Computing Opaque Interior Barriers à la Shermer

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Abstract

The problem of finding a collection of curves of minimum total length that meet all the lines intersecting a given polygon was initiated by Mazurkiewicz in 1916. Such a collection forms an *opaque barrier* for the polygon. In 1991 Shermer proposed an exponential-time algorithm that computes an interior-restricted barrier made of segments for any given convex n-gon. He conjectured that the barrier found by his algorithm is optimal, however this was refuted recently by Provan et al. Here we give a Shermer like algorithm that computes an interior polygonal barrier whose length is at most 1.7168 times the optimal and that runs in $O(n)$ time. As a byproduct, we also deduce upper and lower bounds on the approximation ratio of Shermer's algorithm.

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1 Introduction

The problem of finding small sets that block every line passing through a unit square was first considered by Mazurkiewicz in 1916 [21]; see also [2, 14]. Let B be a convex body in the plane. Following Bagemihl [2], a set Γ is an *opaque set* or a *barrier* for B, if Γ meets every line that intersects B. We restrict our attention to barriers consisting of countably many rectifiable curves. A barrier does not need to be connected; it may consist of one or more rectifiable arcs and its parts may lie anywhere in the plane, including the exterior of B [2].

What is the length of the shortest barrier for a given convex body B? In spite of considerable efforts, the answer to this question is not known even in the simplest instances, such as when B is a square, a disk, or an equilateral triangle; see [3], [4, Problem A30], [10], [11], [12], [13, Section 8.11], [15, Problem 12]. For example, when B is a unit square, the barrier in Figure 1(right) is conjectured to be optimal; on the other hand, the current best lower bound on the length of a barrier was only 2 until very recently; the earliest record for this bound of 2 dates back to Jones in 1964 [16]. For barriers consisting of finitely many straight-line segments and restricted to lie in a concentric square of side 2, Dumitrescu and Jiang [8] established the first lower bound greater than 2, namely $2 + 10^{-12}$ (and $2 + 10^{-5}$ for interior barriers). Kawamura et al. [17] recently obtained the first general lower bound (that does not require finiteness or locality), namely 2.00002, which now holds the current record.

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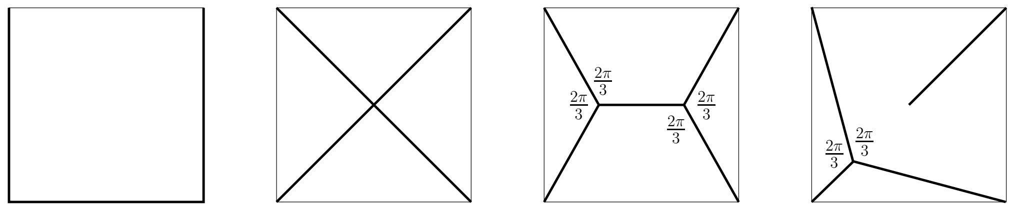

Figure 1 The first three from the left are barriers for the unit square of lengths 3, $2\sqrt{2} = 2.8284\ldots$, and $1+\sqrt{3} = 2.7320\ldots$. Right: The diagonal segment $[(1/2, 1/2), (1, 1)]$ together with three segments connecting the corners $(0, 1)$, $(0, 0)$, $(1, 0)$ to the point $(\frac{1}{2} - \frac{\sqrt{3}}{6}, \frac{1}{2} - \frac{\sqrt{3}}{6})$ yield a barrier of length $\sqrt{2} + \frac{\sqrt{6}}{2} = 2.6389\ldots$.

A barrier blocks any line of sight across the region B or detects any ray that passes through it. Potential applications are in guarding and surveillance [5]. Some applications to the optimization of saving and recovery routes can be found in [19, 20]. The problem of short barriers has attracted researchers for decades. We refer the reader to [7, 9] and the references therein for many of the earlier results in this area.

Types of Barriers. Several variants of the problem have been considered depending on the types of curves allowed in a barrier: the most restricted are the barriers made of single continuous arcs, then connected barriers, and lastly, arbitrary (possibly disconnected) barriers. For the unit square, the shortest known barrier in these three categories have lengths 3, $1 + \sqrt{3} = 2.7320\ldots$ and $\sqrt{2} + \frac{\sqrt{6}}{2} = 2.6389\ldots$, respectively. They are depicted in Figure 1. Obviously, disconnected barriers offer the greatest freedom of design. For instance, Kawohl [18] showed that the barrier in Figure 1(right) is optimal in the class of curves with at most two components restricted to lie in the square. For the unit disk, the shortest known barrier consists of three arcs. See also [11, 13].

Barriers can be also classified by their possible locations. In certain instances, it might be infeasible to construct barriers guarding a specific domain outside the domain, since that part might be controlled by different owners. An *interior barrier* of a body B is a barrier constrained to the interior and the boundary of B. For example, all four barriers for the unit square illustrated in Figure 1 are interior barriers. By slightly relaxing the interior constraint, we call a barrier for B, $(1 + \varepsilon)$*-interior*, if it lies in the interior or on the boundary of $B + D_\varepsilon$, the Minkowski sum of B and a disk D_ε of radius $\varepsilon > 0$ centered at the origin.

On the other hand, certain instances may prohibit barriers lying in the interior of a domain. An *exterior barrier* of B is constrained to exterior and the boundary of B. As an illustration, the first barrier from the left in Figure 1 is exterior (and since it is contained in the boundary of the domain, it is interior as well).

Approximations. In the absence of methods for finding optimal barriers, attention has turned to approximation algorithms. A key fact in establishing a constant approximation ratio is the following lower bound on the length of a barrier: Every barrier Γ of a convex body B in the plane satisfies

$$|\Gamma| \geq \frac{\text{per}(B)}{2}, \tag{1}$$

where $\text{per}(B)$ denotes the perimeter of B. The proof of (1) is folklore, based on Cauchy's integral formula [9, 12, 16]. It follows that the boundary of B, of length $\text{per}(B)$, is always a 2-approximation to the optimal barrier, and a 2-approximation to the optimal *interior* barrier as well.

Recent work focused on obtaining better approximation ratios. Even though we have so little control on the shape or length of optimal barriers, barriers whose lengths are relatively close to optimal can be computed efficiently for a convex polygon P with n vertices. Various approximation algorithms with a smaller constant ratio (below 1.6) have been obtained recently [9]:

(i) A (possibly disconnected) barrier for P, whose length is at most $\frac{1}{2} + \frac{2+\sqrt{2}}{\pi} = 1.5867\ldots$ times the optimal, can be computed in $O(n)$ time.
(ii) A connected polygonal barrier whose length is at most 1.5716 times the optimal can be computed in $O(n)$ time.
(iii) A single-arc polygonal barrier whose length is at most $\frac{\pi+5}{\pi+2} = 1.5834\ldots$ times the optimal can be computed in $O(n)$ time.
(iv) An optimal interior single-arc barrier can be computed in $O(n^2)$ time.
(v) An interior connected barrier whose length is at most $(1+\varepsilon)$ times the optimal can be found in polynomial time.

It is worth noting that none of the above approximations holds for interior barriers, for two reasons: either (i) the barrier found is not guaranteed to be interior, or (ii) the approximation ratio is based on a lower bound for a specific class of barriers, different from that given by (1), or both. In this paper, we present the first nontrivial approximation algorithm with ratio below 2 for computing an interior barrier for a given convex polygon.

▶ **Theorem 1.** *Given a convex polygon P with n vertices, an interior barrier for P, whose length is at most $0.8588\,\mathrm{per}(P) = 1.7168\,\frac{\mathrm{per}(P)}{2}$, hence in particular at most 1.7168 times the optimal, can be computed in $O(n)$ time.*

Shermer's Conjecture. In the late 1980s, Akman [1] soon followed by Dublish [6] had reported algorithms for computing a minimum interior barrier of a given convex polygon. Both algorithms were shown to be suboptimal by Shermer [25] in 1991, who proposed a new exact algorithm. Shermer conjectured that a shortest interior barrier (he calls it an "opaque forest") of a given convex polygon P with n vertices can be generated by an instance of the following procedure:

(a) Find a triangulation T of P.
(b) Remove zero or more diagonals of T, so that at most one nontriangular interior region U is formed. Let the edges of U's Steiner tree be in the opaque forest.
(c) For all triangles of T (other than U, if U is a triangle), let the height of the triangle (using the edge topologically closest to U as the base) be in the opaque forest.

Equivalently, the algorithm proposed by Shermer is the following: For all possible subsets of 3 or more vertices of P, compute the convex hull $U \subset P$. For each such U, include the minimum Steiner tree of U in the barrier Γ and triangulate $P \setminus U$ in all possible ways. For each fixed triangulation, include the height of each triangle (using the edge topologically closest to U as the base) in Γ. Return the shortest interior barrier obtained in this way. (Note that not all choices of U and the triangulation of $P \setminus U$ produce an interior barrier. For example, if the triangulation contains an obtuse triangle with the base incident to the obtuse angle, then the corresponding height is in the exterior of the triangle and the polygon. However, the Shermer's method always produces an interior barrier for $U = P$.)

Recently, Provan et al. [23] refuted Shermer's conjecture with a convex polygon as simple as a rhombus. Specifically, their example shows that Shermer's procedure does not always

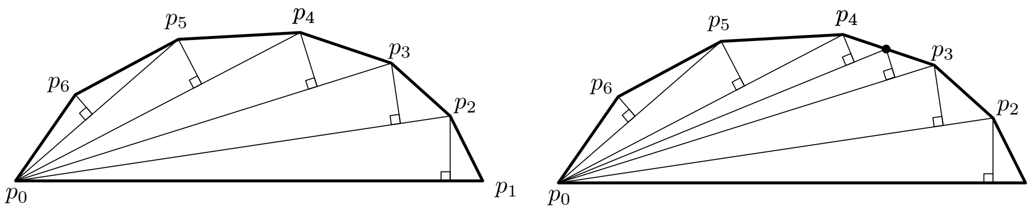

Figure 2 Left: A fan triangulation and the corresponding heights. Right: Edge p_3p_4 is subdivided by a dummy vertex.

compute the shortest interior barrier or the shortest unrestricted barrier. In Section 4, we slightly refine the lower bound for the approximation ratio of Shermer's algorithm offered by the example of Provan et al. Moreover, it is easy to replicate this lower bound with polygons with any number of vertices $n \geq 4$.

▶ **Theorem 2.** *There exist convex polygons (e.g., a rhombus) for which Shermer's algorithm returns an interior barrier that is at least* 1.00769 *times longer than the optimal.*

It is well-known that the number of triangulations of a convex n-gon is C_{n-2}, where $C_n = \frac{1}{n+1}\binom{2n}{n}$ is the the nth Catalan number, hence the proposed algorithm runs in time $O^*(4^n)$ (the $O^*()$ notation hides polynomial factors). However, it is easy to avoid the exponential running time while still achieving an approximation ratio well below 2. The approximation algorithm we will present constructs a barrier by a procedure similar to Shermer's, where some steps are relaxed. Moreover, the approximation ratio we derive also holds for Shermer's original algorithm.

▶ **Corollary 3.** *The approximation ratio of Shermer's procedure is at most 1.7168 and at least 1.00769.*

2 Preliminaries

Our main tool is an upper bound on the sum of heights produced by Shermer's procedure in a specific triangulation of a convex polygon.

Let $P = (p_0, p_1, \ldots, p_m)$ be a convex polygon where the vertices are labeled in counterclockwise order. The *fan triangulation* of P is obtained by inserting the chords p_0p_i, $i = 2, \ldots, m-1$, as shown in Figure 2.

Let h_i denote the distance from p_i to the supporting line of p_0p_{i-1} for $i = 2, \ldots, m$. The shortest segments between p_i and line p_0p_{i-1} are the *heights* corresponding to the fan triangulation. We first give a sufficient condition for the heights to lie in the interior of P.

▶ **Lemma 4.** *Assume that $P = (p_0, p_1, \ldots, p_m)$ lies in a half-disk of diameter p_0p_1. Then every triangle (p_0, p_{i-1}, p_i), $i = 2, \ldots, m$, has a right or obtuse angle at p_i; consequently the heights of the fan triangulation of P lie in the interior or on the boundary of P.*

Proof. By Thales' theorem, we have $\angle p_0p_ip_1 \geq \pi/2$ for $i = 2, \ldots, m$. Hence $\angle p_0p_ip_{i-1} \geq \angle p_0p_ip_1 \geq \pi/2$, and every triangle (p_0, p_{i-1}, p_i) has an obtuse or right angle at p_i, as required. ◀

▶ **Lemma 5.** *Assume that $P = (p_0, p_1, \ldots, p_m)$ lies in a half-disk of diameter p_0p_1. Let $\alpha = \angle p_1p_0p_m < \pi/2$ and $A = \text{area}(P)$. Then $\sum_{i=2}^{m} h_i \leq \sqrt{2\alpha A}$.*

Proof. Put $a_i = |p_0 p_i|$, $i = 1, \dots, m$, and $\alpha_i = \angle p_{i-1} p_0 p_i$, $i = 2, \dots, m$. Refer to Figure 2. Observe that $\sum_{i=2}^m \alpha_i = \alpha$. We have

$$\sum_{i=2}^{m} h_i = \sum_{i=2}^{m} a_i \sin \alpha_i. \tag{2}$$

Since P is subdivided into (the fan of) $m - 1$ triangles with common vertex p_0, we also have

$$\sum_{i=2}^{m} a_{i-1} a_i \sin \alpha_i = 2A. \tag{3}$$

Since P lies in a half-disk of diameter $p_0 p_1$, we have $a_{i-1} < a_i$, for $i = 2, \dots, m$. Consequently

$$\sum_{i=2}^{m} a_i^2 \sin \alpha_i \leq 2A. \tag{4}$$

By the Cauchy-Schwarz inequality in the first step and by Jensen's inequality for the sin function in the second step, (2) and (4) imply that

$$\left(\sum_{i=2}^{m} h_i \right)^2 = \left(\sum_{i=2}^{m} a_i \sin \alpha_i \right)^2 \leq \left(\sum_{i=2}^{m} a_i^2 \sin \alpha_i \right) \left(\sum_{i=2}^{m} \sin \alpha_i \right)$$
$$\leq (2A) \left((m-1) \sin \frac{\alpha}{m-1} \right) \leq (2A) \left((m-1) \frac{\alpha}{m-1} \right) = 2\alpha A. \tag{5}$$

The required inequality follows by taking square roots. ◀

We present an alternative proof for Lemma 5, via an integral formula, which gives a tighter bound when the points p_i lie on an integrable curve.

▶ **Lemma 6.** *Assume that $P = (p_0, p_1, \dots, p_m)$ lies in a half-disk of diameter $p_0 p_1$, such that p_0 is at the origin, p_1 is on the positive x-axis. Let $\alpha = \angle p_1 p_0 p_m < \pi/2$ and $A = \text{area}(P)$. Parametrize the polygonal arc $(p_1, \dots, p_m)$ in polar coordinate by $(\theta, \lambda(\theta))$ for $\theta \in [0, \alpha]$. Then $\sum_{i=2}^m h_i \leq \int_0^\alpha \lambda(\theta)\, d\theta \leq \sqrt{2\alpha A}$.*

Proof. Put $\lambda_i = |p_1 p_i|$, $i = 2, \dots, m$, and $\theta_i = \angle p_1 p_0 p_i$, $i = 2, \dots, m$. Then we have

$$\sum_{i=2}^{m} h_i = \sum_{i=2}^{m} \lambda_i \sin(\theta_i - \theta_{i-1}) \leq \sum_{i=2}^{m} \lambda_i (\theta_i - \theta_{i-1}). \tag{6}$$

By Lemma 4, every triangle (p_0, p_{i-1}, p_i), $i = 2, \dots, m$, has a right or obtuse angle at p_i. If we successively subdivide an edge $p_i p_{i+1}$, $i = 1, \dots, m-1$, with dummy vertices (see Figure 2), then the sum of heights increases. By an infinite refinement of the polygonal arc $(p_1, \dots, p_m)$ with dummy vertices, we obtain

$$\sum_{i=2}^{m} \lambda_i (\theta_i - \theta_{i-1}) \leq \int_0^\alpha \lambda(\theta)\, \mathrm{d}\theta. \tag{7}$$

By the Cauchy-Schwarz inequality,

$$\int_0^\alpha \lambda(\theta)\, \mathrm{d}\theta \leq \left(\int_0^\alpha \mathrm{d}\theta \right)^{1/2} \left(\int_0^\alpha \lambda^2(\theta)\, \mathrm{d}\theta \right)^{1/2} = \sqrt{\alpha} \left(\int_0^\alpha \lambda^2(\theta)\, \mathrm{d}\theta \right)^{1/2}. \tag{8}$$

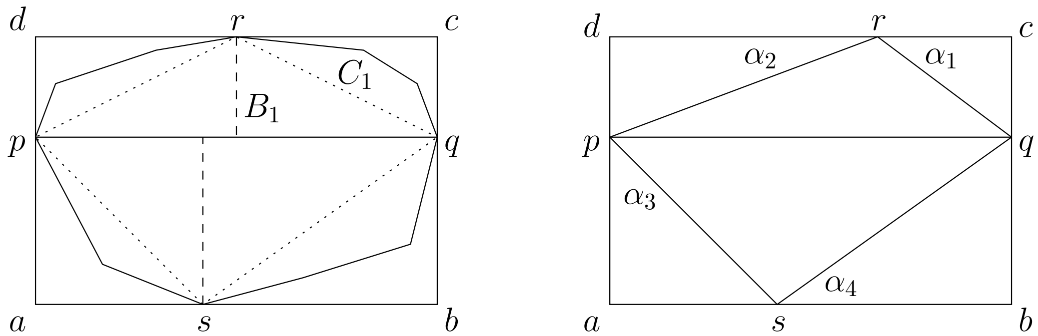

Figure 3 Left: eight areas in a rectangle. Right: four angles in a rectangle.

We have

$$\sum_{i=2}^{m} h_i \leq \int_0^\alpha \lambda(\beta)\, d\beta, \quad 2A = \int_0^\alpha \lambda^2(\beta)\, d\beta, \quad \alpha = \int_0^\alpha d\beta, \tag{9}$$

and consequently, (8) yields $\sum_{i=2}^{m} h_i \leq \sqrt{2\alpha A}$. ◀

3 Proof of Theorem 1

By the classic isoperimetric inequality [26, Exercises 5–8, 7–17] (or see [24]), we have

$$A \leq \frac{L^2}{4\pi}, \text{ and } L \leq \pi D, \tag{10}$$

where $L = \text{per}(P)$, $A = \text{area}(P)$, and $D = \text{diam}(P)$.

We can assume without loss of generality that P has a horizontal unit diameter pq, and let $Q = abcd$ be a minimal axis-parallel rectangle (of unit width) containing P. Denote the height of Q by $y \leq 1$; refer to Figure 3.

Let r and s be two points of P on the top and bottom sides of Q, respectively. Let α_i, $i = 1, 2, 3, 4$, denote the smallest acute angles in each of the four right triangles incident to the vertices of Q: Δqcr, Δrdp, Δpas, Δsbq. If B_i, C_i, $i = 1, 2, 3, 4$, are the areas indicated in the figure, write

$$A_i = B_i + C_i, \quad i = 1, \ldots, 4, \text{ and } B = \sum_{i=1}^{4} B_i, \quad C = \sum_{i=1}^{4} C_i, \text{ so that } A = \sum_{i=1}^{4} A_i.$$

We have $\text{area}(P) = \sum_{i=1}^{4} A_i = A = B + C$. Observe that $C_i \leq B_i$, for each $i = 1, 2, 3, 4$, hence $2C_i \leq B_i + C_i = A_i$, for each $i = 1, 2, 3, 4$. Consequently, $2C \leq A$.

Approximation Algorithm. Given a convex polygon P with n vertices, consider four interior barriers Γ_i, $i = 1, 2, 3, 4$, defined as follows: Γ_i consists of the boundary of P in three “quarters” of Q and the heights of a fan triangulation in the fourth quarter from the smallest angle made at the two endpoints of the convex chain. The algorithm outputs the shortest one.

Note that Γ_i, $i = 1, 2, 3, 4$, is a valid interior barrier for P: the boundary of P in three quarters of Q (say, all quarters but the ith) blocks any line intersecting $\text{conv}(P) \setminus C_i$; and the fan of heights in C_i is a barrier by Shermer’s argument [25] (this can be shown by an easy inductive argument). By Lemma 4, all heights in the fan triangulations lie in the interior or on the boundary of P, so Γ_i, $i = 1, 2, 3, 4$, is an interior barrier. For a given polygon P with n vertices, the four barriers can be computed in $O(n)$ time: indeed, a diameter pair, an axis-aligned bounding box, the fan triangulations, and the heights can all be computed in $O(n)$ time.

First Bound on the Approximation Ratio. In the i-th quarter of Q, the smallest angle made at the two endpoints of the convex chain is at most α_i, for $i = 1, 2, 3, 4$. By Lemma 5, the sum of heights in the fan triangulation in the ith quarter is at most $\sqrt{2\alpha_i C_i}$. The length of the interior barrier Γ returned by the algorithm is no more than the average length of the four candidates:

$$4|\Gamma| \leq \sum_{i=1}^{4} |\Gamma_i| \leq 3L + \sum_{i=1}^{4} \sqrt{2\alpha_i C_i}. \tag{11}$$

By the Cauchy-Schwarz inequality we have

$$\sum_{i=1}^{4} \sqrt{2\alpha_i C_i} \leq \sqrt{\left(\sum_{i=1}^{4} \alpha_i\right)\left(2\sum_{i=1}^{4} C_i\right)} = \sqrt{\Lambda \cdot 2C}, \tag{12}$$

where $\Lambda := \sum_{i=1}^{4} \alpha_i$. Since $\alpha_i \leq \pi/4$, for each $i = 1, 2, 3, 4$, we have $\Lambda = \sum_{i=1}^{4} \alpha_i \leq \pi$. Recall that $2C \leq A$. These two bounds together with the first inequality in (10) yield the following upper bound on the right hand side of (12):

$$\sqrt{\Lambda \cdot 2C} \leq \sqrt{\pi A} \leq \frac{1}{2}L. \tag{13}$$

Hence by using (11), (12), and (13), it follows that the approximation ratio ρ is at most

$$\frac{|\Gamma|}{L/2} \leq \frac{3L + \sqrt{\Lambda \cdot 2C}}{2L} \leq \frac{3L + 0.5L}{2L} = 1.75. \tag{14}$$

A Refined Bound on the Approximation Ratio. We derive sharper bounds on both factors Λ and $2C$ that appear in (12).

The first key component is establishing an upper bound on the sum of angles $\Lambda = \sum_{i=1}^{4} \alpha_i$.

▶ **Lemma 7.** *Define the function*

$$f(y) = \begin{cases} \dfrac{1}{2} + \dfrac{\arctan \frac{y}{1-y}}{\pi}, & y \in [0, 1/2], \\ \dfrac{3}{4} + \dfrac{\arctan \frac{2y-1}{3-2y}}{\pi}, & y \in [1/2, 1]. \end{cases}$$

Then $\sum_{i=1}^{4} \alpha_i \leq f(y)\,\pi$. *This inequality is the best possible for all* $y \in (0, 1]$.

Proof. Observe that $f(1/2) = 3/4$, and that $f(y)$ is continuous at $y = 1/2$. Denote by y_1, y_2 the vertical distances from pq to the top and bottom side of Q, respectively. We can assume without loss of generality that $y_1 \leq y_2$. Put $x_1 = |cr|$, $x_2 = |rd|$, $x_3 = |as|$, $x_4 = |sb|$, so that $x_1 + x_2 = x_3 + x_4 = 1$. Refer to Figure 4. Denote by ℓ the supporting line of pq. We can assume without loss of generality that $|dr| \leq |rc|$ and $|as| \leq |sb|$ (by applying a reflection above and below the line ℓ, independently). This implies that $\alpha_1 = \angle crq$. We distinguish two cases depending on whether y_2 is larger or smaller than $1/2$.

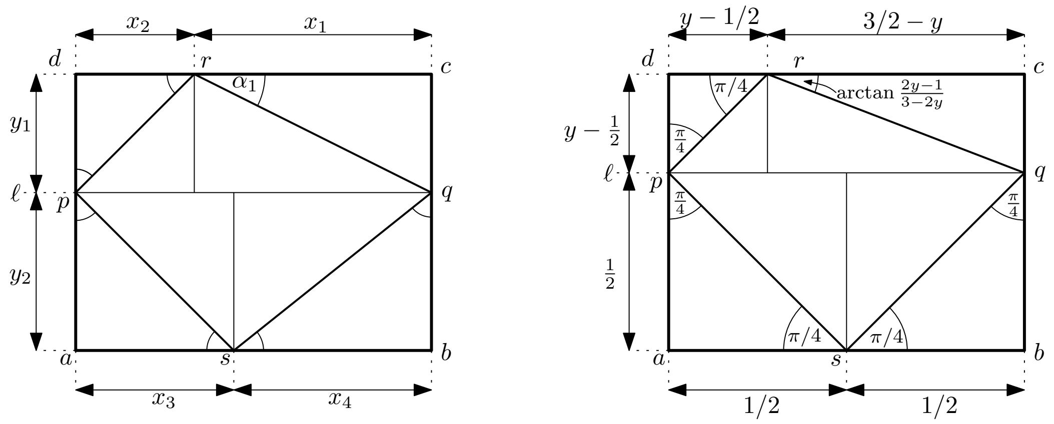

Figure 4 Left: the setup used in the proof of Lemma 7. Right: the canonical configuration in Case 1 where $y_2 \geq 1/2$.

Case 1: $y_2 \geq 1/2$. We first show that $\Lambda = \sum_{i=1}^{4} \alpha_i = \pi/2 + \arctan \frac{2y-1}{3-2y}$ can be attained. Consider the configuration in Figure 4, where $y_1 = x_2 = y - 1/2$, $y_2 = x_3 - x_4 = 1/2$, and $x_1 = 3/2 - y$. We call this the *canonical* configuration in Case 1. In the canonical configuration, Δdpr, Δasp, and Δbqs are isosceles right triangles, hence $\alpha_2 = \alpha_3 = \alpha_4 = \pi/4$, and $\alpha_1 = \arctan \frac{2y-1}{3-2y}$.

It is enough to show that the canonical configuration maximizes Λ under the constraint that $y_2 \geq 1/2$. Consider an arbitrary configuration with $y_2 \geq 1/2$. Keeping the bounding box Q fixed, move the points p, q, r, and s continuously to the canonical configuration such that $\alpha_1 + \alpha_2$ monotonically increases, and $\alpha_3 + \alpha_4$ increases overall during the transformation. Note that α_i, $i = 1, 2, 3, 4$, does not necessarily correspond to the same angle during a continuous transformation: a change can occur when $\alpha_i = \pi/4$. However, the value of α_i does change continuously.

Note that $\alpha_3 + \alpha_4 \leq \pi/4 + \pi/4 = \pi/2$. Therefore $\alpha_3 + \alpha_4$ is nondecreasing over all, no matter how we move p, q, r, and s to the canonical configuration. For changes in $\alpha_1 + \alpha_2$, we distinguish two subcases.

Case 1.1: $\alpha_2 = \angle dpr \leq \pi/4$. Move r to the right until $\angle dpr = \angle drp = \pi/4$. Observe that both α_1 and α_2 monotonically increase, hence $\alpha_1 + \alpha_2$ also monotonically increases. Next, move p, q, and r simultaneously such that pq remains horizontal and (d, p, r) remains an isosceles right triangle until p (hence p and q) reaches their canonical position. Observe that $\alpha_1 + \alpha_2$ monotonically increases. Finally, move s to its canonical position (which affects neither α_1 not α_2).

Case 1.2: $\alpha_2 = \angle drp \leq \pi/4$. This means that p and q are above their canonical position. Move p and q down simultaneously such that pq remains horizontal, until either they reach their canonical position or $\angle drp = \pi/4$. In the latter alternative, the proof can be finalized as in Case 1.1. If p and q are at their canonical position but $\angle drp < \pi/4$, then $x_2 \leq 1/2 \leq x_1$, and $\alpha_2 = \angle drp$. Move r to the left until r reaches its canonical position. Consider the circular arc determined by the three points p, r, q. Since r remains on the left half of cd, $\angle prq$ decreases, and correspondingly $\alpha_1 + \alpha_2$ increases. Finally, move s to its canonical position.

Case 2: $y_2 \leq 1/2$. (Note that Case 2 covers the entire range of y, both $y \leq 1/2$ and $y \geq 1/2$.) We maximize Λ is two steps. First, we assume that the line ℓ is fixed, and determine optimal positions for r and s. In the second step, we optimize over all positions

of ℓ (subject to the constraint $y_2 \geq 1/2$).
Specifically, we first prove that

$$\alpha_1 + \alpha_2 \leq \frac{\pi}{4} + \arctan \frac{y_1}{1-y_1} \tag{15}$$

$$\alpha_3 + \alpha_4 \leq \frac{\pi}{4} + \arctan \frac{y_2}{1-y_2}. \tag{16}$$

We distinguish two cases depending on whether $\alpha_2 = \angle dpr$ or $\alpha_2 = \angle drp$.

Case 2.1: $\alpha_2 = \angle dpr \leq \pi/4$.] Recall that $y_1 \leq y_2$. In addition, we have $x_2 \leq y_1$ and $1 - y_1 \leq x_1$. We need to show that

$$\arctan \frac{x_2}{y_1} + \arctan \frac{y_1}{x_1} \leq \frac{\pi}{4} + \arctan \frac{y_1}{1-y_1}.$$

Similar to Case 1.1, this inequality is obtained by a continuous movement of r to the right until $|dr| = |dp|$.

Case 2.2: $\boldsymbol{\alpha_2 = \angle drp \leq \pi/4}$**.** We have $y_1 \leq x_2$ and $x_1 \leq 1 - y_1$. We need to show that

$$\arctan \frac{y_1}{x_2} + \arctan \frac{y_1}{x_1} \leq \frac{\pi}{4} + \arctan \frac{y_1}{1-y_1}.$$

Similar to Case 1.2, this inequality is obtained by a continuous movement of r to the left until $|dr| = |dp|$.

Cases 2.1 and 2.2 together prove inequality (15). The proof of inequality (16) is analogous.
To conclude the analysis of case 2 for $y \in [0, 1/2]$, we need to verify that:

$$\left(\frac{\pi}{4} + \arctan \frac{y_1}{1-y_1}\right) + \left(\frac{\pi}{4} + \arctan \frac{y_2}{1-y_2}\right) \leq \frac{\pi}{2} + \arctan \frac{y}{1-y}, \text{ or equivalently,}$$

$$\arctan \frac{y_1}{1-y_1} + \arctan \frac{y_2}{1-y_2} \leq \arctan \frac{y}{1-y}. \tag{17}$$

Applying the tangent function to both sides of the inequality, it remains to show that

$$\frac{\frac{y_1}{1-y_1} + \frac{y_2}{1-y_2}}{1 - \frac{y_1}{1-y_1}\frac{y_2}{1-y_2}} \leq \frac{y}{1-y}, \text{ or equivalently,}$$

$$\frac{y - 2y_1y_2}{1-y} \leq \frac{y}{1-y},$$

which obviously holds; moreover, we have equality in the limit when $y_1 \to 0$ and $y_2 \to y$.
To conclude the analysis of case 2 for $y \in [1/2, 1]$, we need to check that

$$\left(\frac{\pi}{4} + \arctan \frac{y_1}{1-y_1}\right) + \left(\frac{\pi}{4} + \arctan \frac{y_2}{1-y_2}\right) \leq \frac{3\pi}{4} + \arctan \frac{2y-1}{3-2y}, \text{ or equivalently,}$$

$$\arctan \frac{y_1}{1-y_1} + \arctan \frac{y_2}{1-y_2} \leq \frac{\pi}{4} + \arctan \frac{2y-1}{3-2y}. \tag{18}$$

Applying the tangent function to both sides of the inequality, it remains to show that

$$\frac{\frac{y_1}{1-y_1} + \frac{y_2}{1-y_2}}{1 - \frac{y_1}{1-y_1}\frac{y_2}{1-y_2}} \leq \frac{1 + \frac{2y-1}{3-2y}}{1 - \frac{2y-1}{3-2y}}, \text{ or equivalently,}$$

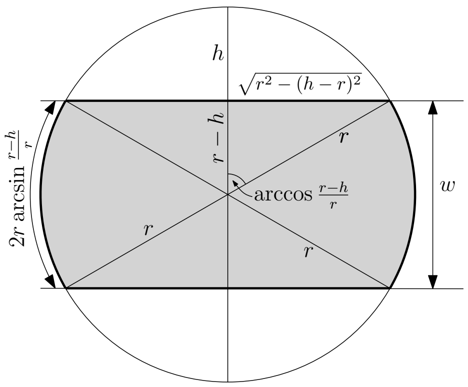

Figure 5 A convex body B of maximum area subject to the constraints that $\mathrm{diam}(B) = 2r$ and the width of B is w.

$$\frac{y - 2y_1y_2}{1-y} \le \frac{1}{2(1-y)}.$$

The last inequality holds since for $y_1 + y_2 = y$ with the above constraints, the product y_1y_2 is minimized when $y_1 = y - 1/2$ and $y_2 = 1/2$. Moreover, we have equality in the limit when $y_1 \to y - 1/2$ and $y_2 \to 1/2$. This completes the analysis of Case 2, and hence the proof of Lemma 7.

◀

Remark. Note that $f(y)$ is an increasing function of y and $f(1) = 1$; hence $f(y) \le 1$ for $y \in [0,1]$, with a strict inequality for $y < 1$. Thus the inequality in Lemma 7 is an improvement of the inequality $\Lambda \le \pi$ used earlier in (13).

The second key component is establishing an upper bound on the sum of areas $C = \sum_{i=1}^{4} C_i$.

▶ **Lemma 8.** *Define the function*

$$g(y) = 2 - \frac{2y}{\frac{\pi}{2} - \arccos y + y\sqrt{1-y^2}}, \quad y \in [0,1].$$

Then $2C \le g(y)\, A$. *This inequality is the best possible for all* $y \in (0,1]$.

It is known [26, Exercise 6–10] that convex body B of maximum area, subject to the constraints that $\mathrm{diam}(B) = D$ and the width of B is w, is is the intersection of a disk of diameter D and a strip of parallel lines at distance w symmetric about the center of the disk, as shown in Figure 5. Denote by $\Psi(r,h)$ the area of a circular cap of height h in a disk of radius r. An easy calculation [27] yields

$$\Psi(r,h) = r^2 \arccos((r-h)/r) - (r-h)\sqrt{r^2-(r-h)^2}.$$

Proof of Lemma 8. Recall that P has unit diameter and is enclosed in between two parallel lines at distance y. As noted above, this implies that

$$A \le \frac{\pi}{4} - 2\Psi\left(\frac{1}{2}, \frac{1-y}{2}\right) = \frac{\pi}{4} - \frac{1}{2}\arccos y + \frac{y}{2}\sqrt{1-y^2}. \tag{19}$$

Consequently, by taking into account that $C = A - y/2$, (19) yields

$$\frac{2C}{A} = \frac{2A-y}{A} = 2 - \frac{y}{A} \leq \frac{2y}{\frac{\pi}{2} - \arccos y + y\sqrt{1-y^2}},$$

as desired. ◀

Remark. Note that $g(y) \leq 1$, for $y \in [0,1]$, with a strict inequality for $y < 1$. Thus the inequality in Lemma 8 is an improvement of the inequality $2C \leq A$ used earlier in (13).

The inequality $\sqrt{\pi A} \leq L/2$, from (10), was used in obtaining the approximating ratio of 1.75. The following lemma refines this inequality in terms of y.

▶ **Lemma 9.** *Define the function*

$$\tau(y) = \frac{1}{4}\sqrt{1 - \frac{\pi^2(1-y)^2}{16(\arcsin y + \sqrt{1-y^2})^2}}, \quad y \in [0,1].$$

Then $\sqrt{\pi A}/(2L) \leq \tau(y)$. This inequality is the best possible for all $y \in (0,1]$.

Recall [26, Exercise 6–9] that for every convex body K in the plane, a centrally symmetric convex body K^* is obtained by the symmetrization $K^* = \frac{1}{2}(K - K)$. It is well known that K^* has the same diameter, width and perimeter as K, and that the area of K^* is greater than or equal to that of K; moreover, along every direction, the distance between the two parallel supporting lines of K^* is the same as that for K. This implies that if K has diameter $2r$ and width w, then K^* is contained in a disk or radius r, and in a strip of width w, where the two parallel lines of the strip are equidistant from the center of the disk; see Figure 5.

Proof of Lemma 9. Let $A^* = \text{area}(P^*)$ and $L^* = \text{per}(P^*)$. As noted above, $A \leq A^*$ and $L = L^*$, hence $\sqrt{\pi A}/(2L) \leq \sqrt{\pi A^*}/(2L^*)$. Therefore, it suffices to prove the lemma for P^*.

Let P be a centrally symmetric convex body of diameter 1 and width y. Then the circumradius R of P is $1/2$, and its inradius r is at most $y/2$. Consequently, $R - r \geq (1-y)/2$. Clearly, we have $L \in [2,\pi]$. As noted above, P is contained in the convex body that is the intersection of a disk of diameter 1 and a strip of width y, where the two parallel lines of the strip are equidistant from the disk center (Figure 5). The boundary of this convex body consists of two circular arcs each of length $\arcsin y$, and two line segments each of length $\sqrt{1-y^2}$, and so its perimeter is $2\arcsin y + 2\sqrt{1-y^2}$. Consequently, the perimeter L of P is bounded above as $L \leq 2\arcsin y + 2\sqrt{1-y^2}$.

It is known [22] that for every planar convex body of area A, circumradius R, inradius r, and perimeter L, we have the following sharpening of the first inequality in (10):

$$4\pi A \leq L^2 - \pi^2(R-r)^2.$$

It follows that

$$\frac{\sqrt{\pi A}}{2L} \leq \frac{\sqrt{\frac{L^2}{4} - \frac{\pi^2}{16}(1-y)^2}}{2L} = \frac{1}{4}\sqrt{1 - \frac{\pi^2(1-y)^2}{4L^2}} =: \delta(L,y).$$

Note that for every fixed $y \in [0,1]$, $\delta(L,y)$ is an increasing function in L for $L \in [2,\pi]$. Recall that $L \leq 2\arcsin y + 2\sqrt{1-y^2} \leq \pi$ for $y \in [0,1]$, and by the monotonicity of $\delta(L,y)$ we get

$$\frac{\sqrt{\pi A}}{2L} \leq \frac{1}{4}\sqrt{1 - \frac{\pi^2(1-y)^2}{16(\arcsin y + \sqrt{1-y^2})^2}}$$

for $y \in [0,1]$; consequently, we have $\sqrt{\pi A}/(2L) \leq \tau(y)$, as claimed. ◀

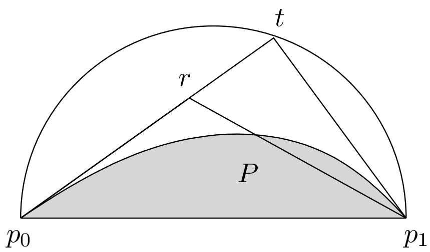

Figure 6 The polygon P lies in a right triangle.

Remark. Note that $\tau(y) \le 1/4$, for $y \in [0,1]$, with a strict inequality for $y < 1$. Thus the inequality in Lemma 9 is an improvement of the inequality $\sqrt{\pi A} \le L/2$ used earlier in (13).

We now finalize the proof of the upper bound in Theorem 1. By using the bounds in Lemmas 7, 8, and 9, we obtain the sharper analogues of (13) and (14):

$$\sqrt{\Lambda \cdot 2C} \le \sqrt{(f(y)\,\pi)(g(y)\,A)} = \sqrt{f(y)g(y)\pi A} \tag{20}$$

$$\rho \le \frac{3L + \sqrt{\Lambda \cdot 2C}}{2L} \le \frac{3L + \sqrt{f(y)g(y)\pi A}}{2L} \le 1.5 + \sqrt{f(y)g(y)}\,\tau(y). \tag{21}$$

A numerical calculation show that $\sqrt{f(y)g(y)}\,\tau(y)$ is maximized to $0.21757\ldots$ at $y = 0.87894\ldots$ and correspondingly $f(y) = 0.92439\ldots$, $g(y) = 0.82244\ldots$, and $\tau(y) = 0.24952\ldots$. Consequently, $\rho \le 1.5 + 0.21757\ldots = 1.71757\ldots < 1.7176$.

The Final Bound on the Approximation Ratio. We refine Lemma 5 using a stronger condition, namely we assume that the polygon $P = (p_0, p_1, \ldots, p_m)$ not only lies in a half-disk of diameter p_0p_1, but also in a right triangle with diameter p_0p_1; refer to Figure 6.

Assume that $P = (p_0, p_1, \ldots, p_m)$ lies in a half-disk of diameter p_0p_1, such that p_0 is at the origin and p_1 is on the positive x-axis. Let $\alpha = \angle p_1p_0p_m < \pi/2$ and $A = \text{area}(P)$. Parametrize the polygonal arc $(p_1, \ldots, p_m)$ in polar coordinate by $(\theta, \lambda(\theta))$ for $\theta \in [0, \alpha]$. Then $\sum_{i=2}^m h_i \le \int_0^\alpha \lambda(\theta)\,\mathrm{d}\theta \le \sqrt{2\alpha A}$.

▶ **Lemma 10.** *Assume that $P = (p_0, p_1, \ldots, p_m)$ is contained in a right triangle Δp_0p_1t with $\angle p_0tp_1 = \pi/2$ and $\angle tp_0p_1 = \alpha$. Let $A = \text{area}(P)$. Then $\sum_{i=2}^m h_i \le \sqrt{2\mu(\alpha)A} \le \sqrt{2\alpha A}$, where*

$$\mu(\alpha) = \ln^2\left(\frac{1+\sin\alpha}{\cos\alpha}\right) \Big/ \tan\alpha.$$

Proof. Let r be a point on the edge p_0t such that the area of the triangle Δp_0p_1r is A. Observe that to extend the lengths $\lambda(\theta)$ of the triangles of angle $d\theta$ in the fan by the same amount, the triangles corresponding to larger values of β require more area. This implies that, with α, A, and $|p_0p_1|$ fixed, $\int_0^\alpha \lambda(\theta)\,\mathrm{d}\theta$ is maximized when P is exactly the triangle Δp_0p_1r. Moreover, with α and A fixed, and with $|p_0p_1|$ variable, $\int_0^\alpha \lambda(\theta)\,\mathrm{d}\theta$ is maximized when Δp_0p_1r is a right triangle with angle $\angle p_0rp_1 = \pi/2$.

Put $x = |p_0p_1|$ in this extreme case. Then $\lambda(\theta) = x\cos\alpha/\cos\theta$. In the right triangle Δp_0p_1r with $\angle p_0rp_1 = \pi/2$, $2A = |p_0r| \cdot |p_1r| = x\sin\alpha \cdot x\cos\alpha$. Thus $x\cos\alpha = \sqrt{2A/\tan\alpha}$. By integral calculus, we have

$$\int_0^\alpha \frac{\mathrm{d}\theta}{\cos\theta} = \ln\left(\frac{1+\sin\alpha}{\cos\alpha}\right).$$

By Lemma 6 it follows that

$$\sum_{i=2}^{m} h_i \leq \int_0^{\alpha} \lambda(\theta)\,\mathrm{d}\theta = x\cos\alpha \int_0^{\alpha} \frac{\mathrm{d}\theta}{\cos\theta} = \sqrt{2A/\tan\alpha}\,\ln\left(\frac{1+\sin\alpha}{\cos\alpha}\right) = \sqrt{2\mu(\alpha)A}.$$

Moreover, with α and A fixed, (9) holds for $\lambda(\theta) = x\cos\alpha/\cos\theta$ too. Thus $\sqrt{2\mu(\alpha)A} \leq \sqrt{2\alpha A}$. ◀

The function $\mu(\alpha)$ is concave for $0 \leq \alpha < \pi/2$. By Jensen's inequality, the constraint $\sum_{i=1}^{4} \alpha_i \leq f(y)\pi$ implies that $\sum_{i=1}^{4} \mu(\alpha_i) \leq 4\mu(f(y)\pi/4)$. Let $\hat{f}(y) = 4\mu(f(y)\pi/4)/\pi$. Then $\sum_{i=1}^{4} \mu(\alpha_i) \leq \hat{f}(y)\cdot\pi$.

Using $\hat{f}(y)$ instead of $f(y)$ in the final analysis, a numerical calculation show that $\sqrt{f(y)g(y)}\,\tau(y)$ is maximized to $0.21674\ldots$ at $y = 0.87256\ldots$ and correspondingly $\hat{f}(y) = 0.91364\ldots$, $g(y) = 0.82615\ldots$, and $\tau(y) = 0.24947\ldots$. Consequently, $\rho \leq 1.5 + 0.21674\ldots = 1.71674\ldots < 1.7168$, as claimed. ◀

Remark. Note that for a disk Ω of unit radius, every interior barrier must have length at least 2π. Indeed, for every point $p \in \partial\Omega$, blocking p from the line ℓ_p tangent to Ω at p requires that $p \in \Gamma$. It follows that $\partial\Omega \subseteq \Gamma$, which in turn yields $|\Gamma| \geq |\partial\Omega| = 2\pi$, as claimed. On the other hand, a length of 2π clearly suffices. In contrast, using a $(1+\varepsilon)$-interior barrier yields a significant length-reduction, as shown in the following.

▶ **Corollary 11.** *For any $\varepsilon > 0$, the unit disk Ω admits a $(1+\varepsilon)$-interior barrier of length at most $(\pi+2)(1+\varepsilon)$. In particular, the unit disk Ω admits a $(1+\varepsilon)$-interior barrier of length at most 5.1416, provided that $\varepsilon > 0$ is sufficiently small.*

Proof. Assume that Ω is centered at the origin. For a given $\varepsilon > 0$, let n be a sufficiently large even integer such that a regular n-gon P_n inscribed in $(1+\varepsilon)\Omega$ contains Ω, and P_n has a horizontal diameter pq. Consider an interior barrier for P_n that consists of the half of the perimeter of P_n below the x-axis and the heights of a fan triangulation for the remainder of P_n above the x-axis.

We use Lemma 6 for bounding the sum of heights, and for this purpose, we parametrize P_n in polar coordinates with respect to vertex p. Observe that $\lambda(\theta) \leq 2(1+\varepsilon)\cos\theta$, for $\theta \in [0, \frac{\pi}{2}]$. The perimeter of P_n is bounded from above by the corresponding perimeter of $(1+\varepsilon)\Omega$, and the sum of heights is at most $\int_0^{\pi/2} \lambda(\theta)\,\mathrm{d}\theta$ by Lemma 6. Hence the length of this barrier is bounded from above by

$$(1+\varepsilon)\pi + \int_0^{\pi/2} 2(1+\varepsilon)\cos\theta\,\mathrm{d}\theta = (\pi+2)(1+\varepsilon),$$

which is at most 5.1416 when ε is sufficiently small. ◀

4 A Lower Bound on the Approximation Ratio of Shermer's Algorithm

In this section we prove the lower bound given in Theorem 2. Refer to Figure 7.

Let ℓ_S and ℓ_P, respectively, be the lengths of the two barriers illustrated on the left and the right. We have

$$\ell_S = 1 + 2a,$$

and

$$\ell_P(b) = b + 2a(1-b) + 2\sqrt{(a^2-1)b^2 + (2-b)^2}.$$

Figure 7 Two barriers for a rhombus with shorter diagonal of length 2 and with sides of equal length $a \geq 4$. Left: The barrier found by Shermer's procedure. Right: The barrier found by Provan et al. [23]. The height from the top vertex to the dashed line in the right barrier is $b \leq 1$.

The derivative of $\ell_P(b)$ is

$$\ell'_P(b) = 1 - 2a + \frac{(a^2-1)\cdot 2b - 2(2-b)}{\sqrt{(a^2-1)b^2 + (2-b)^2}} = 1 - 2a + \frac{2a^2b - 4}{\sqrt{a^2b^2 - 4b + 4}}.$$

Setting $\ell'_P(b) = 0$ yields

$$(4a^2 - 4a + 1)(a^2b^2 - 4b + 4) = 4a^4b^2 - 16a^2b + 16,$$

which simplifies to

$$a^2b^2 - 4b - 4(4a^2 - 4a - 3)/(4a - 1).$$

For $a \geq 4$, this quadratic equation in b has a unique positive real root

$$b_0 = \frac{4 + \sqrt{16 + 16a^2(4a^2 - 4a - 3)/(4a-1)}}{2a^2} = \frac{2 + 2\sqrt{1 + a^2(4a^2 - 4a - 3)/(4a-1)}}{a^2},$$

and the length $\ell_P(b)$ of the barrier of Provan et al. is minimized when $b = b_0$. In particular, $b_0 = 1$ when $a = 4$, and $b_0 < 1$ when $a > 4$.

Assisted by a computer program, we can verify that for $a \geq 4$, the ratio $\ell_S/\ell_P(b_0)$ is maximized to $1.00769\ldots$ when $a = 16.299\ldots$ ($\sqrt{a^2-1} = 16.268\ldots$), and correspondingly $b_0 = 0.49053\ldots$. ◀

5 Concluding Remarks

Observe that when the input is a regular n-gon P_n, our algorithm returns a barrier whose length is at most $1.5\pi + \int_{\pi/4}^{\pi/2} \cos(\theta)\, d\theta = 1.5\pi + 2 - \sqrt{2} = 1.6845\ldots\pi$. On the other hand our algorithm attains a ratio at most 1.7176 for *every* convex polygon. This may indicate that its analysis is quite tight.

The length of the barrier constructed in Corollary 11 for the regular n-gon P_n is at most $5.1416 = 1.6367\ldots\pi$. We believe that this barrier is not too far from the optimal one. This may indicate that the algorithm itself is quite good, at least for polygons similar in shape, fat polygons in particular.

All these estimates however are expressed in terms of the same trivial lower bound (1). The reader can notice that for polygons P that are long and skinny, the lower bound $\text{per}(P)/2$ is not too bad. Indeed we indicate below how to modify our algorithm so that it returns an interior barrier whose length is close to $\text{per}(P)/2$. On the other hand, it is worth pointing out that the bottleneck in the analysis of our algorithm is for large values of y, namely $y \approx 0.88$. For such values, we believe that the lower bound $\text{per}(P)/2$ in (1) is quite loose. The conclusion is that further improvement in the approximation ratio of our algorithm for interior barriers relies on an improved lower bound beyond $\text{per}(P)/2$ for fat polygons (i.e., with large y, in our analysis.), and that the case of small y is not too hard to deal with. We conjecture that the approximation ratio of the following algorithm is below 1.1.

Algorithm. In addition to the four candidate interior barriers Γ_i, $i = 1, 2, 3, 4$, defined earlier, we add two new candidates, Γ_+ and Γ_-. The barrier Γ_+ consists of the boundary of P below the diameter pq, the height h_r from r in Δrpq, and the two fans of Shermer heights to the left and to the right of h_r, corresponding to the minimum angles in Δrdp and Δrcq. The barrier Γ_- is defined analogously, by the boundary of P above the diameter pq, etc. The algorithm returns the shortest of these six barriers.

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