Lower Bounds on Expansions of Graph Powers

Tsz Chiu Kwok and Lap Chi Lau

The Chinese University of Hong Kong
Shatin, Hong Kong
`{tckwok,chi}@cse.cuhk.edu.hk`

──── **Abstract** ────

Given a lazy regular graph G, we prove that the expansion of G^t is at least $\Omega(\sqrt{t})$ times the expansion of G. This bound is tight and can be generalized to small set expansion. We show some applications of this result.

1 Introduction

Let $G = (V, E, w)$ be an undirected weighted graph with $n = |V|$ vertices. The expansion of a set $S \subseteq V$ is defined as

$$\phi(S) := \frac{1}{|S|} \sum_{u \in S, v \notin S} w(u, v),$$

and the expansion of G is defined as

$$\phi(G) := \min_{S \subseteq V, |S| \leq n/2} \phi(S).$$

Graph expansion is a fundamental parameter with diverse applications in theoretical computer science [4].

A well-known operation to improve the graph expansion is by taking the t-th power of G, which has a natural correspondence to simulating the random walk on G for t steps. In our setting, we assume that G is 1-regular, that is, $\sum_{v \in V} w(u, v) = 1$ for every $u \in V$. We also assume that G is lazy, that is, $w(u, u) \geq \frac{1}{2}$ for every $u \in V$. Let A be the adjacency matrix of G with $A_{u,v} = w(u, v)$ for any $u, v \in V$, which corresponds to the transition matrix of the random walk on G. The t-th power of G, denoted by G^t, is defined as the undirected graph with adjacency matrix A^t, which corresponds to the transition matrix of the t-step random walk of G. Note that G^t is also 1-regular if G is.

The question we study is to prove lower bounds on $\phi(G^t)$ in terms of $\phi(G)$. Besides being a basic graph theoretical question, proving lower bounds on $\phi(G^t)$ has applications in hardness of approximation [3, 8]. Our main result is a tight lower bound on the expansion of the graph power of a lazy 1-regular graph.

1.1 Previous Work

There is a spectral method to show that $\phi(G^t)$ is larger than $\phi(G)$ for large enough t. This is based on the connection between the graph expansion and the second eigenvalue of the adjacency matrix. Let $1 = \alpha_1 \geq \alpha_2 \geq \ldots \geq \alpha_n \geq 0$ be the eigenvalues of the adjacency

matrix A_G of G, where $\alpha_1 = 1$ because G is 1-regular and $\alpha_n \geq 0$ because G is lazy. Let $L_G := I - A_G$ be the Laplacian matrix of G and $0 = \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n \leq 1$ be its eigenvalues. Note that $\lambda_i = 1 - \alpha_i$ for $1 \leq i \leq n$. Cheeger's inequality [1] states that

$$\frac{1}{2}\lambda_2 \leq \phi(G) \leq \sqrt{2\lambda_2}.$$

Note that the eigenvalues of A^t is $1 = \alpha_1^t \geq \alpha_2^t \geq \ldots \geq \alpha_n^t \geq 0$, and thus the i-th eigenvalue of the Laplacian matrix of G^t is $1 - \alpha_i^t = 1 - (1 - \lambda_i)^t$. Therefore, by Cheeger's inequality, we have

$$\phi(G^t) \geq \frac{1}{2}(1 - (1 - \lambda_2)^t) \geq \frac{1}{2}(1 - (1 - \frac{1}{2}t\lambda_2)) = \frac{1}{4}t\lambda_2 \geq \frac{1}{8}t \cdot \phi(G)^2 = \Omega(t \cdot \phi(G)^2),$$

where the second inequality follows from Fact 2.1 when $t\lambda_2 < 1/2$.

Recently, the spectral method was extended to prove lower bounds on the small set expansion of a graph. Given $0 < \delta < 1/2$, the small set expansion of G is defined as

$$\phi_\delta(G) := \min_{S \subseteq V, |S| \leq \delta n} \phi(S).$$

Raghavendra and Schramm [8] proved an analog of the above bound for small set expansion:

$$\phi_{\Omega(\delta)}(G^t) = \Omega(t \cdot \phi_\delta(G)^2),$$

when G is a lazy 1-regular graph and $t = O(1/\phi_\delta(G)^2)$. The proof is based on the techniques developed in [2] relating higher eigenvalues to small set expansion. They used this lower bound to amplify the hardness of the small set expansion problem; see Section 3.3 for more discussions.

1.2 Our Results

Our main result is a tight lower bound on $\phi(G^t)$.

▶ **Theorem 1.** *Let G be an undirected 1-regular lazy graph. For any non-negative integer t, we have*

$$\phi(G^t) \geq \frac{1}{20}(1 - (1 - \phi(G))^{\sqrt{t}}) = \Omega(\min(\sqrt{t} \cdot \phi(G), 1)).$$

This is a quadratic improvement of the previous bound. This bound is tight up to a constant factor for all t as we will show examples (e.g. cycles) in Section 2.6.

Observe that the above spectral method only showed that $\phi(G^t) > \phi(G)$ when $t = \Omega(1/\phi(G))$ but did not show that $\phi(G^t) > \phi(G)$ for small t. Theorem 1 implies that $\phi(G^t) > \phi(G)$ for some small constant t. Actually, we can show that $\phi(G^3) > \phi(G)$ when $\phi(G) < 1/2$ by a more careful calculation.

▶ **Theorem 2.** *Let G be an undirected 1-regular lazy graph with even n. We have*

$$\phi(G^3) \geq \frac{3}{2}\phi(G) - 2\phi(G)^3.$$

Theorem 1 can be extended easily to small set expansion.

▶ **Theorem 3.** *Let G be an undirected 1-regular lazy graph. For any non-negative integer t, we have*

$$\phi_{\delta/2}(G^t) \geq \frac{1}{20}(1 - (1 - 2\phi_\delta(G))^{\sqrt{t}}) = \Omega(\min(\sqrt{t} \cdot \phi_\delta(G), 1)).$$

We show some applications of our results in Section 3, including the gap amplification result in [8] for small set expansion and some reductions for proving Cheeger-type inequalities [1, 6].

1.3 Techniques

Instead of using the spectral method, we use the Lovász-Simonovits curve [7] which was designed to analyze the mixing time of random walk using graph expansion. As it turns out, this more combinatorial approach has the advantage of directly reason about graph expansion without having the quadratic loss in the spectral method.

Given an initial probability distribution p on the vertex set, let $C^{(t)}(x)$ be the sum of the probability of the x largest vertices after t steps of random walk on G. First, we observe in Lemma 6 that

$$\phi_{G^t}(S) \geq 1 - C^{(t)}(|S|)$$

when the initial distribution is $\chi_S/|S|$ where χ_S is the characteristic vector of S. Hence, to lower bound $\phi_{G^t}(S)$, we can instead upper bound $C^{(t)}(|S|)$. Imprecisely, with the method developed by Lovász and Simonovits (see Section 2.2), we can essentially argue that for all S with $|S| \leq n/2$,

$$C^{(t)}(|S|) \lesssim \frac{1}{2^t} \sum_{i=0}^{t} \binom{t}{i} C^{(0)}\big((1 - \phi(G))^i (1 + \phi(G))^{t-i}|S|\big)$$

$$= \frac{1}{2^t} \sum_{i=0}^{t} \binom{t}{i} \min\{(1 - \phi(G))^i (1 + \phi(G))^{t-i}, 1\},$$

where the equality holds because $C^{(0)}(x) = \min\{x/|S|, 1\}$ as the initial distribution is $\chi_S/|S|$. Since there is at least a $1/10$ fraction of terms in the summation with $i \geq t/2 + \sqrt{t}$, we have

$$C^{(t)}(|S|) \lesssim \frac{1}{10}(1 - \phi(G))^{\sqrt{t}} + \frac{9}{10} \leq \frac{1}{10}(1 - \frac{1}{2}\sqrt{t} \cdot \phi(G)) + \frac{9}{10},$$

where the last inequality is by Fact 2.1 when $\sqrt{t} \cdot \phi(G) \leq 1/2$. Therefore, for all S with $|S| \leq n/2$, we have

$$\phi_{G^t}(S) \geq \frac{1}{20}\sqrt{t} \cdot \phi(G), \text{ and therefore } \phi(G^t) = \Omega(\sqrt{t} \cdot \phi(G)).$$

We need to be careful to make the arguments in $\lesssim$ precise and this is some technicality of the proof, but the main ideas are pretty accurately summarized in this section.

2 Expansion of Graph Power

2.1 Preliminaries

When G is clear from the context, we use $\phi = \phi(G)$ to denote the conductance of G.

Let χ_S be a column vector such that $\chi_S(u) = 1$ if $u \in S$ and $\chi_S(u) = 0$ otherwise. The expansion of S can be expressed as

$$\phi(S) = \frac{\chi_S^T(I - A)\chi_S}{|S|}.$$

The following fact is used frequently in the proof.

▶ **Fact 2.1.** *For any $z \in [0, 1]$, we have*

$$(1 - z)^t \geq 1 - zt, \quad or \quad 1 - (1 - z)^t \leq zt.$$

For any $zt \in [0, 1/2]$, we have

$$(1 - z)^t \leq \exp(-zt) \leq 1 - \frac{1}{2}zt, \quad or \quad 1 - (1 - z)^t \geq \frac{1}{2}zt.$$

2.2 Lovász-Simonovits Curve

Lovász and Simonovits [7] introduced a curve which is useful in bounding mixing time using graph expansion. Given a probabilistic vector $p : V \to \mathbb{R}_{\geq 0}$, the curve is defined as

$$C(p, x) = \max_{\delta_1 + \cdots + \delta_n = x, 0 \leq \delta_i \leq 1} \sum_{i=1}^{n} \delta_i p_i,$$

for $x \in [0, n]$. When x is an integer, $C(p, x)$ is simply the sum of the largest x values in the vector p, and it is linear between two integral values. Clearly $C(p, x)$ is concave. We use $C^{(t)}(x)$ to denote $C(A^t p, x)$ when p is clear from the context. We use $\overline{x}$ to denote $\min(x, n - x)$ for $x \in [0, n]$. This notation is frequently used and should be interpreted as the distance to the boundary. The following lemma shows that the curves "drop" faster when the expansion of G is larger.

▶ **Lemma 4** ([7]). *If G is a lazy 1-regular graph, then for any integer $t \geq 0$ and any integer $x \in [0, n]$, we have*

$$C^{(t+1)}(x) \leq \frac{1}{2} \left(C^{(t)}(x - 2\phi\overline{x}) + C^{(t)}(x + 2\phi\overline{x}) \right).$$

We remark that Lemma 4 only give bounds on integral values[1]. In our proof, however, we require bounds for all $x \in [0, n]$. The following lemma provides a slightly weaker bound that also holds for fractional x when the graph is lazy 1-regular.

▶ **Lemma 5.** *If G is a lazy 1-regular graph, then for any integer $t \geq 0$ and $x \in [0, n]$, we have*

$$C^{(t+1)}(x) \leq \frac{1}{2} \left(C^{(t)}(x - \phi\overline{x}) + C^{(t)}(x + \phi\overline{x}) \right).$$

Proof. Since $C^{(t)}$ is concave, we have

$$C^{(t)}(x - \beta\overline{x}) + C^{(t)}(x + \beta\overline{x}) \leq C^{(t)}(x - \gamma\overline{x}) + C^{(t)}(x + \gamma\overline{x}) \text{ for } \beta > \gamma. \tag{1}$$

We will prove that

$$C^{(t+1)}(x) \leq \frac{1}{2} \left(C^{(t)}(x - 2\phi'\overline{x}) + C^{(t)}(x + 2\phi'\overline{x}) \right) \tag{2}$$

where $\phi' = \frac{n-1}{n}\phi$, and this would imply the lemma by (1) since $\phi' \geq \frac{1}{2}\phi$.

Note that for any integral $x \in [0, n-1]$ and any $\alpha \in [0, 1]$,

$$
\begin{aligned}
C^{(t+1)}(x + \alpha) &= (1 - \alpha)C^{(t+1)}(x) + \alpha C^{(t+1)}(x + 1) \\
&\leq (1 - \alpha) \left(C^{(t)}(x - 2\phi\overline{x}) + C^{(t)}(x + 2\phi\overline{x}) \right) \\
&\quad + \alpha \left(C^{(t)}(x + 1 - 2\phi\overline{(x+1)}) + C^{(t)}(x + 1 + 2\phi\overline{(x+1)}) \right) \\
&= \left((1 - \alpha)C^{(t)}(x - 2\phi\overline{x}) + \alpha C^{(t)}(x + 1 - 2\phi\overline{(x+1)}) \right) \\
&\quad + \left((1 - \alpha)C^{(t)}(x + 2\phi\overline{x}) + \alpha C^{(t)}(x + 1 + 2\phi\overline{(x+1)}) \right) \\
&\leq C^{(t)}(x + \alpha - 2\phi((1 - \alpha)\overline{x} + \alpha\overline{(x+1)})) \\
&\quad + C^{(t)}(x + \alpha + 2\phi((1 - \alpha)\overline{x} + \alpha\overline{(x+1)})),
\end{aligned}
$$

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[1] It was claimed in [7] that the lemma holds for any $x \in [0, n]$, but later it was pointed out in [10] that the lemma only holds for integral x when the graph is lazy 1-regular.

where the first inequality follows from Lemma 4, and last inequality holds because $C^{(t)}$ is concave. If $(1-\alpha)\overline{x} + \alpha\overline{(x+1)} = \overline{(x+\alpha)}$, then Lemma 4 holds and the lemma follows by (1).

Note that the only case where $(1-\alpha)\overline{x} + \alpha\overline{(x+1)} \neq \overline{(x+\alpha)}$ is when n is odd and $x = (n-1)/2$. At that time, $\overline{x} = \overline{(x+1)} = x$ and thus $(1-\alpha)\overline{x} + \alpha\overline{(x+1)} = x$. Therefore, when n is odd and $x = (n-1)/2$, we have

$$C^{(t+1)}(x+\alpha)$$
$$\leq \frac{1}{2}\left(C^{(t)}(x+\alpha-2\phi x) + C^{(t)}(x+\alpha+2\phi x)\right)$$
$$\leq \frac{1}{2}\left(C^{(t)}\left(x+\alpha-2(\frac{n-1}{n})\cdot\phi\cdot\overline{(x+\alpha)}\right) + C^{(t)}\left(x+\alpha+2(\frac{n-1}{n})\cdot\phi\cdot\overline{(x+\alpha)}\right)\right)$$
$$= \frac{1}{2}\left(C^{(t)}\left(x+\alpha-2\phi'\cdot\overline{(x+\alpha)}\right) + C^{(t)}\left(x+\alpha+2\phi'\cdot\overline{(x+\alpha)}\right)\right),$$

where the later inequality holds because $C^{(t)}$ is concave and $\overline{x+\alpha} \leq x + \frac{1}{2} = n/2$. ◀

2.3 Proof of Theorem 1

As mentioned in the proof outline in Section 1.3, we first show that we can prove a lower bound on $\phi(G^t)$ by proving an upper bound on $C^{(t)}(|S|)$ for the initial distribution $\chi_S/|S|$.

▶ **Lemma 6.** *Suppose that for any set $S \subseteq V$ with $|S| \leq n/2$, we have $C^{(t)}(|S|) \leq 1 - \alpha$ for the initial distribution $p = \chi_S/|S|$, then we can conclude that $\phi(G^t) \geq \alpha$.*

Proof. Let S be the set attaining minimum expansion in G^t, that is, $|S| \leq n/2$ and $\phi_{G^t}(S) = \phi(G^t)$. For the initial distribution $p = \chi_S/|S|$,

$$C^{(t)}(|S|) = C(A^t p, |S|) \geq \chi_S^T A^t p = \frac{\chi_S^T A^t \chi_S}{|S|} = 1 - \frac{\chi_S^T(I-A^t)\chi_S}{|S|} = 1 - \phi_{G^t}(S).$$

Therefore, we have $\phi(G^t) = \phi_{G^t}(S) \geq 1 - C^{(t)}(|S|) \geq \alpha$. ◀

With Lemma 6, it remains to upper bound $C^{(t)}(|S|)$ for the initial distribution $\chi_S/|S|$ for any S with $|S| \leq n/2$. It turns out that there is a good upper bound independent of $|S|$.

▶ **Lemma 7.** *For any S with $|S| \leq n/2$, for the initial distribution $p = \chi_S/|S|$, for any non-negative integer t, we have*
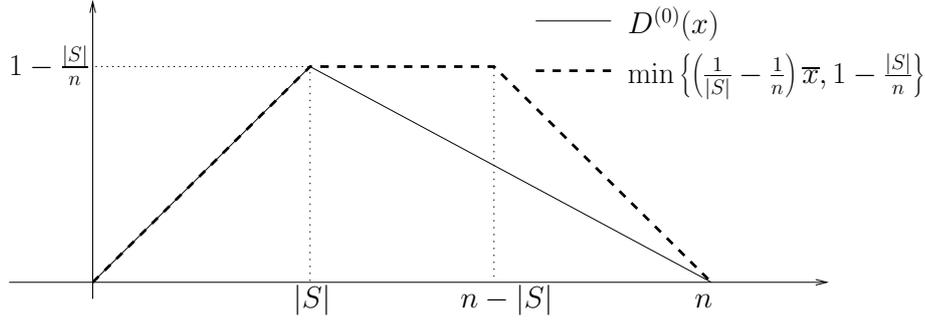
$$C^{(t)}(|S|) \leq 1 - \frac{1}{20}(1-(1-\phi)^{\sqrt{t}}).$$

Proof. For technical reasons, we consider $D^{(t)}(x) = C^{(t)}(x) - x/n$ instead to make the argument more symmetric. See Figure 1 for the definition of $D^{(0)}$. Note that Lemma 5 still holds for $D^{(t)}$ since x/n is linear. So, we have

$$D^{(t+1)}(x) \leq \frac{1}{2}\left(D^{(t)}(x-\phi\overline{x}) + D^{(t)}(x+\phi\overline{x})\right).$$

By applying this equation repeatedly, we have

$$D^{(t)}(x) \leq \frac{1}{2^t}\sum_{T\in\{-1,1\}^t} D^{(0)}(f_T(x)), \tag{3}$$

◼ **Figure 1** The solid line is the curve $D^0(x)$ and the dotted line is the upper bound on $D^{(0)}(x)$ that is stated in (5).

where T is a sequence of t ± 1-bits and f_T is defined recursively as follows. In the base case, when the sequence is empty, we define $f_{()}(x) = x$ for any $x \in [0, n]$. For any partial sequence T', we define

$$f_{(T',+1)}(x) = \begin{cases} f_{T'}(x) - \phi \cdot f_{T'}(x) & \text{if } f_{T'}(x) \le n/2 \\ f_{T'}(x) + \phi \cdot \overline{f_{T'}(x)} & \text{if } f_{T'}(x) > n/2, \end{cases}$$

and

$$f_{(T',-1)}(x) = \begin{cases} f_{T'}(x) + \phi \cdot f_{T'}(x) & \text{if } f_{T'}(x) \le n/2 \\ f_{T'}(x) - \phi \cdot \overline{f_{T'}(x)} & \text{if } f_{T'}(x) > n/2. \end{cases}$$

We can view $+1$ as moving in the direction towards boundary and -1 as moving in the direction towards center. Recall that $\overline{x} = \min\{x, n - x\}$ can be viewed as the distance to the boundary. In the following, we focus on the distance to the boundary of a point rather than its actual location. It follows from the definition that for any $x \in [0, n]$, we have

$$\overline{f_{+1}(x)} = \overline{x} - \phi \overline{x} = (1 - \phi) \cdot \overline{x},$$

and

$$\overline{f_{-1}(x)} \le \overline{x} + \phi \overline{x} = (1 + \phi)\overline{x} \le (1 - \phi)^{-1} \cdot \overline{x}.$$

Therefore, $\overline{f_{T_i}(x)} \le (1 - \phi)^{T_i} \cdot \overline{x}$ where T_i is the i-th bit in the sequence T, and hence

$$\overline{f_T(x)} = \overline{f_{T_t} \circ f_{T_{t-1}} \circ \cdots \circ f_{T_1}(x)} \le (1-\phi)^{T_t} \cdot \overline{f_{T_{t-1}} \circ \cdots \circ f_{T_1}(x)} \le \cdots \le (1-\phi)^{\sum_{i=1}^{t} T_i} \cdot \overline{x}.$$

We call a sequence T good if $\sum_{i=1}^{t} T_i \ge \sqrt{t}$, otherwise we call it bad.

For a good T, we have $\overline{f_T(x)} \le (1 - \phi)^{\sqrt{t}} \cdot \overline{x}$, and thus

$$\overline{f_T(|S|)} \le (1 - \phi)^{\sqrt{t}} \cdot |S| \quad \text{for } |S| \le n/2 \text{ and } T \text{ good.} \tag{4}$$

As the initial distribution is $\chi_S/|S|$, for $t = 0$, we have

$$D^{(0)}(x) \le \min\left\{ \left(\frac{1}{|S|} - \frac{1}{n} \right) \overline{x}, 1 - \frac{|S|}{n} \right\}. \tag{5}$$

See Figure 1 for an illustration of the inequality. The advantage of using $D^{(t)}$ instead of $C^{(t)}$ is that we could bound $D^{(0)}(x)$ using $\overline{x}$ as shown in the above inequality.

Finally, we know that at least a $1/10$ fraction of T are good. So, for S with $|S| \leq n/2$,

$$
\begin{aligned}
D^{(t)}(|S|) &\leq \frac{1}{2^t} \sum_{T \in \{-1,1\}^t} D^{(0)}(f_T(|S|)) && \text{(by (3))} \\
&= \frac{1}{2^t} \sum_{T:\text{good}} D^{(0)}(f_T(|S|)) + \frac{1}{2^t} \sum_{T:\text{bad}} D^{(0)}(f_T(|S|)) \\
&\leq \frac{1}{2^t} \sum_{T:\text{good}} \left(\frac{1}{|S|} - \frac{1}{n} \right) \overline{f_T(|S|)} + \frac{1}{2^t} \sum_{T:\text{bad}} \left(1 - \frac{|S|}{n} \right) && \text{(by (5))} \\
&\leq \frac{1}{2^t} \sum_{T:\text{good}} \left(\frac{1}{|S|} - \frac{1}{n} \right) (1-\phi)^{\sqrt{t}} |S| + \frac{1}{2^t} \sum_{T:\text{bad}} \left(1 - \frac{|S|}{n} \right) && \text{(by (4))} \\
&\leq \frac{1}{10} \left(\left(\frac{1}{|S|} - \frac{1}{n} \right) (1-\phi)^{\sqrt{t}} |S| \right) + \frac{9}{10} \left(1 - \frac{|S|}{n} \right) \\
&= \left(1 - \frac{|S|}{n} \right) - \frac{1}{10} \left(1 - \frac{|S|}{n} - \left(\frac{1}{|S|} - \frac{1}{n} \right) (1-\phi)^{\sqrt{t}} |S| \right) \\
&= \left(1 - \frac{|S|}{n} \right) - \frac{1}{10} \left(1 - \frac{|S|}{n} \right) \left(1 - (1-\phi)^{\sqrt{t}} \right) \\
&\leq \left(1 - \frac{|S|}{n} \right) - \frac{1}{20} (1 - (1-\phi)^{\sqrt{t}}).
\end{aligned}
$$

Therefore,

$$
C^{(t)}(|S|) = D^{(t)}(|S|) + |S|/n \leq 1 - \frac{1}{20}(1 - (1-\phi)^{\sqrt{t}}).
$$

◀

Combining Lemma 6 and Lemma 7, we have

$$
\phi(G^t) \geq \frac{1}{20}(1 - (1-\phi)^{\sqrt{t}}) \geq \frac{1}{40}\sqrt{t} \cdot \phi,
$$

where the last inequality is by Fact 2.1 for $\sqrt{t} \cdot \phi \leq 1/2$. This completes the proof of Theorem 1.

2.4 Proof of Theorem 2

Theorem 1 showed that $\phi(G^t) > \phi(G)$ for a small constant t. To prove that this is true even for $t = 3$, we need to do a more careful calculation. We use the bound

$$
C^{(t+1)}(x) \leq \frac{1}{2} \left(C^{(t)}(x - 2\phi'\overline{x}) + C^{(t)}(x + 2\phi'\overline{x}) \right)
$$

for $\phi' = \frac{n-1}{n}\phi$ as was shown in (2) in the proof of Lemma 5. When $t = 3$, we have

$$
\begin{aligned}
C^{(3)}(|S|) &\leq \frac{1}{8} C^{(0)}((1-2\phi')^3 |S|) + \frac{3}{8} C^{(0)}((1-2\phi')^2(1+2\phi')|S|) + \frac{4}{8} \\
&= \frac{1}{8}(1-2\phi')^3 + \frac{3}{8}(1-2\phi')^2(1+2\phi') + \frac{4}{8} \\
&= 1 - \frac{3}{2}\phi' + 2\phi'^3.
\end{aligned}
$$

Thus we conclude $\phi(G^3) \geq \frac{3}{2}\phi' - 2\phi'^3$. Therefore, for a large graph with small conductance, taking cube increases the conductance by a factor of almost $\frac{3}{2}$. When n is even, we can replace ϕ' by ϕ as was shown in the proof of Lemma 5, and this proves Theorem 2.

2.5 Proof of Theorem 3

Our result can be easily extended to the case of small set expansion with a little loss in size. More precisely, suppose G is an undirected 1-regular lazy graph such that all sets of size at most δn have conductance ϕ_δ, where $\delta \leq 1/2$. In this setting, the following lemma holds in place of Lemma 4.

▶ **Lemma 8.** *If G is a lazy 1-regular graph, then for any integer $t \geq 0$ and any $x \in [0, \delta n]$,*

$$C^{(t+1)}(x) \leq \frac{1}{2} \left(C^{(t)}(x - 2\phi_\delta \cdot \overline{x}) + C^{(t)}(x + 2\phi_\delta \cdot \overline{x}) \right),$$

where $\overline{x} = \min(x, \delta n - x)$ here.

We remark that we do not need to fix the non-integral problem as in Lemma 5 because we only consider $x \leq \delta n \leq n/2$ (see the proof of Lemma 5).

Lemma 6 can be restated as follows with the same proof.

▶ **Lemma 9.** *Suppose that for any set $S \subseteq V$ with $|S| \leq \delta n/2$ with the initial distribution $p = \chi_S/|S|$, we have $C^{(t)}(|S|) \leq 1 - \alpha$, then we can conclude that $\phi_{\delta/2}(G^t) \geq \alpha$.*

Finally, in Lemma 7, we consider $D^{(t)}(x) = C^{(t)}(x) - \frac{x}{\delta n}$ instead, and we use the new $\overline{x}$ in the analysis. Observe that $f_T(x)$ can never leave the range $[0, \delta n]$ when x starts in the range. Therefore the same analysis applies and we have the following lemma.

▶ **Lemma 10.** *For any S with $|S| \leq \delta n/2$, for the initial distribution $p = \chi_S/|S|$, for any non-negative integer t, we have*

$$C^{(t)}(|S|) \leq 1 - \frac{1}{20}(1 - (1 - 2\phi_\delta)^{\sqrt{t}}).$$

Theorem 3 follows by combining Lemma 9 and Lemma 10.

2.6 Tight Examples

We show that the dependence on t in Theorem 1 is tight up to a constant factor. The tight example we use is a lazy cycle. Intuitively, after t steps of random walk on a lazy cycle, the final position with high probability only differs from the initial position by $O(\sqrt{t})$, and therefore the expansion should be bounded by $O(\sqrt{t})$ times the original expansion. It turns out that we can easily justify this intuition through Cheeger's inequality.

▶ **Proposition 2.2.** *Let C_n be the lazy cycle. Then we have $\phi(C_n^t) = O(\sqrt{t} \cdot \phi(C_n))$.*

Proof. As in Section 1.1, we have

$$\lambda_2(C_n^t) = 1 - (1 - \lambda_2(C_n))^t \leq t\lambda_2(C_n) = O(t \cdot \phi(C_n)^2),$$

where the inequality is by Fact 2.1 and the last equality is by the spectrum of the cycle. By Cheeger's inequality, $\phi(C_n^t) = O(\sqrt{\lambda_2(C_n^t)})$, and thus $\phi(C_n^t) = O(\sqrt{t} \cdot \phi(C_n))$. ◀

We remark that tight examples of Theorem 1 must have "high threshold rank". By the improved Cheeger's inequality in [6], we have $\phi(G) = O(k\lambda_2/\sqrt{\lambda_k})$ for any k. Therefore, by the same calculation as in Section 1.1, we have that for any k,

$$\phi(G^t) \geq \frac{1}{4}t\lambda_2 = \Omega(\frac{t \cdot \phi(G) \cdot \sqrt{\lambda_k}}{k}),$$

and therefore a graph G with $\lambda_k(G)$ small for a small k could not be a tight example for Theorem 1.

2.7 Irregular Graphs

Theorem 1 showed that $\phi(G^t) = \Omega(\sqrt{t} \cdot \phi(G))$ for a regular graph. There are different ways to generalize the statement to irregular graphs. In the following, we show that the generalization is not true if we replace expansion by conductance, and we show that the generalization is true if we replace expansion by the escape probability of a t-step random walk.

The conductance of a set $\phi(S)$ is defined as

$$\frac{\sum_{v \in S, u \notin S} w(u,v)}{\text{vol}(S)},$$

where $\text{vol}(S) := \sum_{v \in S} \deg(v)$ and the conductance of a graph $\phi(G)$ is defined as

$$\min_{S \subseteq V, \text{vol}(S) \leq \text{vol}(V)/2} \phi(S).$$

Consider the graph G consisting of a regular complete graph with self loops $(2I + \frac{1}{n}K_n)$ and an extra vertex u. The extra vertex only connects to a single vertex v in the complete graph with edge weight 1 and it has a self loop of weight m. We assume the complete graph is so large that $n > 2m^4$. Then $\phi(G) = \phi(\{u\}) = 1/m + o(1/m)$. Consider G^3. Since $\deg_{G^3}(u) = m^3 + o(m^3) < n/2$, the set achieving minimum conductance is still $\{u\}$. In G^3, the total weight of edges between u and the complete graph is $m^2 + o(m^2)$. Therefore $\phi(G^3) = 1/m + o(1/m)$. Note that the same argument applies for any G^t if we set n to be large enough. Therefore, no matter how small $\phi(G)$ is or how large t is, we cannot argue that $\phi(G^t) > (1 + \epsilon)\phi(G)$ for a positive constant ϵ when we replace expansion by conductance in irregular graphs.

On the other hand, our results can be extend to another natural generalization of expansion. Consider the definition

$$\varphi(G^t) = \min_{S \subseteq V, |S| \leq n/2} \varphi_{G^t}(S) = \min_{S \subseteq V, |S| \leq n/2} (1 - \frac{\chi_S^T (D^{-1}A_G)^t \chi_S}{|S|}),$$

where $\varphi_{G^t}(S)$ is the probability that a t-step random walk starting from a random vertex in S escapes S. With this definition and assuming that the graph does not contain a vertex of degree more than half of the total degrees, we can show that Lemma 5 still holds, with a extended definition for $C^{(t)}$. Therefore, $\varphi(G^t) = \Omega(\min\{\sqrt{t} \cdot \varphi(G), 1\})$ follows.

3 Applications

In this section, we discuss some consequences of our main theorem. We show that proving the general cases of Cheeger's inequalities can be reduced to proving the special cases where the eigenvalues are constants. Similar arguments can be used to deduce the recent result on gap amplification of small set expansion in [8].

3.1 Cheeger's Inequalities

Let G be an undirected 1-regular lazy graph. The following result shows that if one could prove Cheeger's inequality when λ_2 is a constant, then one could prove Cheeger's inequality for all λ_2. One consequence is that if one could prove that say $\phi(G) = O((\lambda_2)^{1/100})$ (so that Cheeger's inequality is true when λ_2 is a constant), then it actually implies that $\phi(G) = O(\sqrt{\lambda_2})$.

▶ **Corollary 11.** *Suppose one could prove that $\lambda_2(H) \geq C$ for some constant $C \leq 1/2$ whenever $\phi(H) \geq 1/40$, then it implies that $\phi(G) \leq \sqrt{\lambda_2(G)/C}$ for any G and any $\lambda_2(G)$.*

Proof. Given G, we assume that $\lambda_2(G) \leq \phi(G)^2/2$, as otherwise the statement is trivial. Consider $H = G^{1/\phi(G)^2}$. By Theorem 1, we have

$$\phi(H) \geq \frac{1}{20}(1 - (1 - \phi(G))^{\sqrt{1/\phi(G)^2}}) \geq \frac{1}{40}.$$

Therefore, if we could prove that $\lambda_2(H) \geq C$, then we could conclude that

$$C \leq \lambda_2(H) = 1 - (1 - \lambda_2(G))^{1/\phi(G)^2} \leq \frac{\lambda_2(G)}{\phi(G)^2},$$

and the corollary follows.   ◀

3.2   Improved Cheeger's Inequality

It was proved in [6] that $\phi(G) = O(k\lambda_2/\sqrt{\lambda_k})$ for any k. Using similar arguments as above, the following result shows that if one could prove this improved Cheeger's inequality when λ_3 is a constant, then one could prove it for all λ_3. For instance, if one could prove that say $\phi(G) = O(\lambda_2/\lambda_3^{100})$, then it actually implies that $\phi(G) = O(\lambda_2/\sqrt{\lambda_3})$.

▶ **Corollary 12.** *Suppose one could prove that $\phi(H) \leq C\lambda_2(H)$ for some $C \geq 1/10$ whenever $\lambda_3(H) \geq 1/2$, then it implies that $\phi(G) \leq 40C\lambda_2(G)/\sqrt{\lambda_3(G)}$ for any G and any $\lambda_3(G)$.*

Proof. We assume that $\phi \leq \sqrt{\lambda_3}/2$, as otherwise, by Cheeger's inequality, $2\lambda_2(G) \geq \phi(G)^2 \geq \frac{1}{2}\phi(G)\sqrt{\lambda_3}$ and the statement is true. Consider $H = G^{1/\lambda_3(G)}$. Then

$$\lambda_3(H) = 1 - (1 - \lambda_3(G))^{1/\lambda_3} \geq 1 - e^{-1} \geq 1/2.$$

Therefore, if one could prove that $\phi(H) \leq C\lambda_2(H)$, then

$$C\lambda_2(H) \geq \phi(H) \geq \frac{1}{20}(1 - (1 - \phi(G))^{\sqrt{1/\lambda_3(G)}}) \geq \frac{\phi(G)}{40\sqrt{\lambda_3(G)}},$$

where the second inequality is by Theorem 1 and the last inequality is by Fact 2.1. On the other hand,

$$\lambda_2(H) = 1 - (1 - \lambda_2(G))^{1/\lambda_3(G)} \leq \frac{\lambda_2(G)}{\lambda_3(G)},$$

and the corollary follows by combining the two inequalities.   ◀

3.3   Gap Amplification for Small Set Expansion

Consider the small set expansion problem $\mathrm{SSE}_{\delta,\delta'}(c,s)$: Given a graph G, distinguish whether $\phi_\delta(G) \leq c$ or $\phi_{\delta'}(G) \geq s$. The small set expansion conjecture [9] states that for any $\epsilon > 0$, there exists $\delta > 0$ such that $\mathrm{SSE}_{\delta,\delta}(\epsilon, 1 - \epsilon)$ is NP-hard.

Let f be a function such that $f(x) = \omega(\sqrt{x})$. Raghavendra and Schramm [8] showed that if for all $\epsilon > 0$ there exists $\delta > 0$ such that $\mathrm{SSE}_{\delta,\delta}(\epsilon, f(\epsilon))$ is NP-hard, then for all $\epsilon > 0$ there exists $\delta > 0$ such that $\mathrm{SSE}_{\delta,\delta/8}(\epsilon, 1/2)$ is NP-hard.

We would show that our techniques can be easily applied to get similar result.

▶ **Theorem 13.** *If for all $\epsilon > 0$ there exists $\delta > 0$ such that $\mathrm{SSE}_{\delta,\delta}(\epsilon, f(\epsilon))$ is NP-hard, then for all $\epsilon > 0$ there exists $\delta > 0$ such that $\mathrm{SSE}_{\delta,\delta/2}(\epsilon, \Omega(1))$ is NP-hard.*

Proof. Given an instance G that we would like to distinguish whether $\phi_\delta(G) \leq \epsilon$ or $\phi_\delta(G) \geq f(\epsilon)$, we consider the graph $H = G^{O(1/f(\epsilon)^2)}$. In the case when $\phi_\delta(G) \geq f(\epsilon)$, by Theorem 3, we have

$$\phi_{\delta/2}(H) = \Omega(\sqrt{1/f(\epsilon)^2} \cdot f(\epsilon)) = \Omega(1).$$

In the case when $\phi_\delta(G) \leq \epsilon$, we have

$$\phi_\delta(H) \leq (1/f(\epsilon)^2) \cdot \epsilon = o_\epsilon(1) \leq \epsilon',$$

where the equality holds because $f(\epsilon) = \omega(\sqrt{\epsilon})$ and the first inequality holds because

$$\phi_{G^t}(S) = 1 - \frac{\chi_S^T A^t \chi_S}{|S|} \leq t \cdot \phi_G(S),$$

where the inequality is proven in [10] by a simple induction. Therefore, if $\mathrm{SSE}_{\delta,\delta}(\epsilon, f(\epsilon))$ is NP-hard, then $\mathrm{SSE}_{\delta,\delta/2}(\epsilon', \Omega(1))$ is NP-hard. ◀

Finally, we remark that it is easier to bound $\phi_\delta(G^t)$ for large t using Lovász-Simonovits curve. Using the techniques in [5], we have the following bound for $C^{(t)}$ when the initial probability vector is $\chi_S/|S|$:

$$C^{(t)}(x) \leq \frac{x}{\delta n} + \sqrt{\frac{x}{|S|}}(1 - \frac{\phi^2}{2})^t.$$

Therefore,

$$\phi_{G^t}(S) \geq 1 - C^{(t)}(|S|) \geq 1 - \frac{|S|}{\delta n} - (1 - \frac{\phi^2}{2})^t,$$

where the first inequality follows from Lemma 6. Set $t = 100/\phi^2$, then for $|S| \leq \delta n/4$, we have $\phi_{G^t}(S) \geq \frac{3}{4} - \exp(-50)$. Therefore, if $\mathrm{SSE}_{\delta,\delta}(\epsilon, f(\epsilon))$ is NP-hard, then $\mathrm{SSE}_{\delta,\delta/4}(\epsilon', 1/2)$ is NP-hard.

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