A “Game Semantical” Intuitionistic Realizability Validating Markov’s Principle

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Abstract

We propose a very simple modification of Kreisel’s modified realizability in order to computationally realize Markov’s Principle in the context of Heyting Arithmetic. Intuitively, realizers correspond to arbitrary strategies in Hintikka-Tarski games, while in Kreisel’s realizability they can only represent winning strategies. Our definition, however, does not employ directly game semantical concepts and remains in the style of functional interpretations. As term calculus, we employ a purely functional language, which is Gödel’s System T enriched with some syntactic sugar.

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1 Introduction

1.1 Markov’s Argument

Given a recursive function $f : \mathbb{N} \to \mathbb{N}$, if it is impossible that for every natural number n, $f(n) \neq 0$, then there exists an n such that $f(n) = 0$. This classically true statement has come to be universally known as Markov’s Principle, and was introduced by Markov in the context of his theory of Constructive Recursive Mathematics (see e.g. [23]). Markov’s original argument for it was simply the following: if it is not possible that for all n, $f(n) \neq 0$, then by computing in sequence $f(0), f(1), f(2), \ldots$, one will eventually hit a number n such that $f(n) = 0$, which can be *effectively* recognized as a witness. For the rest of the paper we shall consider the formalization of Markov’s principle in Heyting Arithmetic, that is the axiom scheme

$$\mathsf{MP} : \neg\forall x^{\mathtt{N}} P \to \exists x^{\mathtt{N}} P^{\perp}$$

where P is a decidable predicate and $P^{\perp}$ its negation.

Markov’s justification of his own principle is hardly satisfying from a constructive point of view; the intuitionistic school of Brouwer, indeed, rejected it. It is true that, following

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Markov's argument, one can recursively realize MP using Kleene's realizability interpretation [13], thus providing a computational interpretation of it. However, a Kleene realizer just blindly searches for a witness of the conclusion, without even considering the possible constructive content of a proof of the premise. In other terms, such a realizer does not embody the meaningful transformation of a proof of $\neg\forall x^{\mathtt{N}} P$ into a proof of $\exists x^{\mathtt{N}} P^{\perp}$ which is demanded by the Brouwer-Heyting-Kolmogorov reading of logical constants [23]. However, when added to Heyting Arithmetic, MP gives rise to a *constructive* system enjoying the disjunction and the existential witness property [21] (if a disjunction is derivable, one of the disjoint is derivable too, and if an existential statement is derivable, so it is one instance of it). So a better interpretation of MP can and must be provided. In this article we shall try to answer, in particular, to the following question: is it possible to realize Markov's Principle just using a functional language and a simple intuitionistic realizability?

1.2 Gödel's Dialectica Interpretation

A much more refined constructive justification of Markov's Principle was in fact introduced by Gödel [10]. Indeed, the idea behind Gödel's Dialectica Interpretation is so refined, that it forms the basis for all subsequent constructive interpretations of MP [5, 11]. As pointed out by Diller [6], a very satisfying constructive justification of MP is indeed hidden in the Dialectica, and is the following. A formal proof of $\neg\forall x^{\mathtt{N}} P$ is a natural deduction of $\perp$ from the hypothesis $\forall x^{\mathtt{N}} P$. If we consider a normal form of this proof, we have actually a deduction of $\perp$ from finitely many instances $P(t_1), \ldots, P(t_n)$; so one of them must be false and we get a t_i such that $P^{\perp}(t_i)$, and t_i reduces to some numeral n. Thus, as required in the BHK semantics, from any proof of the premise of Markov's Principle one can effectively extract a witness for the conclusion, without having to run a blindfold process.

More in detail, Gödel's interpretation of implication allows one to describe a realizer of the premise $\neg\forall x^{\mathtt{N}} P$ of MP as a functional mapping a witness for $\forall x^{\mathtt{N}} P$ (essentially, something void) into a possible counterexample to $\forall x^{\mathtt{N}} P$. If this counterexample works, one witness $\exists x^{\mathtt{N}} P^{\perp}$, otherwise one has refuted the realizer of the premise of MP.

Gödel's Dialectica is thus very interesting and, rather remarkably, allows to computationally interpret any proof in Heyting Arithmetic plus MP with a term in a simple and purely functional language, Gödel's system T. However, in spite of its simple interpretation of MP, the Dialectica is a rather involved translation, which burdens a lot the reading of implication, making it particularly painful to unravel in presence of nested implications in the translated formula, as it is often the case. It is also quite cumbersome to decorate natural deductions with Gödel realizers. Is it really needed all this complication if one wants just to interpret Markov's Principle?

1.3 Kreisel's Modified Realizability

Inspired by Gödel's interpretation, Kreisel put forward his modified realizability [14, 15] as a simplification of the Dialectica, which is actually equivalent to it in the case of formulas without implications (Oliva [17]). In modified realizability, the familiar BHK reading of implication is restored – which originates the main simplification – and the term assignment for proofs can be taken as a pleasant intuitionistic Curry-Howard correspondence. Unfortunately, Kreisel introduced modified realizability with the *specific aim* of showing that Markov's principle is not realizable in the syntactical model made by the terms of Gödel's T. So one is left with a very good intuitionistic realizability, which is not able to concretely realize MP.

1.4 Modified Realizability and Friedman’s Translation

The Friedman translation is a strikingly simple device introduced by Friedman [7] in order to prove closure of intuitionistic systems S under Markov’s rule:

$$S \vdash \neg\forall x^{\mathtt{N}} P \implies S \vdash \exists x^{\mathtt{N}} \neg P$$

where P is any decidable quantifier free formula. While combining Friedman’s translation with modified realizability allows to interpret any *fixed* instance of MP, the situation does not improve too much because it is not possible to validate the full axiom scheme MP. In other terms, if a proof contains more than one instance of MP, combining Friedman’s translation with modified realizability is not enough to interpret it.

Indeed, one possible solution to this issue, due to Coquand-Hofmann [5], is to first make Friedman’s translation more flexible by using a somewhat unusual forcing [3] and then combining the result with modified realizability. We seek however a simpler and less ad hoc modification of modified realizability.

1.5 Game Semantics and Functional Interpretations

What’s wrong with modified realizability? The problem is that it is not a refined game semantics, which is really the framework needed to explain constructively classical principles (see e.g. [4, 1]). Instead, the Dialectica is better suited to represent dialogues among players – i.e. proofs and tests – which arise in classical game semantics.

The standard way to associate a game to an arithmetical formula A is to consider an interaction between two players who debate A; the first player – usually called Eloise – tries to show that it is true, while the second player – usually called Abelard – tries to show that the formula is false. Thus, Eloise wins when true atomic formulas are on the board while Abelard wins with false ones. In the case of formulas of the shape $A \vee B, \exists x^{\mathtt{N}} A$, Eloise moves: in the first case by choosing A or B and in the second case by choosing an instance $A(n)$, where n is intended to be a witness for the existential quantifier. In the case of formulas of the shape $A \wedge B, \forall x^{\mathtt{N}} A$, Abelard moves: in the first case by choosing A or B and in the second case by choosing an instance $A(n)$, where n is intended to be a counterexample to the universal quantifier. This kind of game was introduced by Hintikka [12] and it is also known as Tarski game.

As far as $\rightarrow$-free formulas are concerned, modified realizability and Dialectica agree (Oliva [17]): a realizer represents in both cases a winning strategy for Eloise, that is, a way of selecting moves that allows Eloise to win every play, no matter how Abelard plays. But in the case of formulas of the shape $A \rightarrow B$, according to modified realizability, Abelard should give Eloise a *winning* strategy for A, and then the game for B is played; while according to Dialectica, Abelard should give Eloise *some* strategy for A and then the game for B is played, and either Eloise wins this game, or “temporarily” looses it, but still with the possibility of winning the whole game if she manages to show that the strategy offered by Abelard was not winning. This second way of formulating the game for $\rightarrow$ is much better, since the first one is not concretely playable: how to establish effectively whether the strategy given by Abelard to Eloise is winning? In the case of Dialectica, Abelard is given a chance to play the game for the premise A without necessarily having to play in *the best* way possible, but just at *his best*, as in real life games.

1.6 A Game Semantical Twist of Modified Realizability

The goal of the present paper is to tweak modified realizability in such a way that its game semantical content is improved and made more similar to the one of Dialectica, while retaining the simplicity and the appeal of Kreisel's original definition. One should allow realizers to be not only winning strategies, but arbitrary ones, thus allowing poor Abelard to have more chances to play in the game for the formula $A \to B$. True realizers – among which those extracted from proofs – should be winning strategies, but in the concept of realizability should appear also weaker realizers, that is, arbitrary strategies.

1.7 Plan of the paper

In Section §2 the term calculus $\mathcal{T}$ in which realizers are written and the language of the arithmetical theory $\mathsf{HA}^\omega + \mathsf{MP}$ are introduced. In Section §3 we give our definition of realizability. In Section §4 an extensionality property of $\mathcal{T}$ is introduced and discussed as a crucial tool for studying the realizer of the Markov's Principle, defined in Section §5. Section §6 is devoted to prove our main result, that every theorem of $\mathsf{HA}^\omega + \mathsf{MP}$ is realizable; also the relationship between our notion of realizability and truth is discussed. Conclusions and considerations about future works are in Section §7.

2 The Term Calculus

In this section we introduce the typed lambda calculus $\mathcal{T}$ in which realizers are written. System $\mathcal{T}$ is obtained from Gödel's T (see [8, 9]) by adding a new atomic type U and new operations on it. The basic objects of $\mathcal{T}$ are numerals ($\mathsf{S} \ldots \mathsf{S}0$), booleans ($\mathtt{True}, \mathtt{False}$) and its basic computational constructs are primitive recursion at all types (R), if-then-else (if), pairs, as in Gödel's T. Terms of the form $\mathsf{if}_A\, t_1\, t_2\, t_3$ will be sometimes written in the far more legible form $\mathsf{if}\ t_1\ \mathsf{then}\ t_2\ \mathsf{else}\ t_3$. $\mathcal{T}$, which is formally described in Figure 1, also includes:

- two denumerable sets of constants of type U, namely $\top_0, \top_1, \top_2, \ldots$ and $\bot_0, \bot_1, \bot_2, \ldots$;
- two constants $\mathtt{tt}$ and $\mathtt{ff}$ of type $\mathtt{N} \to \mathsf{U}$: they transform numerals n into, respectively, the constants $\top_n$ and $\bot_n$; these are also the only constructs of the system that can generate constants of type U;
- the constant quote of type $\mathsf{U} \to \mathtt{N}$: quote takes as argument any constant $\top_n$ or $\bot_n$ and transform it into the numeral n. In other terms, quote takes a constant of type U and returns its Gödel number, which is its position in the enumeration. However – and this will be crucial in the following! – quote is not able to tell *from which* enumeration its argument comes from and it just returns its position n, which may thus refer to the ordering $\top_0, \top_1, \top_2, \ldots$ as well as the ordering $\bot_0, \bot_1, \bot_2, \ldots$. Therefore, quote is partially blind with respect to constants of type U: of its argument $\top_n$ or $\bot_n$, it sees only something like $?_n$ – i.e. the index n.

These non-standard features of $\mathcal{T}$ notwithstanding, the type U and all the constants $\mathtt{tt}, \mathtt{ff}, \mathsf{quote}$ are just syntactic sugar. Indeed, the type U can be encoded in Gödel's T as $\mathtt{Bool} \times \mathtt{N}$; then $\top_n$ and $\bot_n$ can be encoded respectively as $\langle \mathtt{True}, n\rangle$ and $\langle \mathtt{False}, n\rangle$; $\mathtt{tt}$ and $\mathtt{ff}$ can be encoded respectively as $\lambda x^{\mathtt{N}}\, \langle \mathtt{True}, x\rangle$ and $\lambda x^{\mathtt{N}}\, \langle \mathtt{False}, x\rangle$, and quote as the term $\lambda x^{\mathtt{Bool}\times\mathtt{N}}\, \pi_1(x)$. The typing rules for $\mathtt{tt}, \mathtt{ff}, \mathsf{quote}$ fully agree with the above encodings. So its clear that $\mathcal{T}$ is still a purely functional language; however, in order to be able to reason about it in a more refined way, we have found necessary to add the new type and constants as primitive constructs.

Types

$$\sigma, \tau ::= \mathtt{N} \mid \mathtt{Bool} \mid \mathsf{U} \mid \sigma \to \tau \mid \sigma \times \tau$$

Constants

$$\begin{gathered} \top_0, \top_1, \top_2, \ldots \\ \bot_0, \bot_1, \bot_2, \ldots \\ c ::= \mathsf{R}_\tau \mid \mathsf{if}_\tau \mid 0 \mid \mathsf{S} \mid \mathtt{True} \mid \mathtt{False} \mid \top_i \ (i \in \mathbb{N}) \mid \bot_i \ (i \in \mathbb{N}) \mid \mathsf{tt} \mid \mathsf{ff} \mid \mathsf{quote} \end{gathered}$$

Terms

$$t, u ::= \ c \mid x^\tau \mid tu \mid \lambda x^\tau u \mid \langle t, u\rangle \mid \pi_0 u \mid \pi_1 u$$

Typing Rules for Variables and Constants

$$\begin{gathered} x^\tau : \ \tau \mid 0 : \ \mathtt{N} \mid \mathsf{S} : \ \mathtt{N} \to \mathtt{N} \mid \mathtt{True} : \ \mathtt{Bool} \mid \mathtt{False} : \ \mathtt{Bool} \mid \\ \top_i : \mathsf{U} \text{ for every } i \in \mathbb{N} \ \mid \bot_i : \mathsf{U} \text{ for every } i \in \mathbb{N} \\ \mathsf{tt} : \mathtt{N} \to \mathsf{U} \mid \mathsf{ff} : \mathtt{N} \to \mathsf{U} \mid \ \mathsf{quote} \ : \mathsf{U} \to \mathtt{N} \\ \mathsf{if}_\tau : \ \mathtt{Bool} \to \tau \to \tau \to \tau \mid \mathsf{R}_\tau : \ \tau \to (\mathtt{N} \to (\tau \to \tau)) \to \mathtt{N} \to \tau \end{gathered}$$

Typing Rules for Composed Terms

$$\frac{t : \sigma \to \tau \qquad u : \sigma}{tu : \tau} \qquad \frac{u : \tau}{\lambda x^\sigma u : \sigma \to \tau} \qquad \frac{u : \sigma \qquad t : \tau}{\langle u, t\rangle : \sigma \times \tau} \qquad \frac{u : \tau_0 \times \tau_1}{\pi_i u : \tau_i}\ i \in \{0, 1\}$$

Reduction Rules All the usual reduction rules for simply typed lambda calculus (see Girard [8]) plus the rules for recursion, if-then-else and projections

$$\mathsf{R}_\tau uv0 \mapsto u \quad \mathsf{R}_\tau uv\mathsf{S}(t) \mapsto vt(\mathsf{R}_\tau uvt) \quad \mathsf{if}_\tau\, \mathtt{True}\, u\, v \mapsto u \quad \mathsf{if}_\tau\, \mathtt{False}\, u\, v \mapsto v \quad \pi_i\langle u_0, u_1\rangle \mapsto u_i, i = 0, 1$$

plus the following ones, assuming n be a numeral:

$$\begin{array}{ll} \mathsf{tt}n \mapsto \top_n & \mathsf{ff}n \mapsto \bot_n \\ \mathsf{quote}\, \top_m \to m & \mathsf{quote}\, \bot_m \to m \end{array}$$

Figure 1 The extension $\mathcal{T}$ of Gödel’s system T.

It is easy provable that $\mathcal{T}$ is strongly normalizing and has the uniqueness-of-normal-form property:

▶ **Theorem 1** (Strong Normalization and Weak Church-Rosser). *The system $\mathcal{T}$ enjoys strong normalization and weak-Church-Rosser (uniqueness of normal forms) for all closed terms of atomic types* $\mathtt{N}, \mathtt{Bool}$ *or* U.

Proof. By the translation of $\mathcal{T}$ into T. ◀

The following normal form theorem for $\mathcal{T}$ also holds.

▶ **Theorem 2** (Normal Form Property for $\mathcal{T}$). *Assume A is either an atomic type* $\mathtt{N}, \mathtt{Bool}, \mathsf{U}$ *or a product type. Then any closed normal term $t \in \mathcal{T}$ of type A is: a numeral $n : \mathtt{N}$, or a boolean* $\mathtt{True}, \mathtt{False} : \mathtt{Bool}$, *or a constant $\top_i : \mathsf{U}$, or a constant $\bot_i : \mathsf{U}$, or a pair $\langle u, v\rangle : B \times C$.*

Proof. As in Lemma 5 in [2]. ◀

From now onwards, for every pair of terms t, u of System $\mathcal{T}$, we shall write $t = u$ if they are the same term modulo the equality rules corresponding to the reduction rules of System $\mathcal{T}$ (equivalently, if they have the same normal form).

Finally, we define two sets of terms:

$$\mathbb{T} := \{t \mid t \text{ is a term of } \mathcal{T} \text{ and } t = \top_i \text{ for some } i \in \mathbb{N}\}$$

and

$$\perp\!\!\!\perp := \{t \mid t \text{ is a term of } \mathcal{T} \text{ and } t = \bot_i \text{ for some } i \in \mathbb{N}\}\,.$$

2.1 Language of $\mathsf{HA}^\omega + \mathsf{MP}$

We now define the language of the arithmetical theory $\mathsf{HA}^\omega + \mathsf{MP}$.

▶ **Definition 3** (Language of $\mathsf{HA}^\omega + \mathsf{MP}$)**.** The language $\mathcal{L}$ of $\mathsf{HA}^\omega + \mathsf{MP}$ is defined as follows.

1. The terms of $\mathcal{L}$ are all $t \in \mathcal{T}$.
2. The atomic formulas of $\mathcal{L}$ are all $Q \in \mathcal{T}$ such that $Q : \mathtt{Bool}$.
3. The formulas of $\mathcal{L}$ are built from atomic formulas of $\mathcal{L}$ by the connectives $\vee, \wedge, \rightarrow \forall, \exists$ as usual, with quantifiers possibly ranging over variables x^τ, y^τ, z^τ of arbitrary finite type τ of T.

We denote with $\bot$ the atomic formula $\mathtt{False}$. With $P^\bot$ we denote the complement of the predicate P, that is, $\mathsf{if}\ P\ \mathsf{then}\ \mathtt{False}\ \mathsf{else}\ \mathtt{True}$. If P is an atomic formula of $\mathcal{L}$ in the free variables $x_1^{\tau_1}, \ldots, x_n^{\tau_n}$ and $t_1 : \tau_1, \ldots, t_n : \tau_n$ are terms of $\mathcal{L}$, with $P(t_1, \ldots, t_n)$ we shall denote the atomic formula $P[t_1/x_1, \ldots, t_n/x_n]$.

3 Realizability

For every formula A of $\mathcal{L}$, we are now going to define what type $|A|$ realizers of A must have.

▶ **Definition 4** (Types for realizers)**.** For each formula A of $\mathcal{L}$ we define a type $|A|$ of $\mathcal{T}$ by induction on A:

$$|P| = \mathtt{U} \text{ if } P \text{ is atomic} \qquad |A \wedge B| = |A| \times |B| \qquad |A \rightarrow B| = |A| \rightarrow |B|$$
$$|A \vee B| = \mathtt{Bool} \times (|A| \times |B|) \qquad |\forall x^\tau A| = \tau \rightarrow |A| \qquad |\exists x^\tau A| = \tau \times |A|$$

We remark that any HA^ω term of type $|A|$, by definition, can be taken to represent an arbitrary strategy for Eloise in the Hintikka-Tarski game for A. For example, a term

$$t : |\forall x^\tau A| = \tau \rightarrow |A|$$

takes a move $u : \tau$ by Abelard, corresponding to the game $A[u/x^\tau]$, and gives Eloise the strategy tu to follow for the continuation. A term

$$t : |\exists x^\tau A| = \tau \times |A|$$

gives Eloise a move to play, $\pi_0 t = u$, and a strategy $\pi_1 t$ for continuing the game $A[u/x^\tau]$. For precise definitions of Hintikka-Tarski games and strategies we refer to [1]. In this paper, however, we do not need to examine these concepts in further detail, because game semantical notions will be just used as guidelines to understand intuitively the realizability that we are going to introduce.

Let now $\mathsf{p}_0 := \pi_0 : \sigma_0 \times (\sigma_1 \times \sigma_2) \rightarrow \sigma_0$, $\mathsf{p}_1 := \pi_0\pi_1 : \sigma_0 \times (\sigma_1 \times \sigma_2) \rightarrow \sigma_1$ and $\mathsf{p}_2 := \pi_1\pi_1 : \sigma_0 \times (\sigma_1 \times \sigma_2) \rightarrow \sigma_2$ be the three canonical projections from $\sigma_0 \times (\sigma_1 \times \sigma_2)$.

We define the realizability relation $t \Vdash F$, where $t \in \mathcal{T}$ and F is a formula:

▶ **Definition 5** (Realizability)**.** For each closed formula F and closed term $t : |F|$ of System $\mathcal{T}$, we define a relation $t \Vdash F$ of HA^ω by induction on F as follows:

1. $t \Vdash Q$ if and only if ($Q = \mathtt{True}$ and $t \in \mathbb{T}$) or ($Q = \mathtt{False}$ and $t \in \mathbb{\bot}$) for Q atomic formula;
2. $t \Vdash A \wedge B$ if and only if $\pi_0 t \Vdash A$ and $\pi_1 t \Vdash B$;
3. $t \Vdash A \vee B$ if and only if $\mathsf{p}_0 t = \mathtt{True}$ and $\mathsf{p}_1 t \Vdash A$ or $\mathsf{p}_0 t = \mathtt{False}$ and $\mathsf{p}_2 t \Vdash B$;
4. $t \Vdash A \rightarrow B$ if and only if for all u, if $u \Vdash A$, then $tu \Vdash B$;
5. $t \Vdash \exists x^\tau A$ if and only if $\pi_0 t = u$ for $u : \tau$ closed term of HA^ω and $\pi_1 t \Vdash A[u/x]$;
6. $t \Vdash \forall x^\tau A$ if and only if for all closed term $u : \tau$ of HA^ω, $tu \Vdash A[u/x]$.

We remark that the clauses 2–6 of our realizability relation coincide exactly with those of modified realizability for the corresponding formulas. Our definition tweaks modified realizability in two other ways. Firstly, instead of considering Gödel’s T as canonical term model, we take $\mathcal{T}$. Secondly, we modify in a crucial way the realizability condition for atomic formulas. In modified realizability P is realizable by any term if it is true, while not realizable if it is false; in our case, a realizer of P is a term which just computes the truth value of P and returns it under the form of a constant belonging to $\mathbb{T}$ or to $\perp\!\!\!\perp$.

In game semantical language, a realizer of P just determines the outcome of the Hintikka-Tarski game for P, returning a constant belonging to $\mathbb{T}$ or to $\perp\!\!\!\perp$ according as to whether Eloise or Abelard wins. The intuition is that, as anticipated in the introduction, we want arbitrary strategies to realize formulas. This forces atomic formulas to be realizable regardless of their truth value, and we just need the truth value to be reflected by realizers. Of course, realizer coming from proofs will have an extra condition that will prevent them from realizing false formulas, as we shall soon see. We shall also show that any closed formula of HA^ω is realizable: any strategy t for A can be mapped into a realizer t_A of A which follows the strategy t. All that implies a crucial change in the meaning with respect to modified realizability also for implication. Since arbitrary strategies can be turned into realizers, a realizer of $A \to B$ will map not only winning strategies for A into winning strategies for B, but also realizers/arbitrary-strategies for A into realizers/arbitrary-strategies for B.

▶ **Definition 6** (Translation of Arbitrary Strategies). Let A be any formula and $t : |A|$ any term of HA^ω containing all the free variables of A. We define by induction on A a term t_A of $\mathcal{T}$ with free variables containing those of A:

- If P is atomic, then

$$t_P := \mathsf{if}\ P\ \mathsf{then}\ \top_0\ \mathsf{else}\ \perp_0$$

- $t_{A\wedge B} := \langle (\pi_0 t)_A, (\pi_1 t)_B \rangle \qquad t_{A\vee B} := \langle \mathsf{p}_0 t, (\mathsf{p}_1 t)_A, (\mathsf{p}_2 t)_B \rangle \qquad t_{A\to B} := \lambda x^{|A|}.\,(tx)_B$
- $t_{\forall x^\tau A} := \lambda x^\tau.\,(tx)_A \qquad t_{\exists x^\tau A} := \langle \pi_0 t, (\pi_1 t)_{A[\pi_0 t/x^\tau]} \rangle$

where x is fresh.

▶ **Proposition 7** (Arbitrary Strategies and Realizability). *Let A be any closed formula and $t : |A|$ any closed term of HA^ω. Then*

$$t_A \Vdash A$$

Proof. We proceed by induction on A. We cover only few representative cases, the others being similar.

1. $A = P$, with P atomic. Then

$$t_P := \mathsf{if}\ P\ \mathsf{then}\ \top_0\ \mathsf{else}\ \perp_0$$

   Now, if $P = \mathtt{True}$, then $t_P = \top_0 \in \mathbb{T}$, so $t_P \Vdash P$; if $P = \mathtt{False}$, then $t_P = \perp_0 \in \perp\!\!\!\perp$, so $t_P \Vdash P$.
2. $A = B \to C$. Then

$$t_A := \lambda x^{|B|}.\,(tx)_C$$

   Now, suppose $u \Vdash B$. We have to show $t_A u \Vdash C$. But it is easy to see that

$$t_A u = (tx)_C[u/x] = (tu)_C$$

   and by inductive hypothesis $(tu)_C \Vdash C$. We thus conclude by Lemma 9 that $t_A \Vdash A$.

3. $A = \exists x^\tau B$. Then
$$t_A := \langle \pi_0 t, (\pi_1 t)_{B[\pi_0 t/x^\tau]} \rangle$$
Since by inductive hypothesis
$$(\pi_1 t)_{B[\pi_0 t/x^\tau]} \Vdash B[\pi_0 t/x^\tau]$$
we conclude by Lemma 9 that $t_A \Vdash A$.

◀

In the following, we will focus on a particular class of terms, called *proof-like*. These are the terms that are extracted from the actual proofs, and that neither contain any constant from the set $\perp\!\!\!\perp$ nor have the possibility of generating them with a constant ff.

▶ **Definition 8** (Proof-like Terms)**.** A proof-like term is a term t of $\mathcal{T}$ which does not contain constants of the form $\perp_i$ $(i \in \mathbb{N})$ or ff.

In the following, the "true" realizers will be proof-like terms. They actually represent winning strategies, that is, they carry sound constructive information about the formula they realize.

The concept of proof-like realizer is also crucial to determine a meaningful interaction between strategies in the definition of realizability for implication. For instance, suppose that some proof-like term t realizes a formula $A \to B$, where for simplicity A and B are $\to$-free. Let u be a realizer of A. Then tu must realize B. Since tu is not necessarily proof-like, tu may not represent a winning strategy for B. For example, assume $B = \forall x^{\mathtt{N}}\, \exists y^{\mathtt{N}}\, P(x, y)$; then there could be a numeral n such that if we let $m = \pi_0(tun)$, then $P(n, m) = \mathtt{False}$. n is a *test* that refutes the realizer tu, when seen as a strategy for B. Now, the term $\pi_1(tun)$, which realizes $P(n, m)$, must reduce to a constant in $\perp\!\!\!\perp$. Since t is proof-like, such a constant must be produced by the term u in the reduction of $\pi_1(tun)$; namely, a test must be produced that refutes u as well, when seen a strategy. For example, if $A = \exists x^{\mathtt{N}}\, \forall y^{\mathtt{N}} Q(x, y)$, in the reduction of $\pi_1(tun)$, $\pi_1 u$ must be applied to some numeral j such that $\pi_0 u = i$ and $Q(i, j) = \mathtt{False}$. In that case, a constant in $\perp\!\!\!\perp$ is produced, and it may actually be the constant which is the normal form of $\pi_1(tun)$.

We point out that this behaviour of realizers of implications is analogous to that of terms witnessing the Dialectica interpretation of implications.

The next Lemma tells that realizability respects the notion of equality of $\mathcal{T}$ terms: if two terms can be proved equal in $\mathcal{T}$, then then they realize the same formulas.

▶ **Lemma 9.** *If $t_1 = t_2$ and $u_1 = u_2$ are valid in $\mathcal{T}$, then $t_1 \Vdash A[u_1/x]$ if and only if $t_2 \Vdash A[u_2/x]$ for each formula A.*

Proof. By induction on the formula A. ◀

4 Extensionality

Proving that Markov's Principle is realizable by a proof-like term is by no means trivial. The goal of this section is to introduce a key tool that will let us describe an important kind of extensionality property of System $\mathcal{T}$. Afterwards, we shall be able to reason in a more sophisticated way about terms of $\mathcal{T}$, and in particular about the realizer of MP that we shall propose.

A basic feature of typed functional lambda calculi is extensionality: in concrete computations, there is no way to discriminate syntactically different terms if, denotationally, they represent the same function. For example, suppose t and u are two terms of T of type $\mathtt{N}^2 \to \mathtt{N}$ implementing in different way the addition function. For instance, t may perform recursion on the first argument and u on the second. The two terms represent the same function, but they are syntactically different normal forms. Nevertheless, any term $\Psi : (\mathtt{N}^2 \to \mathtt{N}) \to \mathtt{N}$ of T will not be able to discriminate t and u: Ψt and Ψu will convert to the same numeral.

Another characteristic of typed lambda calculi is the impossibility of distinguishing different *mute* constants, which are the constants whose associated reduction rules cannot leak any information about their shape. If we take a term t and permute its mute constants obtaining t', the normal form of t' can be obtained from the normal form of t by the same permutation of constants. To put it differently, mute constants can be moved around and duplicated inside a term, but they have no influence whatsoever on the evolution of the computation. Now, while the constants $\mathtt{True}, \mathtt{False}, \mathsf{S}, 0$ can be discriminated (by if and R), the constants of the form $\top_n, \bot_n$ do not. They are not completely mute since their indexes can be recognized by $\mathtt{quote}$, but their main form $(\bot, \top)$ cannot be determined by any reduction rule.

All these considerations lead us to the concept of extensionality modulo a relation $\mathcal{R}$ over the base type U. Here, $\mathcal{R}$ relates terms which should be regarded as *almost*, or *observationally*, equal. If we take the usual definition of extensionality and, instead of fixing it to be equality at type U, we let it to be $\mathcal{R}$, we determine a more flexible concept of extensionality, relating objects which can well be different, but cannot be computationally distinguished. Now, let us consider any reflexive binary relation $\mathcal{R}$ between closed terms of type U of $\mathcal{T}$. $\mathcal{R}$ is said to be *saturated with respect to equality* if for every t_1, t_2, u_1, u_2, if $t_1 \,\mathcal{R}\, t_2$ and $t_1 = u_1$ and $t_2 = u_2$, then $u_1 \,\mathcal{R}\, u_2$.

▶ **Definition 10** (Extensionality Modulo a Relation). Let t and u two closed terms of $\mathcal{T}$ of type ρ and $\mathcal{R}$ a reflexive relation between closed terms of type U of $\mathcal{T}$ saturated with respect to equality. We define the *extensionality relation* $t \sim_{\mathcal{R}} u$ by induction on the type ρ:

- If $\rho = \mathsf{U}$, then $t \sim_{\mathcal{R}} u$ if and only if $t \,\mathcal{R}\, u$;
- If $\rho = \mathtt{N}$, then $t \sim_{\mathcal{R}} u$ if and only if $t = u$;
- If $\rho = \mathtt{Bool}$, then $t \sim_{\mathcal{R}} u$ if and only if $t = u$;
- If $\rho = \tau \to \sigma$, then $t \sim_{\mathcal{R}} u$ if and only if $\forall v : \tau \, \forall w : \tau.\ v \sim_{\mathcal{R}} w$ implies $tv \sim_{\mathcal{R}} uw$;
- If $\rho = \tau \times \sigma$, then $t \sim_{\mathcal{R}} u$ if and only if $\pi_0 t \sim_{\mathcal{R}} \pi_0 u$ and $\pi_1 t \sim_{\mathcal{R}} \pi_1 u$.

Intuitively, a closed term t of $\mathcal{T}$ is extensional modulo $\mathcal{R}$ if $t \sim_{\mathcal{R}} t$. Let us now prove that the relation $\sim_{\mathcal{R}}$ as well is saturated with respect to equality.

▶ **Lemma 11.** *Given u_1, u_2, t_1, t_2 closed terms of $\mathcal{T}$ of type σ, suppose $u_1 = t_1$, $u_2 = t_2$ and $u_1 \sim_{\mathcal{R}} u_2$. Then $t_1 \sim_{\mathcal{R}} t_2$.*

Proof. By induction on the type σ.

- $\sigma = \mathsf{U}$: the thesis follows by saturation of the relation $\mathcal{R}$.
- $\sigma = \mathtt{N}$ or $\rho = \mathtt{Bool}$: by Definition 10, $u_1 = u_2$, so $t_1 = t_2$ and we conclude $t_1 \sim_{\mathcal{R}} t_2$.
- $\sigma = \rho \to \tau$. Let us consider any pair of terms $r : \rho$ and $s : \rho$ such that $r \sim_{\mathcal{R}} s$. By Definition 10 of the extensionality relation and by the fact that $u_1 \sim_{\mathcal{R}} u_2$, it holds that $u_1 r \sim_{\mathcal{R}} u_2 s$. Now, we can apply the inductive hypothesis to the type τ of the terms $u_1 r$ and $u_2 s$: since $u_1 r = t_1 r$ and $u_2 s = t_2 s$, we have $t_1 r \sim_{\mathcal{R}} t_2 s$. Therefore, by Definition 10, $t_1 \sim_{\mathcal{R}} t_2$.
- $\sigma = \rho \times \tau$. By Definition 10, $u_1 \sim_{\mathcal{R}} u_2$ implies that $\pi_0 u_1 \sim_{\mathcal{R}} \pi_0 u_2$ and $\pi_1 u_1 \sim_{\mathcal{R}} \pi_1 u_2$. Since $\pi_0 u_1 = \pi_0 t_1, \pi_0 u_2 = \pi_0 t_2, \pi_1 u_1 = \pi_1 t_1, \pi_1 u_2 = \pi_1 t_2$, by applying the inductive hypothesis on the types ρ and τ one has that $\pi_0 t_1 \sim_{\mathcal{R}} \pi_0 t_2$ and $\pi_1 t_1 \sim_{\mathcal{R}} \pi_1 t_2$. Thus, by Definition 10, $t_1 \sim_{\mathcal{R}} t_2$. ◀

The following proposition says that any closed term of $\mathcal{T}$ in which `quote` does not occur, is extensional modulo $\mathcal{R}$, where $\mathcal{R}$ is any reflexive binary relation between terms. The proof of the extensionality of the constant `quote` requires instead the definition of a particular relation $\mathcal{R}$ and will be formalized in Lemma 14. Since $\sim_{\mathcal{R}}$ can be seen as a *logical relation*, in the sense of Plotkin, our proposition can be seen as yet another incarnation of the usual Fundamental Theorem of logical relations (see e.g. [16]).

▶ **Proposition 12** (Extensionality). *Let t be a term of $\mathcal{T}$ with free variables among $x_1, \ldots, x_k$ and assume that the constant* `quote` *does not occur in t. If $u_1, \ldots, u_k, v_1, \ldots, v_k$ are closed terms of $\mathcal{T}$ such that $u_1 \sim_{\mathcal{R}} v_1, \ldots, u_k \sim_{\mathcal{R}} v_k$, then $t[u_1/x_1 \ldots u_k/x_k] \sim_{\mathcal{R}} t[v_1/x_1 \ldots v_k/x_k]$.*

Proof. By induction on the structure of t.

1. t is a variable x_i for some $i \in [1, k]$. Trivially, $x_i[u_1/x_1 \ldots u_k/x_k] = u_i \sim_{\mathcal{R}} v_i = x_i[v_1/x_1 \ldots v_k/x_k]$.
2. t is an application $t_1 t_2$. Suppose $t_1 : \tau \to \sigma$ and $t_2 : \tau$. By inductive hypothesis, one has $t_1[u_1/x_1 \ldots u_k/x_k] \sim_{\mathcal{R}} t_1[v_1/x_1 \ldots v_k/x_k]$ and $t_2[u_1/x_1 \ldots u_k/x_k] \sim_{\mathcal{R}} t_2[v_1/x_1 \ldots v_k/x_k]$. By Definition 10 of the extensionality relation
$$t_1[u_1/x_1 \ldots u_k/x_k]t_2[u_1/x_1 \ldots u_k/x_k] \sim_{\mathcal{R}} t_1[v_1/x_1 \ldots v_k/x_k]t_2[v_1/x_1 \ldots v_k/x_k]$$
which is to say
$$t_1 t_2[u_1/x_1 \ldots u_k/x_k] \sim_{\mathcal{R}} t_1 t_2[v_1/x_1 \ldots v_k/x_k]\,.$$
3. t is $\lambda z^\sigma w$. Let us consider any two terms r_1, r_2 of type σ such that $r_1 \sim_{\mathcal{R}} r_2$. By inductive hypothesis, it holds that
$$w[u_1/x_1 \ldots u_k/x_k\ r_1/z] \sim_{\mathcal{R}} w[v_1/x_1 \ldots v_k/x_k\ r_2/z]\,.$$
Since
$$(\lambda z^\sigma w)[u_1/x_1 \ldots u_k/x_k]r_1 = w[u_1/x_1 \ldots u_k/x_k\ r_1/z]$$
$$(\lambda z^\sigma w)[v_1/x_1 \ldots v_k/x_k]r_2 = w[v_1/x_1 \ldots v_k/x_k\ r_2/z]$$
by Lemma 11 we obtain
$$(\lambda z^\sigma w)[u_1/x_1 \ldots u_k/x_k]r_1 \sim_{\mathcal{R}} (\lambda z^\sigma w)[v_1/x_1 \ldots v_k/x_k]r_2$$
and thus the thesis.
4. t is a pair $\langle t_1, t_2\rangle$. Then, for $i = 0, 1$, by induction hypothesis
$$\pi_i(t[u_1/x_1 \ldots u_k/x_k]) = t_i[u_1/x_1 \ldots u_k/x_k] \sim_{\mathcal{R}} t_i[v_1/x_1 \ldots v_k/x_k] = \pi_i(t[v_1/x_1 \ldots v_k/x_k])$$
and thus by Lemma 11 we obtain
$$\pi_i(t[u_1/x_1 \ldots u_k/x_k]) \sim_{\mathcal{R}} \pi_i(t[v_1/x_1 \ldots v_k/x_k])$$
and thus the thesis.
5. t is $\pi_i w$, $i = 0, 1$. By inductive hypothesis,
$$w[u_1/x_1 \ldots u_k/x_k] \sim_{\mathcal{R}} w[u_1/x_1 \ldots u_k/x_k]$$
and by Definition 10 of extensionality we have the thesis.
6. t is a constant such as 0 : `N`, `True` : `Bool`, `False` : `Bool`: we conclude $t \sim_{\mathcal{R}} t$ by Definition 10.

7. t is the constant $\mathsf{S} : \mathtt{N} \to \mathtt{N}$. Given two terms $w_1, w_2 : \mathtt{N}$ such that $w_1 \sim_{\mathcal{R}} w_2$, by definition of extensionality relation, $w_1 = w_2$. Then clearly $\mathsf{S}w_1 \sim_{\mathcal{R}} \mathsf{S}w_2$ and we obtain the thesis by Definition 10.
8. t is $\bot_i$, $\top_i$ $(i \in \mathbb{N})$: $\bot_i \sim_{\mathcal{R}} \bot_i$ and $\top_i \sim_{\mathcal{R}} \top_i$ follows by reflexivity of the relation $\mathcal{R}$.
9. t is the constant $\mathsf{tt} : \mathtt{N} \to \mathsf{U}$. Let us consider two terms w_1 and w_2 of type $\mathtt{N}$ such that $w_1 \sim_{\mathcal{R}} w_2$. By Definition 10, $w_1 = w_2$, i.e they have the same numeral, say m, as normal form. Therefore, $\mathsf{tt}w_1 = \top_m \sim_{\mathcal{R}} \top_m = \mathsf{tt}w_2$ and, by Lemma 11 and definition of the extensionality relation, $\mathsf{tt} \sim_{\mathcal{R}} \mathsf{tt}$.
10. t is the constant ff: as for the previous case.
11. t is the constant if_τ. Let us consider $r_1 : \mathtt{Bool}, r_2 : \tau, r_3 : \tau$ and $s_1 : \mathtt{Bool}, s_2 : \tau, s_3 : \tau$ terms of $\mathcal{T}$ such that $r_1 \sim_{\mathcal{R}} s_1$, $r_2 \sim_{\mathcal{R}} s_2$ and $r_3 \sim_{\mathcal{R}} s_3$. We want to prove that $\mathsf{if}_\tau\, r_1\, r_2\, r_3 \sim_{\mathcal{R}} \mathsf{if}_\tau\, s_1\, s_2\, s_3$. By Definition 10, $r_1 \sim_{\mathcal{R}} s_1$ implies that $r_1 = s_1$, i.e. r_1 and s_1 both reduces to either $\mathtt{True}$ or $\mathtt{False}$.
    There are two cases, according to the normal form of r_1 and s_1. If $r_1 = s_1 = \mathtt{True}$, then $\mathsf{if}_\tau\, r_1\, r_2\, r_3 = r_2 \sim_{\mathcal{R}} s_2 = \mathsf{if}_\tau\, s_1\, s_2\, s_3$ and the thesis follows by Lemma 11. If $r_1 = s_1 = \mathtt{False}$: symmetric to the previous case.
12. t is the constant R_τ. Let us consider $r_1, s_1 : \tau, r_2, s_2 : \mathtt{N} \to (\tau \to \tau), r_3, s_3 : \mathtt{N}$ terms of $\mathcal{T}$ such that $r_1 \sim_{\mathcal{R}} s_1$, $r_2 \sim_{\mathcal{R}} s_2$ and $r_3 \sim_{\mathcal{R}} s_3$. We want to prove that $\mathsf{R}_\tau\, r_1\, r_2\, r_3 \sim_{\mathcal{R}} \mathsf{R}_\tau\, s_1\, s_2\, s_3$. By Definition 10, $r_3 \sim_{\mathcal{R}} s_3$ implies that $r_3 = s_3$ and therefore r_3 and s_3 reduce to the same numeral: we argue by induction on it. If $r_3 = s_3 = 0$, then $\mathsf{R}_\tau\, r_1\, r_2\, 0 = r_1 \sim_{\mathcal{R}} s_1 = \mathsf{R}_\tau\, s_1\, s_2\, s_3$ and one can conclude by Lemma 11. If $r_3 = s_3 = \mathsf{S}(m)$, then
    $$\mathsf{R}_\tau\, r_1\, r_2\, r_3 = \mathsf{R}_\tau\, r_1\, r_2\, \mathsf{S}(m) = r_2\, m\, (\mathsf{R}_\tau\, r_1\, r_2\, m)$$
    $$\mathsf{R}_\tau\, s_1\, s_2\, s_3 = \mathsf{R}_\tau\, s_1\, s_2\, \mathsf{S}(m) = s_2\, m\, (\mathsf{R}_\tau\, s_1\, s_2\, m)$$
    By induction hypothesis $\mathsf{R}_\tau r_1\, r_2\, m \sim_{\mathcal{R}} \mathsf{R}_\tau s_1\, s_2\, m$ and Definition 10, $r_2\, m\, (\mathsf{R}_\tau\, r_1\, r_2\, m) \sim_{\mathcal{R}} s_2\, m\, (\mathsf{R}_\tau\, s_1\, s_2\, m)$ and the thesis follows by Lemma 11.

◀

▶ **Corollary 13.** *Let t be any closed term of $\mathcal{T}$. If* $\mathsf{quote} \sim_{\mathcal{R}} \mathsf{quote}$, $t \sim_{\mathcal{R}} t$.

Proof. Clearly, for some fresh variable $z : \mathsf{U}$, $t = (t[z/\,\mathsf{quote}\,])[\,\mathsf{quote}\,/z]$. Thus, by Proposition 12 applied to $t[z/\,\mathsf{quote}\,]$, we obtain $t \sim_{\mathcal{R}} t$. ◀

In Section 6 we will prove that every theorem in $\mathsf{HA}^\omega + \mathsf{MP}$ is realizable and in particular that a *proof-like* realizer $\mathfrak{r}$ of Markov's Principle $\neg\forall x^{\mathtt{N}} P \to \exists x^{\mathtt{N}} P^\bot$ can be defined. In this case, the extensionality relation plays a crucial role. Our realizer $\mathfrak{r}$ of Markov's Principle will have to map a realizer of $\neg\forall x^{\mathtt{N}} P$ into a realizer of $\exists x^{\mathtt{N}} P^\bot$. In other words, given a realizer of $\neg\forall x^{\mathtt{N}} P$, $\mathfrak{r}$ must in some way extract from it either a counterexample for $\forall x^{\mathtt{N}} P$ to be used as a witness of $\exists x^{\mathtt{N}} P^\bot$, or a constant in $\perp\!\!\!\perp$, by which one can realize everything.

So let us examine a realizer of $\forall x^{\mathtt{N}} P \to \bot$. It takes as input a realizer of $\forall x^{\mathtt{N}} P$ and returns a realizer of $\bot$. A tentative first plan to define $\mathfrak{r}$ may thus be to construct a realizer of $\forall x^{\mathtt{N}} P$ in order to obtain a realizer of $\bot$, that is, a constant in $\perp\!\!\!\perp$. A realizer of $\forall x^{\mathtt{N}} P$ is indeed easily definable in $\mathcal{T}$ as follows:

$$\mathbf{test}_{\lambda x.P} ::= \lambda x^{\mathtt{N}}.\ \mathsf{if}\ P\ \mathsf{then}\ \ \mathsf{tt}x\ \mathsf{else}\ \ \mathsf{ff}x$$

It behaves the expected way: when fed with a numeral m it evaluates $P[m/x]$ yielding $\top_m$ if $P[m/x] = \mathtt{True}$ and $\bot_m$ if $P[m/x] = \mathtt{False}$.

Thus we are done... aren't we? Unfortunately, no. Clearly, $\mathbf{test}_{\lambda x.P}$ is not proof-like, since it contains the subterm ff and so it may evaluate to $\bot_i$ for some numeral i. As previously said, only proof-like terms will be considered realizers/winning strategies and $\mathfrak{r}$ is forbidden to contain a term such as $\mathbf{test}_{\lambda x.P}$.

We have thus to formulate a new plan for constructing $\mathfrak{r}$. The idea is to use extensionality. We want to alter $\mathbf{test}_{\lambda x.P}$ in such a way that it behaves extensionally as before but at the same time it is proof-like! With that in mind, we modify the term $\mathbf{test}_{\lambda x.P}$ like this:

$$\mathbf{mtest}_{\lambda x.P} ::= \lambda x^{\mathtt{N}}.\mathsf{if}\ P\ \mathsf{then}\ \mathbb{tt}x\ \mathsf{else}\ \mathbb{tt}x$$

While that may appear like a crazy attempt, it works. The term $\mathbf{mtest}_{\lambda x.P}$ is indeed proof-like, and differs from $\mathbf{test}_{\lambda x.P}$ only for the fact that it returns a constant in $\mathbb{T}$ also when $P[m/x]$ is false. That would be a great difference in another situation, but here it is not the case: $\mathbf{test}_{\lambda x.P}$ and $\mathbf{mtest}_{\lambda x.P}$ are equal up to a subterm of the form $\mathbb{tt}x$ or $\mathbb{ff}x$, which yields mute constants – constants that cannot be discriminated by any term in $\mathcal{T}$. In other words, $\mathbf{mtest}_{\lambda x.P}$ behaves observationally, i.e. extensionally, like $\mathbf{test}_{\lambda x.P}$, provided the relation $\mathcal{R}$ is suitable chosen.

In order to prove that $\mathbf{mtest}_{\lambda x.P} \sim_{\mathcal{R}} \mathbf{test}_{\lambda x.P}$, $\mathcal{R}$ will be defined to hold either on pairs of equal terms (and this captures the case in which the evaluation of P on the given input n yields $\mathtt{True}$ and both $\mathbf{mtest}_{\lambda x.P}$ and $\mathbf{test}_{\lambda x.P}$ evaluates to $\mathbb{tt}n$) or on pair of discordant constants $(\top_k, \bot_k)$, where the index k is a numeral such that $P[k/x] = \mathtt{False}$. These constants are considered to be "equal" by the terms of our system and their index k is a counterexample to the formula $\forall x^{\mathtt{N}} P$ and therefore a correct witness for $\exists x^{\mathtt{N}} P^{\bot}$. Notice that k can be extracted both from $\top_k$ and $\bot_k$ by the constant quote, which is not able to produce any information about the argument but the index itself. The same constant quote is extensional modulo the relation $\mathcal{R}$ just introduced. All these notions are formalized in the following lemma:

▶ **Lemma 14** (Test Equivalence)**.** *Let us consider the terms* $\mathbf{mtest}_{\lambda x.P}$ *and* $\mathbf{test}_{\lambda x.P}$ *defined above and the saturated-with-respect-to-equality relation*

$$\mathcal{R} ::= \{(t_1, t_2) \mid t_1 = t_2 \ \textit{or}\ (t_1 = \top_k, t_2 = \bot_k \ \textit{and}\ P[k/x] = \mathtt{False}\ \textit{for some numeral}\ k)\}$$

where we assume that the only free variable of P *is* x*. Then:*

1. $\mathbf{mtest}_{\lambda x.P} \sim_{\mathcal{R}} \mathbf{test}_{\lambda x.P}$
2. $\mathsf{quote} \sim_{\mathcal{R}} \mathsf{quote}$

Proof.

1. Let us consider two closed term $s : \mathtt{N}$ and $r : \mathtt{N}$ such that $s \sim_{\mathcal{R}} r$. By Theorem 2 and by Definition 10, s and r reduce to the same numeral, say $n : \mathtt{N}$. We want to prove that $\mathbf{mtest}_{\lambda x.P}\, n \sim_{\mathcal{R}} \mathbf{test}_{\lambda x.P}\, n$.
   Two cases occur:
   - $P[n/x] = \mathtt{True}$. Then $\mathbf{mtest}_{\lambda x.P}\, n = \mathbb{tt}n = \top_n = \mathbb{tt}n = \mathbf{test}_{\lambda x.P}\, n$. By definition of $\mathcal{R}$, $\mathbf{mtest}_{\lambda x.P}\, n \ \mathcal{R}\ \mathbf{test}_{\lambda x.P}\, n$, which is to say $\mathbf{mtest}_{\lambda x.P}\, n \sim_{\mathcal{R}} \mathbf{test}_{\lambda x.P}\, n$.
   - $P[n/x] = \mathtt{False}$. Then

   $$\mathbf{mtest}_{\lambda x.P}\, n = \mathbb{tt}n = \top_n \sim_{\mathcal{R}} \bot_n = \mathbb{ff}n = \mathbf{test}_{\lambda x.P}\, n$$

   Therefore, by Lemma 11 $\mathbf{mtest}_{\lambda x.P}\, n \sim_{\mathcal{R}} \mathbf{test}_{\lambda x.P}\, n$.

   Finally, by Definition 10 and Lemma 11, one can conclude $\mathbf{mtest}_{\lambda x.P} \sim_{\mathcal{R}} \mathbf{test}_{\lambda x.P}$.

2. Let us consider two terms u_1 and u_2 of type U such that $u_1 \sim_{\mathcal{R}} u_2$. By Theorem 2 and by Definition 10:
   - either $u_1 = u_2$, and clearly $\mathsf{quote}\, u_1 = \mathsf{quote}\, u_2$ and, by Definition 10, $\mathsf{quote}\, u_1 \sim_{\mathcal{R}} \mathsf{quote}\, u_2$;
   - or $u_1 = \top_k$, $u_2 = \bot_k$ for some k and $P[k/x] = \texttt{False}$. Also in this case $\mathsf{quote}\, u_1 = \mathsf{quote}\, u_2 = k$ and, by Definition 10, $\mathsf{quote}\, u_1 \sim_{\mathcal{R}} \mathsf{quote}\, u_2$.

   Finally, by Definition 10 and Lemma 11, one can conclude $\mathsf{quote} \sim_{\mathcal{R}} \mathsf{quote}$. ◀

5 A Realizer of Markov’s Principle

We are now ready to define the realizer $\mathfrak{r}$ of Markov’s Principle. $\mathfrak{r}$ takes as argument a realizer z of $\neg\forall x^{\mathtt{N}} P$ and we want $\mathfrak{r}$ to pass the term $\mathbf{mtest}_{\lambda x.P}$ as argument to z. Of course, $\mathbf{mtest}_{\lambda x.P}$ is not a realizer of $\forall x^{\mathtt{N}} P$, which is required in order to obtain with certitude a realizer of $\bot$ from z. However, it is extensionally equal to the realizer $\mathbf{test}_{\lambda x.P}$, which is enough. Now, let us consider $z\,\mathbf{mtest}_{\lambda x.P}$. The informal reasoning is the following (for a detailed argument see the proof of the Adequacy Theorem 15 or come back after having read it for intuitive explanations of the formal details). $z\,\mathbf{mtest}_{\lambda x.P}$ is extensionally equal to $z\,\mathbf{test}_{\lambda x.P}$, for $\mathcal{R}$ chosen as in Lemma 14, and $z\,\mathbf{test}_{\lambda x.P}$ must normalize to a constant in $\perp\!\!\!\perp$, say $\bot_k$. That constant is ultimately generated either by $\mathbf{test}_{\lambda x.P}$ or already by z. In this latter case, also $z\,\mathbf{mtest}_{\lambda x.P}$ will be able to produce $\bot_k$, and we are done, we can realize everything. In the former case, $z\,\mathbf{mtest}_{\lambda x.P}$ should reduce to $\top_k$, with k witness for $\exists x^{\mathtt{N}} P^{\perp}$, because $z\,\mathbf{mtest}_{\lambda x.P}\ \mathcal{R}\ z\,\mathbf{test}_{\lambda x.P}$; then k can be extracted by quote applied to $z\,\mathbf{mtest}_{\lambda x.P}$.

For those reasons, we are lead to define $\mathfrak{r}$ as:

$$\lambda z^{(\mathtt{N}\to\mathsf{U})\to\mathsf{U}} \langle\, \mathsf{quote}\,(z\,\mathbf{mtest}_{\lambda x.P}), \mathsf{if}\ P^{\perp}[\mathsf{quote}\,(z\,\mathbf{mtest}_{\lambda x.P})/x]\ \mathsf{then}\ \mathsf{tt}0\ \mathsf{else}\ z(\mathbf{mtest}_{\lambda x.P})\rangle$$

$\mathfrak{r}$ just tests whether the numeral $k = \mathsf{quote}\,(z\,\mathbf{mtest}_{\lambda x.P})$ is a witness for $\exists x^{\mathtt{N}} P^{\perp}$; if it is the case, then $\mathsf{tt}0 = \top_0$ realizes $P^{\perp}[k/x]$, otherwise $z\,\mathbf{mtest}_{\lambda x.P}$ realizes $\bot$ and thus $P^{\perp}[k/x]$.

5.1 Curry-Howard Correspondence for $\mathsf{HA}^{\omega} + \mathsf{MP}$

In Figure 2, we define a standard natural deduction system for $\mathsf{HA}^{\omega} + \mathsf{MP}$ (see [19], for example) together with a term assignment in the spirit of Curry-Howard correspondence for intuitionistic logic.

We replace purely universal axioms (i.e., Π^0_1-axioms) with *sound Post rules*, which are inferences of the form

$$\frac{\Gamma \vdash A_1\ \Gamma \vdash A_2\ \cdots\ \Gamma \vdash A_n}{\Gamma \vdash A}$$

where $A_1, \ldots, A_n, A$ are atomic formulas of $\mathcal{T}$ such that for every substitution

$$\sigma = [t_1/x_1, \ldots, t_k/x_k]$$

of closed terms $t_1, \ldots, t_k$ of $\mathcal{T}$, $A_1\sigma = \ldots = A_n\sigma = \texttt{True}$ implies $A\sigma = \texttt{True}$. Any other axiomatic presentation of HA^{ω} would have worked just fine, but Post rules allows to define in a uniform way a more flexible deduction system, which is very useful when coding actual

Contexts With Γ we denote contexts of the form $x_1 : A_1, \ldots, x_n : A_n$, with $x_1, \ldots, x_n$ proof variables and $A_1, \ldots, A_n$ formulas of $\mathcal{T}$.

Axioms $\Gamma, x : A \vdash x^{|A|} : A$

Conjunction $\dfrac{\Gamma \vdash u : A \quad \Gamma \vdash t : B}{\Gamma \vdash \langle u, t\rangle : A \wedge B}$ $\quad \dfrac{\Gamma \vdash u : A \wedge B}{\Gamma \vdash \pi_0 u : A}$ $\quad \dfrac{\Gamma \vdash u : A \wedge B}{\Gamma \vdash \pi_1 u : B}$

Implication $\dfrac{\Gamma \vdash u : A \to B \quad \Gamma \vdash t : A}{\Gamma \vdash ut : B}$ $\quad \dfrac{\Gamma, x : A \vdash u : B}{\Gamma \vdash \lambda x^{|A|} u : A \to B}$

Disjunction Intro. $\dfrac{\Gamma \vdash u : A}{\Gamma \vdash \langle \mathtt{True}, u, d^{|B|}\rangle : A \vee B}$ $\quad \dfrac{\Gamma \vdash u : A}{\Gamma \vdash \langle \mathtt{False}, d^{|A|}, u\rangle : A \vee B}$

Disjunction Elim. $\dfrac{\Gamma \vdash u : A \vee B \quad \Gamma \vdash w_1 : A \to C \quad \Gamma \vdash w_2 : B \to C}{\Gamma \vdash \mathsf{if}\ \mathsf{p}_0 u\ \mathsf{then}\ w_1(\mathsf{p}_1 u)\ \mathsf{else}\ w_2(\mathsf{p}_2 u) : C}$

Universal Quantification $\dfrac{\Gamma \vdash u : \forall \alpha^\tau A}{\Gamma \vdash ut : A[t/\alpha^\tau]}$ $\quad \dfrac{\Gamma \vdash u : A}{\Gamma \vdash \lambda \alpha^\tau u : \forall \alpha^\tau A}$

where t is a term of $\mathcal{T}$ and $\alpha^{\mathtt{N}}$ does not occur free in any formula B occurring in Γ.

Existential Quantification $\dfrac{\Gamma \vdash u : A[t/\alpha^\tau]}{\Gamma \vdash \langle t, u\rangle : \exists \alpha^\tau . A}$ $\quad \dfrac{\Gamma \vdash u : \exists \alpha^\tau . A \quad \Gamma \vdash t : \forall \alpha^\tau . A \to C}{\Gamma \vdash t(\pi_0 u)(\pi_1 u) : C}$

where α^τ is not free in C.

Induction $\dfrac{\Gamma \vdash u : A(0) \quad \Gamma \vdash v : \forall \alpha^{\mathtt{N}} . A(\alpha) \to A(\mathsf{S}(\alpha))}{\Gamma \vdash \lambda \alpha^{\mathtt{N}} \mathsf{R} u v \alpha : \forall \alpha^{\mathtt{N}} A}$

Booleans $\dfrac{\Gamma \vdash u : A(\mathtt{True}) \quad \Gamma \vdash v : A(\mathtt{False})}{\Gamma \vdash \lambda \alpha^{\mathtt{Bool}}\ \mathsf{if}\ x\ \mathsf{then}\ u\ \mathsf{else}\ v : \forall \alpha^{\mathtt{Bool}} A}$

Post Rules $\dfrac{\Gamma \vdash u_1 : A_1 \ \Gamma \vdash u_2 : A_2 \ \cdots \ \Gamma \vdash u_n : A_n}{\Gamma \vdash \mathsf{if}\ A\ \mathsf{then}\ \mathtt{tt}0\ \mathsf{else}\ \mathsf{if}\ A_1^\perp\ \mathsf{then}\ u_1\ \mathsf{else}\ \ldots \mathsf{if}\ A_n^\perp\ \mathsf{then}\ u_n\ \mathsf{else}\ \mathtt{tt}0 : A}$

where $n > 0$ and $A_1, A_2, \ldots, A_n, A$ are atomic formulas and the rule is a sound Post rule.

Post Rules with no Premises $\dfrac{}{\Gamma \vdash \mathtt{tt}0 : A}$

where A is an atomic formula of $\mathcal{T}$ and an axiom of equality or a classical propositional tautology.

MP $\dfrac{}{\Gamma \vdash \mathfrak{r} : \neg \forall x^{\mathtt{N}} P \to \exists x^{\mathtt{N}} P^\perp}$

where $\mathfrak{r} = \lambda z^{(\mathtt{N} \to \mathtt{U}) \to \mathtt{U}} \langle \mathsf{quote}\,(z\, \mathbf{mtest}_{\lambda x.P}), \mathsf{if}\ P^\perp [\mathsf{quote}\,(z\, \mathbf{mtest}_{\lambda x.P})/x]\ \mathsf{then}\ \mathtt{tt}0\ \mathsf{else}\ z\, \mathbf{mtest}_{\lambda x.P}\rangle$

Figure 2 Terms Assignment Rules for $\mathsf{HA}^\omega + \mathsf{MP}$.

mathematical proofs. Let now $\mathsf{eq} : \mathtt{N}^2 \to \mathtt{Bool}$ a term of Gödel's system T representing equality between natural numbers. Among the Post rules, we have the Peano axioms

$$\frac{\Gamma \vdash \mathsf{eq}\ \mathsf{S}(x)\,\mathsf{S}(y)}{\Gamma \vdash \mathsf{eq}\ x\, y} \qquad \frac{\Gamma \vdash \mathsf{eq}\ 0\,\mathsf{S}(x)}{\Gamma \vdash \bot}$$

and axioms of equality

$$\frac{}{\Gamma \vdash \mathsf{eq}\ x\, x} \qquad \frac{\Gamma \vdash \mathsf{eq}\, x\, y \quad \Gamma \vdash \mathsf{eq}\, y\, z}{\Gamma \vdash \mathsf{eq}\ x\, z} \qquad \frac{\Gamma \vdash A(x) \quad \Gamma \vdash \mathsf{eq}\, x\, y}{\Gamma \vdash A(y)}$$

and for every A_1, A_2 such that $A_1 = A_2$ is an equation of system $\mathcal{T}$ (equivalently, A_1, A_2 have the same normal form in T), we have the rule

$$\frac{\Gamma \vdash A_1}{\Gamma \vdash A_2}.$$

We also have a Post rule

$$\frac{\Gamma \vdash A_1 \; \Gamma \vdash A_2 \; \cdots \; \Gamma \vdash A_n}{\Gamma \vdash A}$$

for every classical propositional tautology $A_1 \to \ldots \to A_n \to A$, where for $i = 1, \ldots, n$, A_i, A are atomic formulas obtained as combination of other atomic formulas by the Gödel's system T closed terms representing boolean connectives. For example, given terms $\Rightarrow_{\mathtt{Bool}}$ $, \wedge_{\mathtt{Bool}}, \vee_{\mathtt{Bool}} : \mathtt{Bool} \to \mathtt{Bool} \to \mathtt{Bool} \ldots$ representing boolean connectives, one can form, out of atomic formulas A and B, the atomic formulas $\Rightarrow_{\mathtt{Bool}} AB$ and $\wedge_{\mathtt{Bool}} AB$. Using infix notations, we have for example the rules

$$\frac{\Gamma \vdash \bot}{\Gamma \vdash P}, \qquad \frac{\Gamma \vdash B}{\Gamma \vdash A \Rightarrow_{\mathtt{Bool}} B}, \qquad \frac{\Gamma \vdash A \wedge_{\mathtt{Bool}} B}{\Gamma \vdash A}.$$

Finally, we have a rule of case reasoning for booleans. For any formula $A(\alpha^{\mathtt{Bool}})$ he have the axiom:

$$\frac{\Gamma \vdash A(\mathtt{True}) \quad \Gamma \vdash A(\mathtt{False})}{\Gamma \vdash \forall \alpha^{\mathtt{Bool}} A}.$$

We remark that some of the Post rules, for example many of those for $\mathtt{eq}$, are derivable from others. We remark that the negations $^\bot$ and $\neg$, and the disjunctions $\vee_{\mathtt{Bool}}$ and $\vee$ have the same meaning but they are syntactically different: for every atomic formula P, we consider $P^\bot$ and $P \vee_{\mathtt{Bool}} P^\bot$ as atomic formulas, while $\neg P$ and $P \vee P^\bot$ as compound formulas. But one can show that, for every atomic formula P, $\mathsf{HA}^\omega \vdash P^\bot \leftrightarrow \neg P$: it is enough to derive $\mathsf{HA}^\omega \vdash \mathtt{True}^\bot \leftrightarrow \neg \mathtt{True}$ and $\mathsf{HA}^\omega \vdash \mathtt{False}^\bot \leftrightarrow \neg \mathtt{False}$, then use the rule of case reasoning for booleans to obtain $\mathsf{HA}^\omega \vdash \forall \alpha^{\mathtt{Bool}} \alpha^\bot \leftrightarrow \neg \alpha$ and conclude with the elimination of $\forall$ applied to P. We can derive $\mathsf{HA}^\omega \vdash \mathtt{True}^\bot \to \neg \mathtt{True}$ as follows:

$$\frac{\dfrac{\mathtt{True}^\bot, \mathtt{True} \vdash \mathtt{True}^\bot = \mathsf{if}\ \mathtt{True}\ \mathsf{then}\ \mathtt{False}\ \mathsf{else}\ \mathtt{True}}{\mathtt{True}^\bot, \mathtt{True} \vdash \mathtt{False}}}{\mathtt{True}^\bot \vdash \mathtt{True} \to \mathtt{False} = \neg \mathtt{True}}$$

and $\mathsf{HA}^\omega \vdash \neg \mathtt{True} \to \mathtt{True}^\bot$ as follows:

$$\frac{\dfrac{\neg \mathtt{True} \vdash \neg \mathtt{True} \qquad \neg \mathtt{True} \vdash \mathtt{True}}{\neg \mathtt{True} \vdash \mathtt{False}}}{\neg \mathtt{True} \vdash \mathtt{True}^\bot = \mathsf{if}\ \mathtt{True}\ \mathsf{then}\ \mathtt{False}\ \mathsf{else}\ \mathtt{True}}$$

$\mathsf{HA}^\omega \vdash \mathtt{False}^\bot \leftrightarrow \neg \mathtt{False}$ can be derived even more easily, since $\neg \mathtt{False} = \mathtt{False} \to \mathtt{False}$ is derivable and

$$\frac{\vdash \mathtt{True}}{\vdash \mathtt{False}^\bot = \mathsf{if}\ \mathtt{False}\ \mathsf{then}\ \mathtt{False}\ \mathsf{else}\ \mathtt{True}}$$

Moreover, $P \vee_{\mathtt{Bool}} P^\bot$ is an axiom, while we may derive $\mathsf{HA}^\omega \vdash P \vee P^\bot$ again by case reasoning for booleans.

If τ is any type of $\mathcal{T}$, we denote with d^τ a dummy term of type τ, defined by $d^{\mathtt{N}} = 0$, $d^{\mathtt{Bool}} = \mathtt{False}$, $d^{\mathtt{U}} = \top_0$, $d^{\sigma \to \rho} = \lambda z^\sigma . d^\rho$ (with z^σ any variable of type σ), $d^{\sigma \times \rho} = \langle d^\sigma, d^\rho \rangle$.

6 Main Results

6.1 The Adequacy Theorem

We now prove our main result, namely, that every theorem of $\mathsf{HA}^\omega + \mathsf{MP}$ is realizable by a proof-like term. This derives as an easy corollary from the Adequacy Theorem 15. In the Adequacy Theorem we will exploit the extensionality relation defined in Section 4.

As usual in adequacy proofs for realizability, we prove a stronger version of the theorem, suitable to be proved by induction.

▶ **Theorem 15** (Adequacy). *Assume that $\Gamma \vdash w : A$ in $\mathsf{HA}^\omega + \mathsf{MP}$, with $\Gamma = x_1 : A_1, \ldots, x_n : A_n$ and suppose that all the free variables occurring in Γ and $w : A$ are among $\alpha_1 : \tau_1, \ldots, \alpha_k : \tau_k$. For any choice of closed terms $r_1 : \tau_1, \ldots, r_k : \tau_k$ of system $\mathcal{T}$, if there are terms $t_1, \ldots, t_n$ such that, for $i = 1, \ldots, n$*

$$t_i \Vdash A_i[r_1/\alpha_1, \ldots, r_k/\alpha_k]$$

then

$$w[t_1/x_1^{|A_1|}, \ldots, t_n/x_n^{|A_n|}, r_1/\alpha_1, \ldots, r_k/\alpha_k] \Vdash A[r_1/\alpha_1, \ldots, r_k/\alpha_k]\,.$$

Proof.

▶ Notation 1. For any term v and formula B, we denote $v[t_1/x_1^{|A_1|} \cdots t_n/x_n^{|A_n|}\; r_1/\alpha_1 \cdots r_k/\alpha_k]$ with $\overline{v}$ and $B[r_1/\alpha_1 \cdots r_k/\alpha_k]$ with $\overline{B}$. We have $|\overline{B}| = |B|$ for all formulas B.

We proceed by induction on the derivation of $\Gamma \vdash w : A$. Let r be the last rule applied in the derivation.

1. r is an axiom for variables. For some i, $w = x_i^{|A_i|}$ and $A = A_i$. So $\overline{w} = t_i \Vdash \overline{A_i} = \overline{A}$.
2. r is the $\wedge I$ rule, then $w = \langle u, t\rangle$, $A = B \wedge C$, $\Gamma \vdash u : B$ and $\Gamma \vdash t : C$. Therefore, $\overline{w} = \langle \overline{u}, \overline{t}\rangle$. By induction hypothesis, $\pi_0\overline{w} = \overline{u} \Vdash \overline{B}$ and $\pi_1\overline{w} = \overline{t} \Vdash \overline{C}$; so, by Lemma 9, $\overline{w} \Vdash \overline{B} \wedge \overline{C} = \overline{A}$.
3. r is a $\wedge E$ rule, say left, then $\Gamma \vdash u : A \wedge B$, $w = \pi_0 u$. Since $\overline{u} \Vdash \overline{A} \wedge \overline{B}$ by induction hypothesis, if $\overline{w} = \pi_0\overline{u}$ we can conclude $w \Vdash \overline{A}$.
4. r is the $\to E$ rule, then $\Gamma \vdash u : B \to A$ and $\Gamma \vdash t : B$ $w = ut$, . So $\overline{w} = \overline{u}\overline{t} \Vdash \overline{A}$, for $\overline{u} \Vdash \overline{B} \to \overline{A}$ and $\overline{t} \Vdash \overline{B}$ by induction hypothesis.
5. r is the $\to I$ rule, then $w = \lambda x^{|B|}u$, $A = B \to C$ and $\Gamma, x : B \vdash u : C$. Suppose now that $t \Vdash \overline{B}$; we have to prove that $\overline{w}t \Vdash \overline{C}$. By induction hypothesis on u, $\overline{u} \Vdash \overline{C}$. One has

$$\begin{aligned}
\overline{w}t &= (\lambda x^{|B|}u)[t_1/x_1^{|A_1|} \cdots t_n/x_n^{|A_n|}\; r_1/\alpha_1 \cdots r_k/\alpha_k]t \\
&= (\lambda x^{|B|}u)t[t_1/x_1^{|A_1|} \cdots t_n/x_n^{|A_n|}\; r_1/\alpha_1 \cdots r_k/\alpha_k] \\
&= u[t/x^{|B|}][t_1/x_1^{|A_1|} \cdots t_n/x_n^{|A_n|}\; r_1/\alpha_1 \cdots r_k/\alpha_k] \\
&= \overline{u}\,.
\end{aligned}$$

   Then since $\overline{u} = \overline{w}t$, by Lemma 9, $\overline{w}t \Vdash \overline{C}$.
6. r is a $\vee I$ rule, say left (the other case is symmetric), then $w = \langle \mathtt{True}, u, d^{|C|}\rangle$, $A = B \vee C$ and $\Gamma \vdash u : B$. So, $\overline{w} = \langle \mathtt{True}, \overline{u}, d^{|C|}\rangle$ and hence $\pi_0\overline{w}[s] = \mathtt{True}$. $\overline{u} \Vdash \overline{B}$ follows with the help of induction hypothesis.
7. r is a $\vee E$ rule, then

$$w = \text{if } \mathsf{p}_0 u \text{ then } w_1(\mathsf{p}_1 u) \text{ else } w_2(\mathsf{p}_2 u)$$

   and $\Gamma \vdash u : B \vee C$, $\Gamma \vdash w_1 : B \to A$, $\Gamma \vdash w_2 : C \to A$.
   Assume $\mathsf{p}_0\overline{u} = \pi_0\overline{u} = \mathtt{True}$. By inductive hypothesis $\overline{u} \Vdash \overline{B} \vee \overline{C}$, $\overline{w_1} \Vdash \overline{B} \to \overline{A}$ and $\overline{w_2} \Vdash \overline{C} \to \overline{A}$. Therefore, $\mathsf{p}_1\overline{u} \Vdash \overline{B}$. Hence $\overline{w} = \overline{w_1}(\mathsf{p}_1\overline{u})$.

Since $\overline{w_1} \Vdash \overline{B} \to \overline{A}$ and $\mathsf{p}_1\overline{u} \Vdash \overline{B}$, by definition of realizability, $\overline{w_1}(\mathsf{p}_1\overline{u}) \Vdash \overline{A}$. By $\overline{w} = \overline{w_1}((\mathsf{p}_1\overline{u}))$ and Lemma 9, also $\overline{w} \Vdash \overline{A}$.
Symmetrically, if $\mathsf{p}_0\overline{u} = \mathtt{False}$, we obtain again $\overline{w} \Vdash \overline{A}$.

8. r is the $\forall E$ rule, then $w = ut$, $A = B[t/\alpha^\tau]$ and $\Gamma \vdash u : \forall\alpha^\tau B$. So, $\overline{w} = \overline{ut}$. By inductive hypothesis $\overline{u} \Vdash \forall\alpha^\tau \overline{B}$ and so we can conclude that $\overline{ut} \Vdash \overline{B}[\overline{t}/\alpha^\tau]$.

9. r is the $\forall I$ rule, then $w = \lambda\alpha^\tau u$, $A = \forall\alpha^\tau B$ and $\Gamma \vdash u : B$ (with α^τ not occurring free in the formulas of Γ). So, $\overline{w} = \lambda\alpha^\tau\overline{u}$, since $\alpha \neq \alpha_1, \ldots, \alpha_k$. Let $t : \tau$ be a closed term of HA^ω; by Lemma 9, it is enough to prove that $\overline{w}t = \overline{u}[t/\alpha^\tau]$, $\overline{u}[t/\alpha^\tau] \Vdash \overline{B}[t/\alpha^\tau]$, which amounts to show that the induction hypothesis can be applied to u. We observe that, since $\alpha \neq \alpha_1, \ldots, \alpha_k$, for $i = 1, \ldots, n$ we have

$$t_i \Vdash \overline{A_i} = \overline{A_i}[t/\alpha^\tau]\,.$$

10. r is the $\exists E$ rule, then $w = t(\pi_0 u)(\pi_1 u)$, $\Gamma \vdash t : \forall\alpha^\tau : B \to C$ and $\Gamma \vdash u : \exists\alpha^\tau.B$.
By inductive hypothesis $\overline{u} \Vdash \exists\alpha^{\mathtt{N}}.\overline{B}$, $\pi_0\overline{u} = v$ for v term in HA^ω and hence $\pi_1\overline{u} \Vdash \overline{B}[v/\alpha^\tau]$. Then

$$\overline{t}v(\pi_1\overline{u}) \Vdash \overline{C}[v/\alpha^\tau] = \overline{C}\,.$$

We thus obtain by $\overline{w} = \overline{t}(\pi_0\overline{u})(\pi_1\overline{u})$ and by Lemma 9 that $\overline{w} \Vdash \overline{C}$.

11. r is the $\exists I$ rule, then $w = \langle t, u\rangle$, $A = \exists\alpha^\tau B$, $\Gamma \vdash u : B[t/\alpha^\tau]$. So, $\overline{w} = \langle \overline{t}, \overline{u}\rangle$; and, indeed, $\pi_1\overline{w} = \overline{u} \Vdash \overline{B}[\overline{t}/\alpha^\tau]$ by induction hypothesis. By Lemma 9 we conclude the thesis.

12. r is the induction rule. Therefore $w = \lambda\alpha^{\mathtt{N}}\mathsf{R}uv\alpha$, $A = \forall\alpha^{\mathtt{N}}B$, $\Gamma \vdash u : B(0)$ and $\Gamma \vdash v : \forall\alpha^{\mathtt{N}}.B(\alpha) \to B(\mathsf{S}(\alpha))$. So, $\overline{w} = \lambda\alpha^{\mathtt{N}}\mathsf{R}\overline{u}\overline{v}\alpha$.
We have to prove that $\overline{w}u \Vdash \overline{B}[n/\alpha]$ for all closed term u of type $\mathtt{N}$.
Let n be the normal form of u: by Lemma 2 n is a numeral. A plain induction shows that

$$\overline{w}n = \mathsf{R}\overline{u}\overline{v}n \Vdash \overline{B}[n/\alpha]$$

for $\overline{u} \Vdash \overline{B}(0)$ and $\overline{v}i \Vdash \overline{B}(i) \to \overline{B}(\mathsf{S}(i))$ for all numerals i by induction hypothesis. If we set $i = n$, the thesis follows by Lemma 9 and $\overline{w}u = \overline{w}n$.

13. r is the rule for booleans, then $w = \lambda\alpha^{\mathtt{Bool}}\,\mathsf{if}\ \alpha\ \mathsf{then}\ u\ \mathsf{else}\ v$, $\Gamma \vdash u : B(\mathtt{True})$, $\Gamma \vdash v : B(\mathtt{False})$ and $A = \forall\alpha^{\mathtt{Bool}}B$. By inductive hypothesis, $\overline{u} \Vdash \overline{B}(\mathtt{True})$ and $\overline{v} \Vdash \overline{B}(\mathtt{False})$. So, $\overline{w} = \lambda\alpha^{\mathtt{Bool}}\mathsf{if}\ \alpha\ \mathsf{then}\ \overline{u}\ \mathsf{else}\ \overline{v}$. Let $t : \mathtt{Bool}$ be a closed term of HA^ω; by Lemma 9, it is enough to prove that

$$\overline{w}t = (\mathsf{if}\ t\ \mathsf{then}\ \overline{u}\ \mathsf{else}\ \overline{v}) \Vdash \overline{B}[\overline{t}/\alpha^{\mathtt{Bool}}]\,.$$

By Lemma 2, there are two cases:

- the normal form of t is $\mathtt{True}$. Then $\overline{w}t = (\mathsf{if}\ \mathtt{True}\ \mathsf{then}\ \overline{u}\ \mathsf{else}\ \overline{v})$ reduces to $\overline{u}$: the thesis follows by Lemma 9 and the inductive hypothesis on u.
- the normal form of t is $\mathtt{False}$. Then $\overline{w}t$ reduces to $\overline{v}$: the thesis follows by Lemma 9 and the inductive hypothesis on v.

14. r is a Post rule, then $w = \mathsf{if}\ A\ \mathsf{then}\ \mathsf{tt}0\ \mathsf{else}\ \mathsf{if}\ A_1^\perp\ \mathsf{then}\ u_1\ \mathsf{else}\ \ldots\ \mathsf{if}\ A_n^\perp\ \mathsf{then}\ u_n\ \mathsf{else}\ \mathsf{tt}0$. By inductive hypothesis, for $i = 1, \ldots, n$, $\overline{u_i} \Vdash \overline{A_i}$. There are two cases:

- if $\overline{A} = \mathtt{True}$, then $\overline{w} = \mathsf{tt}0 = \top_0 \in \mathbb{T}$ and thus $\overline{w} \Vdash \overline{A}$.
- if $\overline{A} = \mathtt{False}$, then there exists $j \in [1, n]$ such that $\overline{A}_j = \mathtt{False}$ and $\overline{u}_j \in \perp\!\!\!\perp$. Thus $\overline{w} = \overline{u}_j$ and the thesis follows by Lemma 9 and the inductive hypothesis.

15. r is the MP axiom, then for some atomic formula Q

$$\overline{w} = \lambda z^{(\mathtt{N}\to\mathsf{U})\to\mathsf{U}}\langle\, \mathsf{quote}\,(z\,\mathbf{mtest}_{\lambda x.Q}), \mathsf{if}\ Q^{\perp}[\,\mathsf{quote}\,(z\,\mathbf{mtest}_{\lambda x.Q})/x]\ \mathsf{then}\ \mathtt{tt}0\ \mathsf{else}\ z\,\mathbf{mtest}_{\lambda x.Q}\rangle$$

and $\overline{A} = \neg\forall x^{\mathtt{N}} Q \to \exists x^{\mathtt{N}} Q^{\perp}$. Let $u : (\mathtt{N} \to \mathsf{U}) \to \mathsf{U}$ be a closed term of $\mathcal{T}$ such that $u \Vdash (\forall x^{\mathtt{N}} Q) \to \perp$. We have to prove that

$$\overline{w}u = \langle\, \mathsf{quote}\,(u\,\mathbf{mtest}_{\lambda x.Q}), \mathsf{if}\ Q^{\perp}[\,\mathsf{quote}\,(u\,\mathbf{mtest}_{\lambda x.Q})/x]\ \mathsf{then}\ \mathtt{tt}0\ \mathsf{else}\ u\,\mathbf{mtest}_{\lambda x.Q}\rangle \Vdash \exists x^{\mathtt{N}} Q^{\perp}$$

By Theorem 2, assume $\mathsf{quote}\,(u\,\mathbf{mtest}_{\lambda x.Q}) = m$, with m numeral. There are two cases:

- m is a witness for $\exists x^{\mathtt{N}} Q^{\perp}$, that is, $Q^{\perp}[m/x] = \mathtt{True}$. Then

$$\pi_1(\overline{w}u) = \mathsf{if}\ Q^{\perp}[m/x]\ \mathsf{then}\ \mathtt{tt}0\ \mathsf{else}\ u\,\mathbf{mtest}_{\lambda x.Q} = \top_0 \in \mathbb{T}$$

  and by Lemma 9 we can conclude $\overline{w}u \Vdash \exists x^{\mathtt{N}} Q^{\perp}$.
- m is not a witness for $\exists x^{\mathtt{N}} Q^{\perp}$, that is, $Q^{\perp}[m/x] = \mathtt{False}$ and

$$\pi_1(\overline{w}u) = \mathsf{if}\ Q^{\perp}[m/x]\ \mathsf{then}\ \mathtt{tt}0\ \mathsf{else}\ u\,\mathbf{mtest}_{\lambda x.Q} = u\,\mathbf{mtest}_{\lambda x.Q}$$

  In order to obtain the thesis, we have to prove that $u\,\mathbf{mtest}_{\lambda x.Q} \Vdash Q^{\perp}[m/x]$. We have that $\mathbf{test}_{\lambda x.Q} \Vdash \forall x^{\mathtt{N}} Q$ and so $u\,\mathbf{test}_{\lambda x.Q} \Vdash \perp$. Therefore $u\,\mathbf{test}_{\lambda x.Q} = \perp_n$, for some numeral n. Let us define the saturated relation $\mathcal{R}$ defined as in Lemma 14

$$\mathcal{R} ::= \{(t_1, t_2) \mid t_1 = t_2 \text{ or } (t_1 = \top_i, t_2 = \perp_i \text{ and } Q[i/x] = \mathtt{False} \text{ for some } i)\}$$

  By the Test Equivalence Lemma 14, $\mathbf{mtest}_{\lambda x.Q} \sim_{\mathcal{R}} \mathbf{test}_{\lambda x.Q}$, $\mathsf{quote} \sim_{\mathcal{R}} \mathsf{quote}$; therefore, by Corollary 13, $u \sim_{\mathcal{R}} u$ and by Definition 10, $u\,\mathbf{mtest}_{\lambda x.Q} \sim_{\mathcal{R}} u\,\mathbf{test}_{\lambda x.Q}$, which implies $u\,\mathbf{mtest}_{\lambda x.Q}\ \mathcal{R}\ u\,\mathbf{test}_{\lambda x.Q}$. Now, $u\,\mathbf{test}_{\lambda x.Q} = \perp_n$ and it cannot be that $u\,\mathbf{mtest}_{\lambda x.Q} = \top_n$, because by assumption $\mathsf{quote}\,(u\,\mathbf{mtest}_{\lambda x.Q}) = m$ and we would thus have $m = n$, with again by assumption

$$Q[m/x] = \mathtt{True}$$

  By definition of $\mathcal{R}$, this forces $u\,\mathbf{mtest}_{\lambda x.Q} = u\,\mathbf{test}_{\lambda x.Q}$. Therefore, $u\,\mathbf{mtest}_{\lambda x.Q} \in \perp\!\!\!\perp$. We conclude that $u\,\mathbf{mtest}_{\lambda x.Q} \Vdash Q^{\perp}[m/x]$.

◀

Since all the terms decorating the inference rules of $\mathsf{HA}^{\omega} + \mathsf{MP}$ are proof-like, as an easy corollary of Theorem 15 we obtain the main theorem:

▶ **Theorem 16.** *If A is a closed formula and $\mathsf{HA}^{\omega} + \mathsf{MP} \vdash t : A$, then $t \Vdash A$, with t proof-like term of $\mathcal{T}$.*

6.2 Realizability and Truth

We now want to investigate the relationship between realizability and truth. We have already seen in Proposition 7 that any formula is realizable. Here, we want to show that our notion of realizability is consistent at least when realizers come from proofs in $\mathsf{HA}^{\omega} + \mathsf{MP}$: whenever a formula not containing $\to$ is realized by a proof-like term, it is also true, for a suitable notion of truth. Intuitively, we consider a formula of HA^{ω} to hold if it is true in the canonical syntactical model in which quantifiers of type τ range over the closed terms of HA^{ω} of type τ. In particular, the truth of arithmetical formulas is exactly the standard arithmetical truth over $\mathbb{N}$. We now give the obvious definition.

▶ **Definition 17** (Truth in the Syntactical Model)**.** Given a closed formula F of HA^ω, we define by induction over F its truth value $[\![F]\!] \in \{\mathtt{True}, \mathtt{False}\}$.

- If P is atomic, $[\![P]\!] = \mathtt{True}$ if $P = \mathtt{True}$, $[\![P]\!] = \mathtt{False}$ otherwise.
- $[\![A \wedge B]\!] = \mathtt{True}$ if $[\![A]\!] = [\![B]\!] = \mathtt{True}$, $[\![A \wedge B]\!] = \mathtt{False}$ otherwise.
- $[\![A \vee B]\!] = \mathtt{True}$ if $[\![A]\!] = \mathtt{True}$ or $[\![B]\!] = \mathtt{True}$, $[\![A \vee B]\!] = \mathtt{False}$ otherwise.
- $[\![A \to B]\!] = \mathtt{True}$ if $[\![A]\!] = \mathtt{True}$ implies $[\![B]\!] = \mathtt{True}$, $[\![A \to B]\!] = \mathtt{False}$ otherwise.
- $[\![\forall x^\tau A]\!] = \mathtt{True}$ if for all closed terms $t : \tau$ of HA^ω, $[\![A[t/x]]\!] = \mathtt{True}$, $[\![\forall x^\tau A]\!] = \mathtt{False}$ otherwise.
- $[\![\exists x^\tau A]\!] = \mathtt{True}$ if there exists a closed term $t : \tau$ of HA^ω such that $[\![A[t/x^\tau]]\!] = \mathtt{True}$, $[\![\exists x^\tau A]\!] = \mathtt{False}$ otherwise.

We are now ready show the consistency of our notion of realizability.

▶ **Proposition 18** (Consistency of Realizability)**.** *Let F be a closed $\to$-free formula and let t be a proof-like term such that $t \Vdash F$. Then $[\![F]\!] = \mathtt{True}$.*

Proof. By induction on F.

1. $F = P$, with P atomic. Since t is proof-like, no term in the reduction tree of t can contain a constant in $\perp\!\!\!\perp$. Therefore, $t \notin \perp\!\!\!\perp$, and since $t \Vdash P$, it must be that $P = \mathtt{True}$.
2. $F = A \wedge B$. Since $t \Vdash A \wedge B$, we have that $\pi_0 t \Vdash A$ and $\pi_1 t \Vdash B$. By induction hypothesis, $[\![A]\!] = \mathtt{True}$ and $[\![B]\!] = \mathtt{True}$. Therefore, $[\![A \wedge B]\!] = \mathtt{True}$.
3. $F = A \vee B$. Since $t \Vdash A \vee B$, we have that $\mathsf{p}_1 t \Vdash A$ or $\mathsf{p}_2 t \Vdash B$. By induction hypothesis, $[\![A]\!] = \mathtt{True}$ or $[\![B]\!] = \mathtt{True}$. Therefore, $[\![A \vee B]\!] = \mathtt{True}$.
4. $F = \forall x^\tau A$. Since $t \Vdash \forall x^\tau A$, we have that for all closed terms u of HA^ω, $tu \Vdash A[u/x^\tau]$. By induction hypothesis, for all closed terms u of HA^ω, $[\![A[u/x^\tau]]\!] = \mathtt{True}$. Therefore, $[\![\forall x^\tau A]\!] = \mathtt{True}$.
5. $F = \exists x^\tau A$. Since $t \Vdash \exists x^\tau A$, we have that for $\pi_0 t = u$ for some closed term u of HA^ω, and $\pi_1 t \Vdash A[u/x^\tau]$. By induction hypothesis, $[\![A[u/x^\tau]]\!] = \mathtt{True}$. Therefore, $[\![\exists x^\tau A]\!] = \mathtt{True}$. ◀

Proposition 18 is very important since ensure that proof-like realizers produce *correct* constructive content for the formulas they realize. For instance, if $t \Vdash \exists x^\tau A$, then $\pi_0 t = u$ for some closed term u of HA^ω and $[\![A[u/x]]\!] = \mathtt{True}$. Thus, our realizability can be used to extract in an effective way sound witnesses from proofs in $\mathsf{HA}^\omega + \mathsf{MP}$ of $\to$-free formulas. Proposition 18 is not true for all formulas, since the Axiom of Choice is realizable, as in Kreisel's modified realizability, but not true in the syntactical model. But we conjecture that Proposition 18 can be strengthened further and that many kind of formulas containing implications are true when realized. However, for reasons of space and complexity we do not address this matter here.

7 Concluding Remarks and Further Works

As remarked in the introduction, there are several constructive interpretations of Markov's Principle [10, 5, 11]. While the semantics are quite different from each other, it is quite clear that the computational mechanisms employed by the extracted programs are essentially the same. Our realizability is no exception and exploits, as all the other interpretations, a proof of $\neg\forall x^{\mathtt{N}} P$ in order to get a witness for $\exists x^{\mathtt{N}} P^\perp$.

However, it is clear that our realizability is intensionally different from the Dialectica, it is simpler and the term assignment for extracting programs is much lighter. It remains to establish the exact relationship between the two notions: are they equivalent? We conjecture that in most cases there is a translation between realizers of formulas in our sense and terms witnessing their Dialectica interpretation.

Our realizability appears also less ad hoc then Avigad's smooth version [3] of Coquand-Hofmann translation, which requires an usual forcing style definition, with conditions being set of purely universal formulas. With that approach one must always refer to these conditions, which are used to interpret Markov's Principle, even when considering other formulas or axiom schemes (for example, one may like to interpret countable choice, which has nothing to do with MP).

We also remark that our realizability has not been formulated as a syntactical formula translation. Indeed it is not trivial to formalize it in such a way, since we have employed several syntactical tools, as the notion of proof-like term and the normalization theorem. However, we claim to be able to formulate realizability as a formula translation in the style of modified realizability. Once formalized, we also claim that our realizability can be used to obtain with new methods some conservativity results, for example the one stating that $\mathsf{HA}^\omega + \mathsf{MP}$ is conservative over HA^ω for $\rightarrow$-free arithmetical formulas.

Finally, compared with Herbelin [11], we employ a purely functional language, while he uses exception handling mechanisms.

Another way of extending this work is to interpret the generalized Markov's Principle:

$$\mathsf{GMP} : \neg\forall x^\tau P \rightarrow \exists x^\tau P^\perp .$$

It is indeed reasonable that the methods of this paper can be refined in order to interpret also this axiom.

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