

A Strong Direct Product Theorem for the Tribes Function via the Smooth-Rectangle Bound

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Abstract

The main result of this paper is an optimal strong direct product result for the two-party public-coin randomized communication complexity of the Tribes function. This is proved by providing an alternate proof of the optimal lower bound of $\Omega(n)$ for the randomised communication complexity of the Tribes function using the so-called **smooth-rectangle bound**, introduced by Jain and Klauck [6]. The optimal $\Omega(n)$ lower bound for Tribes was originally proved by Jayram, Kumar and Sivakumar [10], using a more powerful lower bound technique, namely the **information complexity bound**. The information complexity bound is known to be at least as strong a lower bound method as the **smooth-rectangle bound** [13]. On the other hand, we are not aware of any function or relation for which the **smooth-rectangle bound** is (asymptotically) smaller than its public-coin randomized communication complexity. The optimal direct product for Tribes is obtained by combining our **smooth-rectangle bound** for tribes with the strong direct product result of Jain and Yao [8] in terms of **smooth-rectangle bound**.

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1 Introduction

Study of lower bounds for various natural functions and relations has been a major theme of research in communication complexity from its advent; both for its own intrinsic value and for applications of these bounds towards other areas of theoretical computer science [15]. Several lower bound techniques have been developed over the years in communication complexity such as fooling sets, discrepancy method, rectangle bound, information complexity bound, partition bound etc. It is interesting to understand the relative power of these techniques and rank them against each other. Sometimes, we would like to understand what is the weakest technique required to prove a particular lower bound.

An important and extensively used technique in communication complexity is the so called **rectangle bound** (a.k.a. the **corruption bound**). In this technique, one argues that for some output value z , and all large rectangles, a constant fraction of inputs in the rectangle have a function value different from z . This helps to lower bound the distributional communication complexity of the function, which then translates to a lower bound on the public-coin communication complexity via Yao's minmax principle [20]. This technique has been successfully applied to obtain optimal lower bounds for several problems; Razborov's lower bound proof [18] for the set-disjointness function [11] is arguably the most well-known application of this technique.



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Another technique that has been extremely useful is the information complexity bound [17, 4]. In this method, one lower bounds the distributional communication complexity by the amount of information the transcript of the protocol reveals about the inputs of Alice and Bob. The tools from information theory then come handy to lower bound the information cost of the protocol. Bar-Yossef, Jayram, Kumar and Sivakumar [1] successfully used this technique¹ to give an alternate proof of the linear lower bound for the set-disjointness function. This method has also been useful to give an optimal linear lower bound for the Tribes function [10].

Jain and Klauck [6], using tools from linear programming and semi-definite programming gave a uniform treatment to several of the existing lower bound techniques and proposed two additional lower bound techniques, the so-called **partition bound** and the **smooth-rectangle bound**. These bounds are stronger than almost all other known lower bound techniques including the **rectangle bound**. The **partition bound**, as the name suggests, is a linear programming formulation of the number of partitions in a randomized protocol. The **smooth-rectangle bound**, a weakening of the **partition bound**, is a robust version of the **rectangle bound** in the following informal sense: **smooth-rectangle bound** for a function f under a distribution μ , is the maximum over all functions g , which are close to f under the distribution μ , of the **rectangle bound** of g . In other words, a function f is said to have a large **smooth-rectangle bound**, if it is close to some other function g (under the distribution μ) which has a large **rectangle bound**, even though f itself might not have a large **rectangle bound**. This suffices to lower bound the communication complexity of f . These new lower bound methods have been successfully applied, for example to obtain an optimal lower bound for the Gap-Hamming problem [3]. In fact we are not aware (to the best of our knowledge) of any function or relation for which the **partition bound** or **smooth-rectangle bound** is (asymptotically) smaller than its public-coin randomized communication complexity. To determine how tight these new lower bounds are, remains an important open question in communication complexity.

Recently, Kerenidis *et al.* [13] showed that the information complexity is at least as powerful as the **relaxed-partition-bound**, which is a bound intermediate between the **partition bound** and the **smooth-rectangle bound**. The relative strengths of the information complexity and **partition bound** is not yet well understood.

Another important theme in communication complexity has been the study of the so called strong direct-product and (the weaker) direct-sum conjectures; again for their own intrinsic value and also for important applications of such results in other areas of theoretical computer science [12]. A strong direct-product conjecture for the public-coin communication complexity of a relation f would state the following. Let c be the public-coin communication complexity of f (with constant error). Suppose k independent instances of f are being solved using communication less than kc , then the overall success would be exponentially small in k . In fact, the information complexity was introduced initially [4] as a tool to resolve the direct sum/product question. However, despite the considerable progress made over the last few years [2, 7], the direct product question has not yet been resolved. On the other hand, we are not aware of any function or relation for which this conjecture is false. Settling this conjecture for all relations, again is an important open question in communication complexity.

¹ The notion of **information complexity** was formalized by Chakrabarti, Shi, Wirth and Yao [CSWY01] in the direct sum context, however has been used by earlier works as well for example by Ponzio, Radhakrishnan and Venkatesh [17] for showing optimal lower bounds on the communication complexity of the pointer-chasing problem. Chakrabarti *et al.* [4] defined and used, what in today's language is called, "external information cost" while Bar-Yossef, Jayram, Kumar and Sivakumar [1] defined and used "internal information cost" in their proof of the disjointness lower bound.

Recently, Jain and Yao [8] proved a direct-product result for all relations in terms of the smooth-rectangle bound (srec). They show that for any relation f , if less than $k \cdot \log \text{srec}(f)$ communication (c.f., Definition 2.2) is provided for solving k independent copies of f , then the overall success is exponentially small in k . This provides a recipe to arrive at strong direct-product results for any relation f : by exhibiting that $\log \text{srec}(f)$ provides optimal lower bound for the public-coin communication complexity of f . Jain and Yao's result implies (and in some cases reproves) strong direct product result for many interesting functions and relations including that for the set-disjointness function (a strong direct-product result for set-disjointness was first shown by Klauck [14], again via showing that the smooth-rectangle bound of a related function is large). This also strongly motivates the search of functions for which their smooth-rectangle bound is asymptotically smaller than their public-coin communication complexity. This leads us to the study of the Tribes function as described below.

1.1 Our result

In this work we are concerned with the Tribes : $\{0, 1\}^n \times \{0, 1\}^n \rightarrow \{0, 1\}$ function, defined as follows.

$$\text{Tribes}(x, y) \stackrel{\text{def}}{=} \bigwedge_{i=1}^{\sqrt{n}} \left(\bigvee_{j=1}^{\sqrt{n}} (x_{(i-1)\sqrt{n}+j} \wedge y_{(i-1)\sqrt{n}+j}) \right).$$

As mentioned earlier, an optimal linear lower bound for Tribes was shown by Jayram, Kumar and Sivakumar [10] using the information complexity technique. It is to be noted that the rectangle bound proves only a $\Theta(\sqrt{n})$ lower bound and thus fails to provide an optimal lower bound for Tribes. In fact, the primary motivation for Jayram *et al.* [10] to study the Tribes function was the fact that it provided the first example where information complexity techniques were provably stronger than the then known ‘‘combinatorial’’ lower bound techniques. Therefore it is natural to ask if Tribes also provides a separation between smooth-rectangle bound and public-coin communication complexity, in the process also implying separation between information complexity bound and smooth-rectangle bound. We consider this question in this work and answer it in the negative.

► **Theorem 1.1** (smooth-rectangle bound for Tribes).

For sufficiently small $\varepsilon \in (0, 1)$, $R_\varepsilon^{\text{pub}}(\text{Tribes}) \geq \log \text{srec}_\varepsilon(\text{Tribes}) \geq \Omega(n)$.

Here, $R_\varepsilon^{\text{pub}}(f)$ refers to the ε -error public-coin randomized communication complexity of f .

Another important motivation for our work (besides answering the above question) is its consequence to strong direct product. As indicated in the recipe outlined above, combining our smooth-rectangle bound for Tribes with the result of Jain and Yao [8], we obtain the following.

► **Corollary 1.2** (strong direct product for Tribes). $R_{1-2^{-\Omega(k)}}^{\text{pub}}(\text{Tribes}^{(k)}) = \Omega(kn)$.

Here, $f^{(k)}$ refers to the k -wise direct product of the function f . Our result (Theorem 1.1) also exhibits for the first time, an asymptotic separation between the smooth-rectangle bound and the rectangle bound for a total function (previously a quadratic separation was known however for the Gap-Hamming partial function [3]).

It is to be noted that the information complexity lower bound for Tribes was generalised to constant depth read-once trees functions [9, 16]. Given our results, it is interesting to ask if these lower bounds can be obtained using the smooth-rectangle bound instead, which would imply a direct product for these functions. These alternate lower bounds might also help to obtain bounds for super-constant depth read-once formulae.

1.2 Our techniques

It will be convenient for us to view the Tribes function as the conjunction of \sqrt{n} set-disjointness functions over \sqrt{n} sized inputs². We refer to the \sqrt{n} sized inputs to each of the disjointness functions as a block. We consider a distribution μ on the inputs for the Tribes function which has support only on the following type of inputs: in every block, except for one block (say j), the inputs to the two parties Alice and Bob are NO instances of the disjointness function (the sets corresponding to the blocks intersect at exactly one location) and in block j , there could be 0, 1 or 2 intersections which occur at locations k_j and l_j . Let's refer to the three types of subsets of inputs based on the number of intersections as U_0, U_1 , and U_2 respectively. Recall that to show that the smooth-rectangle bound of Tribes is large, we need to demonstrate a function g , close to Tribes (under μ), whose rectangle bound is large. This function g is constructed as follows: g takes value 0 in $U_0 \cup U_2$ and value 1 in U_1 . Note that Tribes takes value 0 in U_0 and value 1 in $U_1 \cup U_2$. I.e., Tribes and g disagree on the inputs in U_2 . For our choice of distribution μ , this disagreement set U_2 will have weight $\mu(U_2) \approx 1/16$ while the weight of the 1-inputs will be approximately $\mu(U_1) \approx 6/16$ (i.e., U_1 is 6 times larger than U_2).

Observe that for Tribes, there are large rectangles (of size $\approx 2^{-\sqrt{n}}$ under μ) which are monochromatic. We can just fix any one coordinate in each block and force intersection there to create large 1-monochromatic rectangle. Similarly we can choose any one block and force non-intersection in that entire block to create large 0-monochromatic rectangle. Hence the rectangle bound of Tribes is at most $O(\sqrt{n})$. However, note that the 1-monochromatic rectangles described above are not monochromatic in g . Indeed, we show that there exists constants C and D such that for every large rectangle W (with $\mu(W) \geq 2^{-\Omega(n)}$), $\mu(U_1 \cap W)$ is either dominated by $C \cdot \mu(U_0 \cap W)$ (this is similar to the rectangle bound) or is dominated by $D \cdot \mu(U_2 \cap W)$. This immediately implies the rectangle bound of g is $\Omega(n)$. We will prove the above statement for D strictly smaller than 6. This fact implies that whenever $\mu(U_1 \cap W)$ is not dominated by $C \cdot \mu(U_0 \cap W)$ in W , the ratio of U_2 -inputs to U_1 -inputs in the rectangle W is considerably more than the similar ratio globally (which is $\approx 1/6$). This fact lets us translate the $\Omega(n)$ rectangle bound for g to a similar smooth-rectangle bound for Tribes.

We consider an exhaustive collection of sub-events such that conditioned on any such sub-event, the non-product distribution μ becomes a product distribution. Such handling of non-product distributions, by decomposing them into several product distributions, has been done several times before, for instance in Razborov's proof [18] of the optimal lower bound for the set-disjointness function. Assume such a conditioning exists for the rest of this proof outline.

How does one prove that for all large rectangles W , either $\mu(U_1 \cap W) \leq C\mu(U_0 \cap W)$ or $\mu(U_1 \cap W) \leq D\mu(U_2 \cap W)$ for some D strictly smaller than 6. Note that one cannot prove for all rectangles W , $\mu(U_1 \cap W) \leq D\mu(U_2 \cap W)$ for some D strictly less than 6, since this is false globally (i.e., $\mu(U_1) \approx 6\mu(U_2)$). Hence, one needs to do a case analysis³. And we do this based on the values of $\Pr[X_{l_j} = Y_{l_j} = 1]$ and $\Pr[X_{k_j} = Y_{k_j} = 1]$.

Consider the case when $\Pr[X_{l_j} = Y_{l_j} = 1] \geq \frac{3}{4}\mu(U_1 \cap W)$. Since the rectangle is large, using an entropy argument, we can argue that in most cases, conditioned on the sub-event

² By the disjointness function, we refer to the function $\bigvee_{j=1}^{\sqrt{n}} (x_j \wedge y_j)$. Strictly speaking, this is the set-intersection problem, but as is common in this literature, we will abuse notation and refer to this problem as the set-disjointness problem.

³ Such a case analysis is not required to prove rectangle bound (c.f., proof of disjointness [18]), but is necessary while proving a smooth-rectangle bound.

$(X_{l_j} = Y_{l_j} = 1)$, both $\Pr[X_{k_j} = 1]$ and $\Pr[Y_{k_j} = 1]$ are large enough ($\approx 1/2$). Now since the distribution is product it means that conditioned on $(X_{l_j} = Y_{l_j} = 1)$, $\Pr[X_{k_j} = Y_{k_j} = 1]$ is large enough and hence $\mu(U_2 \cap W)$ is a required fraction of $\mu(U_1 \cap W)$. Similar arguments hold for the case with the roles of l and k reversed.

In the third case, when $\max\{\Pr[X_{l_j} = Y_{l_j} = 1], \Pr[X_{k_j} = Y_{k_j} = 1]\} \leq \frac{3}{4}\mu(U_1 \cap W)$, again using the same entropy argument, we can show that $\Pr[X_{l_j} = Y_{l_j} = 1, X_{k_j} = Y_{k_j} = 0]$ and $\Pr[X_{l_j} = Y_{l_j} = 0, X_{k_j} = Y_{k_j} = 1]$ are large. Now, since W is a rectangle, we can show that $\Pr[X_{l_j} = 1, Y_{l_j} = 0, X_{k_j} = 0, Y_{k_j} = 1]$ and $\Pr[X_{l_j} = 0, Y_{l_j} = 1, X_{k_j} = 1, Y_{k_j} = 0]$ are large using a *cut-and-paste* argument. This implies that $\mu(U_0 \cap W)$ is a required fraction of $\mu(U_1 \cap W)$. This concludes our proof outline.

We note that our distribution is similar to (and in fact inspired from) the distribution used by Jain and Klauck [6] while analyzing the query complexity of the Tribes function. We also note that the distribution used by Jayram, Kumar and Sivakumar [10] in their information complexity lower bound for Tribes is different from our distribution, in particular, their distribution does not put any support on U_2 inputs which have intersections of size 2 within block j . However, we do add that they also use similar in spirit, albeit different cut-and-paste arguments in their lower bound proof.

2 Preliminaries

Communication Complexity

We begin by recalling the Yao's two-party communication model [19] (see Kushilevitz and Nisan [15] for an excellent introduction to the area). Let \mathcal{X} , \mathcal{Y} and \mathcal{Z} be finite non-empty sets, and let $f : \mathcal{X} \times \mathcal{Y} \rightarrow \mathcal{Z}$ be a function. A two-party protocol for computing f consists of two parties, Alice and Bob, who get inputs $x \in \mathcal{X}$ and $y \in \mathcal{Y}$ respectively, and exchange messages in order to compute $f(x, y) \in \mathcal{Z}$.

For a distribution μ on $\mathcal{X} \times \mathcal{Y}$, let the ε -error distributional communication complexity of f under μ (denoted by $D_\varepsilon^\mu(f)$), be the number of bits communicated (for the worst-case input) by the best deterministic protocol for f with average error at most ε under μ . Let $R_\varepsilon^{\text{pub}}(f)$, the public-coin randomized communication complexity of f with worst case error ε , be the number of bits communicated (for the worst-case input) by the best public-coin randomized protocol, that for each input (x, y) computes $f(x, y)$ correctly with probability at least $1 - \varepsilon$. Randomized and distributional complexity are related by the following celebrated result of Yao [20].

► **Theorem 2.1** (Yao's minmax principle [20]). $R_\varepsilon^{\text{pub}}(f) = \max_\mu D_\varepsilon^\mu(f)$.

Given a function $f : \mathcal{X} \times \mathcal{Y} \rightarrow \mathcal{Z}$, the k -wise direct product of f , denoted by $f^{(k)}$ is the function $f : \mathcal{X}^k \times \mathcal{Y}^k \rightarrow \mathcal{Z}^k$ defined as follows: $f^{(k)}((x_1, \dots, x_k), (y_1, \dots, y_k)) = (f(x_1, y_1), \dots, f(x_k, y_k))$. The direct product/sum question involves relating $R^{\text{pub}}(f^{(k)})$ to $R^{\text{pub}}(f)$. More precisely, the strong direct product conjecture states that $R_{1-2^{-\Omega(k)}}^{\text{pub}}(f^{(k)}) = \Omega(k \cdot R_{1/3}^{\text{pub}}(f))$.

The smooth rectangle bound

The smooth rectangle bound was introduced by Jain and Klauck [6], as a generalization of the rectangle bound. Informally, the smooth-rectangle bound for a function f under a distribution μ , is the maximum over all functions g , which are close to f under the distribution μ , of the rectangle bound of g . However, it will be more convenient for us to work with the following

linear programming formulation of smooth-rectangle bound. Please see [6, Lemma 2] and [8, Lemma 6] for the relations between the LP formulation and the more “natural” formulation in terms of rectangle bound. A broad connection between the two definitions is that the variable φ in the dual of the linear programming definition takes non-zero values precisely at the inputs (x, y) where f and g differ.

► **Definition 2.2** (smooth-rectangle bound). For a total Boolean function f , the ε -smooth rectangle bound of f denoted $\text{srec}_\varepsilon(f)$ is defined to be $\max\{\text{srec}_\varepsilon^z(f) : z \in \{0, 1\}\}$, where $\text{srec}_\varepsilon^z(f)$ is given by the optimal value of the following linear program (below \mathcal{W} represents the set of all rectangles in $\mathcal{X} \times \mathcal{Y}$).

$$\begin{array}{ccc}
 \text{Primal} & & \text{Dual} \\
 \min \sum_{W \in \mathcal{W}} v_W & & \max \sum_{(x,y) \in f^{-1}(z)} ((1-\varepsilon)\lambda_{x,y} - \varphi_{x,y}) - \sum_{(x,y) \notin f^{-1}(z)} \varepsilon \cdot \lambda_{x,y} \\
 \sum_{W \ni (x,y)} v_W \geq 1 - \varepsilon, & \forall (x,y) \in f^{-1}(z) & \sum_{(x,y) \in W_z} (\lambda_{x,y} - \varphi_{x,y}) - \sum_{(x,y) \in W_{-z}} \lambda_{x,y} \leq 1, \quad \forall W \\
 \sum_{W \ni (x,y)} v_W \leq 1, & \forall (x,y) \in f^{-1}(z) & \lambda_{x,y}, \varphi_{x,y} \geq 0, \quad \forall (x,y) \\
 \sum_{W \ni (x,y)} v_W \leq \varepsilon, & \forall (x,y) \notin f^{-1}(z) & \text{where } W_z = W \cap f^{-1}(z) \text{ and } W_{-z} = W \setminus f^{-1}(z). \\
 v_W \geq 0, & \forall W. &
 \end{array}$$

► **Theorem 2.3** ([6, Theorem 1]). For all functions $f : \mathcal{X} \times \mathcal{Y} \rightarrow \{0, 1\}$ and $\varepsilon \in (0, 1)$, we have $\mathbb{R}_\varepsilon^{\text{pub}}(f) \geq \log(\text{srec}_\varepsilon(f))$.

Jain and Yao [8] proved the following strong direct product theorem in terms of the smooth rectangle bound.

► **Theorem 2.4** ([8, Theorem 1 and Lemma 6]). Let $f : \mathcal{X} \times \mathcal{Y} \rightarrow \{0, 1\}$ be a Boolean function. For every $\varepsilon \in (0, 1)$, there exists small enough $\eta \in (0, 1/3)$ such that the following holds. For all integers k ,

$$\mathbb{R}_{1-(1-\eta)^{\lfloor \eta^2 k / 32 \rfloor}}^{\text{pub}}(f^{(k)}) \geq \frac{\eta^2}{32} \cdot k \cdot \left(11\eta \cdot \log \text{srec}_\varepsilon(f) - 3 \log \frac{1}{\varepsilon} - 2 \right).$$

Information theory

We need the following basic facts from information theory. Let μ be a (probability) distribution on a finite set \mathcal{X} and X be a random variable distributed according to μ . Let $\mu(x)$ represent the probability of $x \in \mathcal{X}$ according to μ . The entropy of X is defined as $H(X) \stackrel{\text{def}}{=} \sum_x \mu(x) \cdot \log \frac{1}{\mu(x)}$. Entropy satisfies subadditivity: $H(XY) \leq H(X) + H(Y)$.

3 The smooth rectangle bound for Tribes

In this section, we prove a linear lower bound on the randomized communication of Tribes via the smooth-rectangle bound.

First we introduce some notation. We will prove the result for n of the form $(2r + 1)^2$, where $r \geq 2$ is even. Assume the input indices $[n]$ to the Tribes function are partitioned into

\sqrt{n} blocks $s_1, \dots, s_{\sqrt{n}}$, where the i^{th} block $s_i = \{(i-1)\sqrt{n} + 1, \dots, i\sqrt{n}\}$. Thus,

$$\text{Tribes}(x, y) = \bigwedge_{i=1}^{\sqrt{n}} \left(\bigvee_{j \in s_i} (x_j \wedge y_j) \right).$$

A string $x \in \{0, 1\}^n$ can be viewed both as an n -bit string and as a subset $x \subseteq [n]$. We will use both these interpretations.

Consider the distribution $\mu(x, y)$ on the inputs of the Tribes function defined by the following (informal) description. As mentioned earlier, this distribution is inspired by the distribution used by Jain and Klauck [6] while analyzing the query complexity of the Tribes function. Among the \sqrt{n} blocks, one of the blocks is chosen as a special block, say block j . Alice's and Bob's inputs are then chosen such that their inputs when restricted to any of the blocks (special or non-special) have exactly $(r/2 + 1)$ ones each. Furthermore, for each of the non special blocks, Alice's and Bob's input are chosen such that their inputs, restricted to this block, have a unique intersection (this is identical to the yes instances of Razborov's distribution for disjointness) while for the special block j , Alice's and Bob's inputs are chosen such that their inputs, restricted to the special block, have an intersection of size 0, 1 or 2. As in the case of Razborov's distribution, the variable t is used to denote the random variable containing the index of the special block j and other relevant information such that conditioned on t , the distribution (X, Y) is a product distribution. The formal description of the distribution μ is as follows:

1. Choose $j \in [\sqrt{n}]$ uniformly.
 For each $i \in [\sqrt{n}] \setminus \{j\}$, randomly partition the indices in s_i as follows: $s_i = (t_i^A, t_i^B, \{l_i\})$ into 3 disjoint sets such that $|t_i^A| = |t_i^B| = r$ and $l_i \in s_i$.
 For index j , randomly partition the indices in s_j as follows: $s_j = (\tilde{t}_j^A, \tilde{t}_j^B, \{k_j\}, \{l_j\}, \{d_j\})$ into 5 disjoint sets such that $|\tilde{t}_j^A| = |\tilde{t}_j^B| = r - 1$ and $k_j, l_j, d_j \in s_j$. Set $t_j^A = \tilde{t}_j^A \cup \{k_j\}$ and $t_j^B = \tilde{t}_j^B \cup \{k_j\}$.
 Let $t = (j, k_j, (t_i^A, t_i^B, l_i)_{i \in [\sqrt{n}]})$.
2. For each $i \neq j \in [\sqrt{n}]$, set the variables in block s_i as follows:
 - Set $x_{l_i} \leftarrow 1$ and $x_{s_i \setminus (t_i^A \cup \{l_i\})} \leftarrow \bar{0}$. Let $x_{t_i^A}$ be a random string of exactly $r/2$ ones.
 - Set $y_{l_i} \leftarrow 1$ and $y_{s_i \setminus (t_i^B \cup \{l_i\})} \leftarrow \bar{0}$. Let $y_{t_i^B}$ be a random string of exactly $r/2$ ones.
3. Set the variables in block s_j as follows:
 - Let $x_{t_j^A \cup \{l_j\}}$ be a random string of exactly $r/2 + 1$ ones and $x_{s_j \setminus (t_j^A \cup \{l_j\})} \leftarrow \bar{0}$.
 - Let $y_{t_j^B \cup \{l_j\}}$ be a random string of exactly $r/2 + 1$ ones and $y_{s_j \setminus (t_j^B \cup \{l_j\})} \leftarrow \bar{0}$.

Let (X, Y) be distributed according to μ , where X represents the input to Alice and Y represents the input to Bob. Let $T = (J, K_J, (T_i^A, T_i^B, L_i)_{i \in [\sqrt{n}]})$ be the random variable (correlated with (X, Y)) representing t distributed as above. Observe that though (X, Y) is not a product distribution, the conditional distribution $((X, Y) \mid T = t)$ is product for each t .

Partition the set of inputs (in the support of μ) into 3 sets U_0, U_1 and U_2 as follows:

$$U_i = \{(x, y) \mid \mu(x, y) > 0 \text{ and sets } x \text{ and } y \text{ have exactly } \sqrt{n} - 1 + i \text{ intersections}\}.$$

Note that U_0 are the 0-inputs and $U_1 \cup U_2$ the 1-inputs of the Tribes function while $U_0 \cup U_2$ and U_1 are the 0- and 1-inputs respectively of the function g described in Section 1.2.

Let $\beta \stackrel{\text{def}}{=} \frac{r+2}{r+1}$. The following facts can be easily verified from the definition of the distribution μ . For all t ,

$$\Pr[X_{l_j} = 1 \mid T = t] = \frac{\beta}{2}; \quad \Pr[X_{l_j} = X_{k_j} = 1 \mid T = t] = \Pr[X_{l_j} = 1, X_{k_j} = 0 \mid T = t] = \frac{\beta}{4}.$$

Given this, it can be easily checked that the weights of the sets U_0, U_1 and U_2 are as follows: $\mu(U_0) = 1 - 7\beta^2/16$, $\mu(U_1) = 6\beta^2/16$, and $\mu(U_2) = \beta^2/16$.

Our main lemma is the following (we have not optimized the constants).

► **Lemma 3.1.** *There exists a constant $\delta \in (0, 1)$ such that for sufficiently large n , the following holds: for every rectangle $W = A \times B$, we have*

$$0.99\mu(U_1 \cap W) \leq \frac{16}{3(0.99)^2} \cdot \mu(U_2 \cap W) + \frac{16}{(0.99)^2} \mu(U_0 \cap W) + 2^{-\delta n/2+1}.$$

In other words, in any rectangle which contains a significant fraction of inputs from U_1 (i.e., at least $2^{-\delta n/2+1}$), the weight of the U_1 inputs is dominated by some linear function of the weights of U_0 and U_2 inputs. Before proving this lemma, let us first see how this lemma implies the smooth-rectangle bound for Tribes, which implies our Main Theorem 1.1

► **Theorem 3.2** (smooth-rectangle bound for Tribes). *There exists $\gamma \in (0, 1)$ such that for all sufficiently large n and $\varepsilon < 1/1000$, we have: $\text{srec}_\varepsilon^1(\text{Tribes}) \geq 2^{\gamma n}$.*

Proof. We will prove the bound using the dual formulation for smooth-rectangle bound given in Definition 2.2. Define the dual variables $\lambda_{x,y}$ and $\varphi_{x,y}$ as follows:

$$\lambda_{x,y} = \begin{cases} 0 & \text{if } (x,y) \in U_2 \\ 0.99\mu(x,y)2^{\delta n/2-1} & \text{if } (x,y) \in U_1 \\ \frac{16}{(0.99)^2}\mu(x,y)2^{\delta n/2-1} & \text{if } (x,y) \in U_0. \end{cases}$$

$$\varphi_{x,y} = \begin{cases} \frac{16}{3(0.99)^2}\mu(x,y)2^{\delta n/2-1} & \text{if } (x,y) \in U_2 \\ 0 & \text{if } (x,y) \in U_1 \cup U_0. \end{cases}$$

From Lemma 3.1 we get

$$\forall \text{ rectangles } W : \sum_{(x,y) \in \text{Tribes}^{-1}(1) \cap W} (\lambda_{x,y} - \varphi_{x,y}) - \sum_{(x,y) \in (W \setminus \text{Tribes}^{-1}(1))} \lambda_{x,y} \leq 1.$$

The objective of the LP can be bounded as follows:

$$\begin{aligned} & \sum_{(x,y) \in \text{Tribes}^{-1}(1)} ((1 - \varepsilon)\lambda_{x,y} - \varphi_{x,y}) - \sum_{(x,y) \notin \text{Tribes}^{-1}(1)} \varepsilon \cdot \lambda_{x,y} \\ & \geq \left((0.999)(0.99)\mu(U_1) - \frac{16}{3(0.99)^2}\mu(U_2) - \frac{16}{1000(0.99)^2}\mu(U_0) \right) 2^{\delta n/2-1} \\ & \geq 0.02 \cdot 2^{\delta n/2-1} \quad (\text{for sufficiently large } n). \end{aligned}$$

Thus, proved. ◀

Corollary 1.2 follows by combining the above theorem and Jain-Yao's strong direct product theorem in terms of the smooth-rectangle bound (Theorem 2.4).

3.1 Proof of Lemma 3.1

Let $W = A \times B$ be the rectangle. For each $t = (j, k_j, (t_i^A, t_i^B, l_i)_{i \in [\sqrt{n}]})$ and $a, b \in \{0, 1\}$, define,

$$\begin{aligned} R(t, a, b) &= \Pr[X \in A \mid T = t, X_{l_j} = a, X_{k_j} = b], R(t, a) = \Pr[X \in A \mid T = t, X_{l_j} = a], \\ C(t, a, b) &= \Pr[Y \in B \mid T = t, Y_{l_j} = a, Y_{k_j} = b], C(t, a) = \Pr[Y \in B \mid T = t, Y_{l_j} = a]. \end{aligned}$$

Define the following random variables (we will set δ later):

$$\text{BAD}_A(t) = 1 \text{ iff } \min\{R(t, 1, 1), R(t, 1, 0)\} < 0.99 (R(t, 1) - 2^{-\delta n}),$$

and symmetrically,

$$\text{BAD}_B(t) = 1 \text{ iff } \min\{C(t, 1, 1), C(t, 1, 0)\} < 0.99 (C(t, 1) - 2^{-\delta n}).$$

For a given t , let t' denote a partition identical to t except that the role of the indices l_j and k_j are exchanged (i.e., $k'_j = l_j, l'_j = k_j, (t'_j)^A = \tilde{t}_j^A \cup \{l_j\}$ and $(t'_j)^B = \tilde{t}_j^B \cup \{l_j\}$). To define $\text{BAD}(t)$, we need the following two quantities.

$$\begin{aligned} \rho_l(t) &= \Pr[X_{l_j} = Y_{l_j} = 1, X \in A, Y \in B, (X, Y) \in U_1 \mid T = t], \\ \rho_k(t) &= \Pr[X_{k_j} = Y_{k_j} = 1, X \in A, Y \in B, (X, Y) \in U_1 \mid T = t]. \end{aligned}$$

Observe that $\mu(U_1 \cap W \mid T = t) = \rho_l(t) + \rho_k(t)$. Hence, it must be the case that exactly one of the following happens: (1) $\rho_l(t) > 3\mu(U_1 \cap W \mid T = t)/4$, (2) $\rho_k(t) > 3\mu(U_1 \cap W \mid T = t)/4$ or (3) $\max\{\rho_l(t), \rho_k(t)\} \leq 3\mu(U_1 \cap W \mid T = t)/4$ (equivalently, $\min\{\rho_l(t), \rho_k(t)\} \geq \mu(U_1 \cap W \mid T = t)/4$). We define $\text{BAD}(t)$ based on these cases as follows.

$$\text{BAD}(t) = \begin{cases} \text{BAD}_A(t) \vee \text{BAD}_B(t), & \text{if } \rho_l(t) > \frac{3\mu(U_1 \cap W \mid T=t)}{4} \\ \text{BAD}_A(t') \vee \text{BAD}_B(t'), & \text{if } \rho_k(t) > \frac{3\mu(U_1 \cap W \mid T=t)}{4} \\ \text{BAD}_A(t) \vee \text{BAD}_B(t) \vee \text{BAD}_A(t') \vee \text{BAD}_B(t'), & \text{otherwise.} \end{cases} \quad (3.1)$$

The following claim shows that the probability that $\text{BAD}_A(T)$ and $\text{BAD}_B(T)$ occurs is small.

► **Claim 3.3.** *There exists a small fixed constant $\delta > 0$ such that for sufficiently large n , the following holds: for any $(t_i^A, l_i)_{i \in [\sqrt{n}]}$, we have*

$$\Pr[\text{BAD}_A(T) = 1 \mid T_i^A = t_i^A, L_i = l_i, \text{ for each } i \in [\sqrt{n}]] < \frac{1}{6400}.$$

(Symmetrically, for any $(t_i^B, l_i)_{i \in [\sqrt{n}]}$, $\Pr[\text{BAD}_B(T) = 1 \mid T_i^B = t_i^B, L_i = l_i, \text{ for each } i \in [\sqrt{n}]] < \frac{1}{6400}$.)

Proof. We prove the inequality involving $\text{BAD}_A(T)$. The other inequality is proved similarly. We first consider the easy case when $(t_i^A, l_i)_{i \in [\sqrt{n}]}$ satisfies

$$\Pr[X \in A \mid X_{l_i} = 1, T_i^A = t_i^A, L_i = l_i, \text{ for each } i \in [\sqrt{n}]] < 2^{-\delta n}.$$

It follows from the definition of the distribution μ , that the above probability is unchanged on further conditioning by $T = t$ for any t consistent with $(t_i^A, l_i)_{i \in [\sqrt{n}]}$. In other words, this probability is equal to $R(t, 1) = \Pr[X \in A \mid T = t, X_{l_j} = 1]$ for any t consistent with $(t_i^A, l_i)_{i \in [\sqrt{n}]}$. Hence, for any such t we have that $R(t, 1) < 2^{-\delta n}$. Thus, in this case $\text{BAD}_A(t) = 0$ for all such t and we are done.

Now consider the other case when

$$\Pr[X \in A \mid X_{l_i} = 1, T_i^A = t_i^A, L_i = l_i, \text{ for each } i \in [\sqrt{n}]] \geq 2^{-\delta n}. \quad (3.2)$$

Consider a $t = (j, k_j, (t_i^A, t_i^B, l_i)_{i \in [\sqrt{n}]})$ consistent with $(t_i^A, l_i)_{i \in [\sqrt{n}]}$. We know that the bit $(X_{k_j} \mid T = t, X_{l_j} = 1)$ is a unbiased bit. Now, suppose $\text{BAD}_A(t) = 1$. Then, for some $a \in \{0, 1\}$, we have

$$\Pr[X \in A \mid T = t, X_{l_j} = 1, X_{k_j} = a] < 0.99 (\Pr[X \in A \mid T = t, X_{l_j} = 1]).$$

By a simple rewriting of the above inequality, we have

$$\Pr[X_{k_j} = a \mid X \in A, T = t, X_{l_j} = 1] < 0.99 (\Pr[X_{k_j} = a \mid T = t, X_{l_j} = 1]) = 0.99/2. \quad (3.3)$$

In other words, the unbiased bit $(X_{k_j} \mid T = t, X_{l_j} = 1)$ when conditioned on the event “ $X \in A$ ” is now more likely to be $1 - a$ than a .

Suppose, for contradiction, that

$$\Pr[\text{BAD}_A(T) = 1 \mid T_i^A = t_i^A, L_i = l_i, \text{ for each } i \in [\sqrt{n}]] \geq \frac{1}{6400}.$$

Consider the random variable

$$Z \stackrel{\text{def}}{=} (X \mid X_{l_i} = 1, T_i^A = t_i^A, L_i = l_i, \text{ for each } i \in [\sqrt{n}]).$$

Note that the distribution of Z is uniform and each string has probability $\left(\frac{1}{\binom{r}{r/2}}\right)^{\sqrt{n}}$.

Consider the event $E \stackrel{\text{def}}{=} “X \in A \mid T = t, X_{l_j} = 1”$, which by (3.2) has probability at least $2^{-\delta n}$. Therefore the probability of each string in the distribution $(Z|E)$ would be at most $2^{\delta n} \cdot \left(\frac{1}{\binom{r}{r/2}}\right)^{\sqrt{n}}$. Therefore, using standard estimates on binomial coefficients,

$$H(Z|E) \geq \sqrt{n} \cdot \log \binom{r}{r/2} - \delta n \geq \sqrt{n} \cdot r(1 - o(1)) - \delta n.$$

Observe that conditioned on $T_i^A = t_i^A, L_i = l_i$, for each $i \in [\sqrt{n}]$, the index K_J can equally likely be any one of the $r\sqrt{n}$ indices in $\bigcup_i t_i^A$ (each resulting in a different value for T). Furthermore, from (3.3), we have that whenever $\text{BAD}_A(T) = 1$ (which by assumption happens with probability at least $1/6400$), conditioning on E causes X_{K_J} to be a biased bit and hence $H(X_{K_J}) \leq H(0.99/2)$. When $\text{BAD}_A(T) = 0$, which occurs with probability at most $1 - 1/6400$ by assumption, $H(X_{K_J})$ can be trivially bounded from above by 1. Using these facts, we can upper bound the entropy of $(Z|E)$ as follows:

$$\begin{aligned} H(Z|E) &\leq \sum_i H(Z_i|E) && \text{[By subadditivity of entropy]} \\ &\leq r\sqrt{n} \left(\frac{H(0.99/2)}{6400} + \left(1 - \frac{1}{6400}\right) \right). \end{aligned}$$

Combining the upper and lower bounds on $H(Z|E)$, we get

$$\delta n \geq (1 - H(0.99/2) - o(1)) \cdot \frac{r\sqrt{n}}{6400}.$$

Thus, if $\delta > 0$ is small enough we get a contradiction. \blacktriangleleft

The following claim shows that a version of Lemma 3.1 is true when $\text{BAD}(t) = 0$. The proofs of this claim and the subsequent claim differ significantly from the proofs of the corresponding claims in Razborov’s result [18] of linear lower bound for set-disjointness. This is because we need to consider several sub-events of U_1 . Our arguments are more general and in fact can also be used in the context of set-disjointness.

► **Claim 3.4.** *Let n be large enough. If $\text{BAD}(t) = 0$, then,*

$$\mu(U_1 \cap W \mid T = t) \leq \frac{16}{3(0.99)^2} \mu(U_2 \cap W \mid T = t) + \frac{16}{(0.99)^2} \mu(U_0 \cap W \mid T = t) + 2^{-\delta n/2}.$$

The following claim argues that not much probability is lost when $\text{BAD}(T) = 1$.

► **Claim 3.5.** *Let n be large enough. Then,*

$$\mathbb{E}_{t \leftarrow T} [\mu(U_1 \cap W \mid T = t) \cdot \text{BAD}(t)] \leq \frac{1}{100} \cdot \mathbb{E}_{t \leftarrow T} [\mu(W \cap U_1 \mid T = t)] + 2^{-\delta n+3}.$$

For want of space, the proofs of Claims 3.4–3.5 are deferred to the full version of the paper [5].

Lemma 3.1 follows by combining Claim 3.4 and Claim 3.5 as follows.

$$\begin{aligned} & 0.99\mu(U_1 \cap W) \\ &= 0.99 \mathbb{E}_{t \leftarrow T} [\mu(U_1 \cap W \mid T = t)] \\ &\leq \mathbb{E}_{t \leftarrow T} [\mu(U_1 \cap W \mid T = t) \cdot (1 - \text{BAD}(t))] + 2^{-\delta n+3} \quad (\text{from Claim 3.5}) \\ &\leq \mathbb{E}_{t \leftarrow T} \left[\left(\frac{16\mu(U_2 \cap W \mid T = t)}{3(0.99)^2} + \frac{16\mu(U_0 \cap W \mid T = t)}{(0.99)^2} + 2^{-\delta n/2} \right) (1 - \text{BAD}(t)) \right] \\ &\quad + 2^{-\delta n+3} \quad (\text{from Claim 3.4}) \\ &\leq \frac{16}{3(0.99)^2} \cdot \mu(U_2 \cap W) + \frac{16}{(0.99)^2} \cdot \mu(U_0 \cap W) + 2^{-\delta n/2+1} \end{aligned}$$

◀

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