AK})}+\frac{k-1}{a}$, the instance is a YES-instance. Otherwise, $|V'''| \leq 2dj \leq c_3jk^2$, where $c_3$ is a constant which depends only on $\Pi$, which means that $|V(G)| \leq \frac{6k}{1-\lambda}+(c_1+c_2(j-1)+c_3j)k^2$. ◀

## 5 Kernel when $\lambda \neq \frac{1}{2}$, $K_3 \in \Pi$ or $\Pi$ hereditary

We first show that if $\lambda \neq \frac{1}{2}$ or $K_3 \in \Pi$, then APT($\Pi$) has a kernel with $O(k^2)$ vertices. This follows from Theorem 18 together with the following lemmas.

▶ **Lemma 19.** *Let $\Pi$ be a strongly $\lambda$-extendible property. If $\lambda \neq \frac{1}{2}$ or $K_3 \in \Pi$, then $\Pi$ diverges on cliques.*

▶ **Lemma 20.** *Let $\Pi$ be a strongly $\lambda$-extendible property. If $\lambda \neq \frac{1}{2}$ or $K_3 \in \Pi$, then $ex(K_i) > 0$ for all $i \geq 2$.*

▶ **Theorem 21.** *Let $\Pi$ be a strongly $\lambda$-extendible property. If $\lambda \neq \frac{1}{2}$ or $K_3 \in \Pi$, then APT($\Pi$) has a kernel with $\mathcal{O}(k^2)$ vertices.*

We now consider the case when $\Pi$ is a hereditary property. Here we limit ourselves to properties defined on simple or oriented graphs - i.e. we assume that membership in $\Pi$ is independent of any edge-labelling.

Consider first simple graphs. We know by Theorem 21 that if $K_3 \in \Pi$ then we have a polynomial kernel. So consider the case when $K_3 \notin \Pi$.

▶ **Theorem 22.** *Let $\Pi$ be a strongly $\lambda$-extendible property with $\lambda = \frac{1}{2}$. Suppose $\Pi$ is hereditary and $G \notin \Pi$ for any $G \in \mathcal{G}$ such that $U(G) = K_3$. Then $\Pi = \{G \in \mathcal{G} : G \text{ is bipartite}\}$.*

**Proof.** First, assume for the sake of contradiction that $\Pi$ contains a non-bipartite graph $H$. Then $H$ contains an odd cycle $C_l$. By choosing $l$ as small as possible we may assume that $C_l$ is a vertex-induced subgraph of $H$. Then, since $\Pi$ is hereditary, $C_l$ is in $\Pi$. Note that if $l=3$, then $U(C_3)=K_3$, so this is not the case. Consider the graph $H'$ obtained from $C_l$ adding a new vertex $v$ and an edge from $v$ to every vertex of $C_l$. Since both $C_l$ and $K_1=\{v\}$ are in $\Pi$, by the strong $\lambda$-subgraph extension property we can find a subgraph of $H'$ which contains $C_l$, $v$ and at least half of the edges between $v$ and $C_l$. Since $l$ is odd, for any choice of $\frac{l+1}{2}$ edges there are two of them, say $e_1=vx$ and $e_2=vy$, such that the edge $xy$ is in $C_l$. Therefore, since $\Pi$ is hereditary, $H'[v,x,y] \in \Pi$, which leads to a contradiction, as $U(H'[v,x,y])=K_3$.

Now, we will show that all connected bipartite graphs are in $\Pi$, independently from any possible labelling and/or orientation. We will proceed by induction. The claim is trivially true for $j=1,2$. Assume $j \geq 3$ and that every bipartite graph with $l<j$ vertices is in $\Pi$. Consider any connected bipartite graph $H$ with $j$ vertices. Consider a vertex $v$ such that $H'=H-\{v\}$ is connected. By induction hypothesis, $H' \in \Pi$. Consider the graph $H''$ obtained from $H'$ and $G_2$, where $G_2$ is any graph in $\mathcal{G}$ with $U(G_2)=K_2$ (let $V(G_2)=\{v_1,v_2\}$), adding an edge from $v_i$ to $w \in V(H')$ if and only if in $H$ there is an edge from $v$ to $w$. Colour red the edges from $v_1$ and blue the edges from $v_2$.

Since $G_2 \in \Pi$ by inclusiveness and $H' \in \Pi$, by the strong $\lambda$-subgraph extension property there must exist a subgraph $\widetilde{H}$ of $H''$ which contains $G_2$, $H'$ and at least half of the edges between them. Note that the red edges are exactly half of the edges and that if $\widetilde{H}$ contains all of them and no blue edges, then we can conclude that $H$ is in $\Pi$ by block additivity. The same holds if $\widetilde{H}$ contains every blue edge and no red edge.

If, on the contrary, $\widetilde{H}$ contains one red and one blue edge, we will show that it contains a vertex-induced cycle of odd length, which leads to a contradiction according to the first part of the proof. First, suppose that both these edges contain $w \in V(H')$: if this happens, $\widetilde{H}$ contains a cycle of length 3 as a vertex-induced subgraph.

Now, suppose $\widetilde{H}$ contains a red edge $v_1w_1$ and a blue edge $v_2w_2$. Note that $w_1$ and $w_2$ are in the same partition and, since $H'$ is connected, there is a path from $w_1$ to $w_2$ which has even length. Together with $v_1w_1$, $v_2w_2$ and $v_1v_2$, this gives a cycle of odd length. Choosing the shortest path between $w_1$ and $w_2$, we may assume that the cycle is vertex-induced.

Thus, we conclude that the only possible choices for $\widetilde{H}$ are either picking the red edges or picking the blue edges, which concludes the proof. ◀

When $\Pi$ is the property of being bipartite, APT($\Pi$) is exactly the problem Max-Cut ATLB. Crowston et. al. [5] showed that Max-Cut ATLB has a kernel with $\mathcal{O}(k^3)$ vertices. Thus, Theorem 21 together with Theorem 22 give us the following result:

▶ **Theorem 23.** *Let $\Pi$ be a strongly $\lambda$-extendible property on simple graphs, with $\lambda = \frac{1}{2}$, and suppose $\Pi$ is hereditary. Then APT($\Pi$) has a kernel with $\mathcal{O}(k^2)$ or $\mathcal{O}(k^3)$ vertices.*

We now consider oriented graphs. Observe that, up to isomorphism, there are two possible oriented graphs whose underlying graphs are $K_3$. Let $\overrightarrow{K}_3$ denote the oriented graph $G$ with $U(G) = K_3$ whose edges form an oriented cycle. We will call $\overrightarrow{K}_3$ the *oriented* triangle. Let $\overset{\nrightarrow}{K}_3$ denote the unique acyclic oriented graph $G$ with $U(G) = K_3$. We will call $\overset{\nrightarrow}{K}_3$ the *non-oriented* triangle.

If neither $\overrightarrow{K}_3$ or $\overset{\nrightarrow}{K}_3$ have property $\Pi$ then Theorem 22 shows that APT($\Pi$) is equivalent to Max-Cut ATLB and again we have a $O(k^3)$-vertex kernel. If on the other hand both $\overrightarrow{K}_3$ and $\overset{\nrightarrow}{K}_3$ have property $\Pi$ then we have a $O(k^2)$ kernel by Theorem 21. So it remains to consider the case when exactly one of $\overrightarrow{K}_3, \overset{\nrightarrow}{K}_3$ has property $\Pi$. By Lemma 24, this is only possible if $\overset{\nrightarrow}{K}_3 \in \Pi, \overrightarrow{K}_3 \notin \Pi$.

▶ **Lemma 24.** *Let $\Pi$ be a strongly $\lambda$-extendible property on oriented graphs, with $\lambda = \frac{1}{2}$, and suppose $\Pi$ is hereditary. If $\overrightarrow{K}_3 \in \Pi$, then $\overset{\nrightarrow}{K}_3 \in \Pi$.*

So now suppose $\overset{\nrightarrow}{K}_3 \in \Pi, \overrightarrow{K}_3 \notin \Pi$.

▶ **Lemma 25.** *Let $\Pi$ be a strongly $\lambda$-extendible property on oriented graphs, with $\lambda = \frac{1}{2}$, and suppose $\Pi$ is hereditary. If $\overset{\nrightarrow}{K}_3 \in \Pi$, then $ex(K_4) > \frac{1}{4}$. In particular, $\Pi$ diverges on cliques.*

▶ **Lemma 26.** *Let $\Pi$ be a strongly $\lambda$-extendible property on oriented graphs, with $\lambda = \frac{1}{2}$, and suppose $\Pi$ is hereditary, $\overrightarrow{K}_3 \notin \Pi$ and $\overset{\nrightarrow}{K}_3 \in \Pi$. Then $ex(K_j) > 0$ for every $j \neq 3$.*

Using Lemma 26 together with Corollary 13 and Lemmas 12, 15, 16 and 17, we have a bound on the size and number of all blocks in $G - S$, except for those block in $\mathcal{B}_2$ which have excess exactly 0 and have no internal vertices adjacent to $S$. Let $\mathcal{B}_2^0$ be this set of blocks. By Lemma 26 and the fact that $K_2, \overset{\nrightarrow}{K}_3 \in \Pi$, all blocks in $\mathcal{B}_2^0$ are oriented triangles. In order to bound the number of these triangles, we require the following lemma and reduction rule. Lemma 27 bounds the number of blocks in $\mathcal{B}_2^0$ which can contain a neighbor of $S$ (since by definition such a vertex must be a cut vertex); Reduction Rule 2 allows us to limit the remaining blocks by ensuring that no two of them can be adjacent.

▶ **Lemma 27.** *Let $Q_0$ denote the set of cut vertices of $G - S$ which only appear in blocks in $\mathcal{B}_2^0$. Let $\Pi$ be a strongly $\lambda$-extendible property on oriented graphs, with $\lambda = \frac{1}{2}$, and suppose $\Pi$ is hereditary, $\overrightarrow{K}_3 \notin \Pi$ and $\overleftrightarrow{K}_3 \in \Pi$. Let $(G, k)$ be an instance of APT($\Pi$) reduced by Reduction Rule 1. For any $s \in S$, if $|Q_0 \cap N(s)| \geq ((32 + \frac{2}{\inf_{AK}})k - 2)48k - \frac{4k}{\inf_{AK}} + 4k$, then the instance is a* Yes*-instance.*

▶ Reduction Rule 2. Let $B_1, B_2 \in \mathcal{B}_2$ be such that $V(B_1) \cap V(B_2) = \{v\}$, $B_1 = \overrightarrow{K}_3 = B_2$, $\{v\} \cap N(S) = \emptyset$ and $(B_i)_{\mathsf{int}} \cap N(S) = \emptyset$ for $i = 1, 2$. Let $\{w_i\} = (B_i)_{\mathsf{int}}$ and $\{u_i\} = V(B_i) \setminus \{v, w_i\}$ for $i = 1, 2$. If $G - \{v\}$ is disconnected, delete $v, w_1, w_2$, identify $u_1$ and $u_2$ and set $k' = k$. Otherwise, delete $v, w_1, w_2$ and set $k' = k - \frac{1}{4}$.

Putting everything together, we can prove the following theorem.

▶ **Theorem 28.** *Let $\Pi$ be a strongly $\lambda$-extendible property on oriented graphs, with $\lambda = \frac{1}{2}$, and suppose $\Pi$ is hereditary. Then APT($\Pi$) has a kernel with $\mathcal{O}(k^2)$ or $\mathcal{O}(k^3)$ vertices.*

Putting together Theorem 21, Theorem 23, and Theorem 28, we get our main result, Theorem 1.

## 6 Conclusion

We have succeeded in showing that APT($\Pi$) has a polynomial kernel for nearly all strongly $\lambda$-extendible $\Pi$. The only cases in which the polynomial kernel question remains open are those in which $\lambda = \frac{1}{2}$ and either $\Pi$ is not hereditary, or membership in $\Pi$ depends on the labellings on edges. For the cases when $\lambda \neq \frac{1}{2}$ or $\Pi$ contains all triangles, we can improve our kernel from $\mathcal{O}(k^3)$ vertices to $\mathcal{O}(k^2)$ vertices. It would be nice to show a $O(k^2)$ kernel in all cases.

The bound of Poljak and Turzík extends to edge-weighted graphs - for any strongly $\lambda$-extendible property $\Pi$ and any connected graph $G$ with weight function $w : E(G) \to \mathbb{R}^+$, there exists a subgraph $H$ of $G$ such that $H \in \Pi$ and $w(H) \geq \lambda w(G) + \frac{(1-\lambda)w(T)}{2}$, where $T$ is a minimum weight spanning tree of $G$. The natural question following from our results is whether the weighted version of APT($\Pi$) affords a polynomial kernel.

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