Circuit Obfuscation Using Braids

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Abstract

An obfuscator is an algorithm that translates circuits into functionally-equivalent similarly-sized circuits that are hard to understand. Efficient obfuscators would have many applications in cryptography. Until recently, theoretical progress has mainly been limited to no-go results. Recent works have proposed the first efficient obfuscation algorithms for classical logic circuits, based on a notion of indistinguishability against polynomial-time adversaries. In this work, we propose a new notion of obfuscation, which we call partial-indistinguishability. This notion is based on computationally universal groups with efficiently computable normal forms, and appears to be incomparable with existing definitions. We describe universal gate sets for both classical and quantum computation, in which our definition of obfuscation can be met by polynomial-time algorithms. We also discuss some potential applications to testing quantum computers. We stress that the cryptographic security of these obfuscators, especially when composed with translation from other gate sets, remains an open question.

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1 Introduction

1.1 Past work on circuit obfuscation

Informally, an obfuscator is an algorithm that accepts a circuit as input, and outputs a hard-to-understand but functionally equivalent circuit. In this subsection, we briefly outline the state of current research in classical circuit obfuscation. To our knowledge, quantum circuit obfuscation has not been considered in any prior published work.

Methods used for obfuscating logic circuits in practice have so far been essentially ad hoc [11, 41]. Until recently, theoretical progress has primarily been in the form of no-go theorems for various strong notions of obfuscation [7, 21]. The ability to efficiently obfuscate certain circuits would have important applications in cryptography. For instance, sufficiently strong obfuscation of circuits of the form "encrypt with a hard-wired private key" could turn a private-key encryption scheme into a public-key encryption scheme. As this example illustrates, one undesirable outcome is when the input circuit can be recovered completely

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from the obfuscated circuit. In this case, we say that the obfuscator *completely failed* on that circuit [7]. Unfortunately, every obfuscator will completely fail on some circuits (e.g., learnable circuits.) On the other hand, there are trivial obfuscators which will erase at least some information from some circuits, e.g., by removing all instances of $X^{-1}X$ for some invertible gate X.

In order to give a useful formal definition of obfuscation, one must decide on a reasonable definition of "hard-to-understand." The most stringent definition in the literature demands *black-box obfuscation*, i.e., that the output circuit is computationally no more useful than a black box that computes the same function. Barak et al. [8] gave an explicit family of circuits that are not learnable and yet cannot be black-box obfuscated. They also showed that there exist (non-learnable) private-key encryption schemes that cannot be turned into a public-key cryptosystem by obfuscation. Their results do not preclude the possibility of black-box obfuscation for specific families of circuits, or of applying obfuscation to produce public-key systems from private ones in a non-generic fashion. It is an open problem whether quantum circuits can be black-box obfuscated.

A weaker but still quite natural notion is called *best-possible obfuscation*; in this case, we ask that the obfuscated circuit reveals no more information than any other circuit that computes the same function. Goldwasser and Rothblum [21] showed that for efficient obfuscators, best-possible obfuscation is equivalent to *indistinguishability obfuscation*, which is defined as follows. For any circuit C, let $|C|$ be the number of elementary gates, and let f_C be the Boolean function that C computes.

▶ **Definition 1.** A probabilistic algorithm $\mathcal{O}$ is an *indistinguishability obfuscator* for the collection $\mathcal{C}$ of circuits if the following three conditions hold:

1. (functional equivalence) for every $C \in \mathcal{C}$, $f_{\mathcal{O}(C)} = f_C$;
2. (polynomial slowdown) there is a polynomial p such that $|\mathcal{O}(C)| \leq p(|C|)$ for every $C \in \mathcal{C}$;
3. (indistinguishability obfuscation) For any $C_1, C_2 \in \mathcal{C}$ such that $f_{C_1} = f_{C_2}$ and $|C_1| = |C_2|$, the two distributions $\mathcal{O}(C_1)$ and $\mathcal{O}(C_2)$ are indistinguishable.

In the third part of the above definition, one must choose a notion of indistinguishability for probability distributions. Goldwasser and Rothblum [21] consider three such notions: perfect (exact equality), statistical (total variation distance bounded by a constant), and computational (no probabilistic polynomial-time Turing Machine can distinguish samples with better than negligible probability). They show that the existence of an efficient statistical indistinguishability obfuscator would result in a collapse of the polynomial hierarchy to the second level. This result also applies if the condition $|C_1| = |C_2|$ in property (3) of Definition 1 is relaxed to $|C_1| = k|C_2|$ for any fixed constant k [21].

A recent breakthrough has shown that computational indistinguishability may be achievable in polynomial time. Combining a new obfuscation scheme for NC1 circuits with fully homomorphic encryption, Sahai et al. gave an efficient obfuscator which achieves the computational indistinguishability condition under plausible hardness conjectures [19]. Subsequent work outlined a number of cryptographic applications of computational indistinguishability [38].

1.2 Outline of present work

1.2.1 New notion of obfuscation

An exact deterministic indistinguishability obfuscator would yield a solution to the circuit equivalence problem. For general Boolean circuits, this problem is co-NP hard. Therefore, exact deterministic indistinguishability obfuscation of general Boolean circuits cannot be achieved in polynomial time under the assumption $\mathrm{P} \neq \mathrm{NP}$. We propose an alternative route to weakening the exactness condition, by pursuing a notion of "partial-indistinguishability". In partial-indistinguishability obfuscation, we relax condition (3) so that it need only hold for C_1 and C_2 that are related by some fixed, finite set of relations on the underlying gate set.[1]

▶ **Definition 2.** Let G be a set of gates and Γ a set of relations satisfied by the elements of G. An algorithm $\mathcal{O}$ is a (G, Γ)-*indistinguishability obfuscator* for the collection $\mathcal{C}$ of circuits over G if the following three conditions hold:

1. (functionality) for every $C \in \mathcal{C}$, $f_C = f_{\mathcal{O}(C)}$;
2. (polynomial slowdown) there is a polynomial p such that $|\mathcal{O}(C)| \leq p(|C|)$ for every $C \in \mathcal{C}$;
3. ((G, Γ)-indistinguishability) for any $C_1, C_2 \in \mathcal{C}$ that differ by some sequence of applications of the relations in Γ, $\mathcal{O}(C_1) = \mathcal{O}(C_2)$.

The power of the obfuscation is now determined by the power of the relations Γ. If Γ is a complete set of relations, generating all circuit equivalences over G, then a (G, Γ)-indistinguishability obfuscator is a perfect indistinguishability obfuscator according to Definition 1. (Complete sets of relations for $\{\text{Toffoli}\}$ and $\{\text{AND}, \text{OR}, \text{NOT}\}$ are given in [27, 26].) If Γ is the empty set then even the identity map fits the definition, and no obfuscation is taking place. With different sets of relations, one can interpolate between these extremes. The intermediate obfuscators form a partially ordered set, where a (G, Γ')-indistinguishability obfuscator is strictly stronger than a (G, Γ)-indistinguishability obfuscator if Γ' is a strict superset of Γ. We remark that partial-indistinguishability is no stronger than perfect indistinguishability, and appears to be incomparable with statistical and computational indistinguishability. This is part of our motivation in considering this new definition.

In the context of quantum computation, we make only a few minor changes to Definitions 1 and 2. First, the obfuscators will still be classical algorithms. On the other hand, the gates will be unitary and the circuits to be obfuscated will be unitary quantum circuits. Finally, the notion of functional equivalence now simply means that the operator-norm distance between the unitary implemented by C and the unitary implemented by $\mathcal{O}(C)$ is bounded by a small constant $\epsilon > 0$.

1.2.2 Group normal forms

A finitely generated group can be specified by a presentation. This is a list of generators $\sigma_1, \ldots, \sigma_n$ and a list of relations obeyed by these generators. (A relation is simply an identity such as $\sigma_1 \sigma_3 = \sigma_3 \sigma_1$.) All group elements are obtained as products of the generators and their inverses. However, by applying the relations, we can get multiple words in the generators and their inverses that encode the same group element. A normal form specifies, for each group element, a unique decomposition as a product of generators and their inverses. For certain groups, including the braid groups, polynomial time algorithms are known which, given a product of generators and their inverses, can reduce it to a normal form. The word problem is, given two words in the alphabet $\{\sigma_1, \ldots, \sigma_n, \sigma_1^{-1}, \ldots, \sigma_n\}$, to decide whether they specify the same group element. If a normal form can be computed, then this solves the word problem: just reduce both words to normal form and check whether the results are identical. However, an efficient solution for the word problem does not in general imply an efficiently computable normal form.

[1] Our construction for satisfying this definition uses *reversible* gates. The definition of functional equivalence becomes more technical in that context, as discussed in Section 3.1.

1.2.3 Efficient constructions from group representations

In this paper, we propose a general method of designing partial-indistinguishability obfuscators based on groups with efficiently computable normal forms. If a set of gates G obeys the relations Γ of the generators of a group with an efficiently computable normal form, then the reduction to normal form is an efficient (G, Γ)-indistinguishablity obfuscator. The gates may obey additional relations beyond Γ, which is why the obfuscator does not solve the circuit-equivalence problem, which is believed to be intractable for both classical and quantum circuits.

To demonstrate this method, we discuss an implementation using the braid groups B_n, for both classical reversible circuits and unitary quantum circuits. The number of strands n in the braid group depends linearly on the number of dits or qudits on which the circuit acts. In Section 3, we describe a computationally universal reversible classical gate obeying the braid group relations, which was constructed in [34, 37, 31] from the quantum double of A_5. In Section 4.1, we describe a computationally universal quantum gate obeying the braid group relations, which was constructed in [18] from the Fibonacci anyons. Our obfuscation scheme is similar in spirit to previously-proposed obfuscation schemes based on applying local circuit identities [41], but the uniqueness of normal forms adds a qualitatively new feature. One consequence of this feature is that we can satisfy Definition 2 and guarantee the partial-indistinguishability property against computationally unbounded adversaries. The running time of the obfuscator is the same as the running time of the the normal form algorithms, which take time $O(l^2 m \log m)$ for m-strand braids of length l [14].

We remark that these gate sets that obey the braid group relations are not artificial constructions; in fact, they are the most natural choice in many contexts, some of which we list here. In the quantum case, these gates are native to certain proposed physical implementations of quantum computers [31], where the topological braiding property provides inherent fault-tolerance. The problem of approximating the Jones Polynomial invariant of links is complete for polynomial-time quantum computation [2]; an analogous fact is true for a restricted case of quantum computations motivated by NMR implementations [40]. Both of these facts are naturally expressed in the gate set constructed from the Fibonacci representation. In the classical case, the gate set derived from quantum doubles of finite groups was recently used to show BPP-completeness for approximation of certain link invariants [32].

We remark that another potential group family for constructing partial-indistinguishability obfuscators are the mapping class groups $\mathrm{MCG}(\Sigma_g)$ of unpunctured surfaces of genus g. These groups also have quantumly universal representations [5] and an efficiently solvable word problem [23]. It is not known if there are also classically universal permutation representations, or if there are efficiently computable normal forms.

1.2.4 Other gate sets

In some applications the native gate set will be different than the ones used in our construction. It is natural to ask if our obfuscators can be used in these settings as well. By universality (quantum or classical), one has an efficient algorithm B which translates circuits from the native gate set to the braiding gate set, as well as an efficient algorithm C for translation in the opposite direction. We also let N denote the partial-indistinguishability obfuscator. One might then attempt to obfuscate by applying the following:

▶ Algorithm 1.

1. *input: a circuit C on n (qu)dits*
2. *output: The circuit* $\mathsf{C}(\mathsf{N}(\mathsf{B}(C)))$

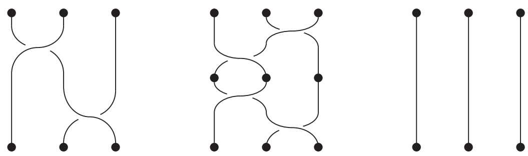


Figure 1 The generator σ_i represents the (clockwise) crossing of strands i and $i+1$ connecting a bottom row of "pegs" to a top row. Multiplication of group elements corresponds to composition of braids. As an example, we show the 3-strand braid $\sigma_1^{-1}\sigma_2$ (left), and the same braid composed with its inverse $\sigma_2^{-1}\sigma_1$ (middle), which is equivalent to the identity element of B_3 (right).

We stress that, unlike the map N, the composed map $\mathsf{N} \circ \mathsf{B}$ does not necessarily satisfy Definition 2. As we discuss in Section 5.1, careless choice of the map B can partially or completely break the security of the obfuscator. Finding translation algorithms securely composable with partial-indistinguishability obfuscators is an area of current investigation.

2 Relevant Properties of the Braid Group

The braid group B_n is the infinite discrete group with generators $\sigma_1, \ldots, \sigma_{n-1}$ and relations

$$\begin{array}{rclr} \sigma_i\sigma_j & = & \sigma_j\sigma_i & \forall\ |i-j| \geq 2 \\ \sigma_i\sigma_{i+1}\sigma_i & = & \sigma_{i+1}\sigma_i\sigma_{i+1} & \forall\ i. \end{array} \tag{1}$$

The group B_n is thus the set of all words in the alphabet $\{\sigma_1, \ldots, \sigma_{n-1}, \sigma_1^{-1}, \ldots, \sigma_{n-1}^{-1}\}$, up to equivalence determined by the above relations. In 1925 Artin proved that the abstract group defined above precisely captures the topological equivalence of braided strings [6], as illustrated in Fig. 1. A charming exposition of this subject can be found in [30].

In the word problem on B_n, we are given words w and z, and our goal is to determine if they are equal as elements of B_n. One solution is to put both w and z into a *normal form*, and then check if they are equal as words. For our purposes, it is enough to describe the normal form and specify the complexity of the algorithm for computing it. The details of the algorithm, along with a thorough and accessible presentation of the relevant facts about braids, can be found in [14].

We first observe that the word problem is easily shown to be decidable if we restrict our attention to an important subset of B_n. Note that the presentation (1) can also be viewed as a presentation of a monoid, which we denote by B_n^+. The elements of B_n^+ are called *positive braids*, and are words in the generators σ_i only (no inverses), up to equivalence determined by the relations in (1). Since all the relations of B_n preserve word length, and there are only finitely many words of any given length, we can decide the word problem (albeit very inefficiently) simply by trying all possible combinations of the relations.

Building upon this, one can give an (inefficient) algorithm for the word problem on B_n itself [22]. First, given two elements a, b of B_n^+, we write $a \preccurlyeq b$ if there exists $z \in B_n^+$ such that $b = az$; in this case we say that a is a *left divisor* of b. Similarly, we write $a \succcurlyeq b$ if there exists $y \in B_n^+$ such that $b = ya$; in this case we say that a is a *right divisor*[2] of b. The center of B_n is the cyclic group generated by Δ_n^2, where

$$\Delta_n := \Delta_{n-1}\sigma_{n-1}\sigma_{n-2}\cdots\sigma_1 \in B_n^+$$

[2] The terminology is not accidental; it turns out that we can also define l.c.m.s and g.c.d.s in B_n^+, and that B_n is the group of fractions of B_n^+. These facts are some of the achievements of Garside theory [20].

(see p.30 of [22] for a simple proof). Geometrically, Δ_n implements a twist by π in the z-plane as the strands move from $z = 0$ to $z = 1$. One can show that $\sigma_i \preccurlyeq \Delta_n$ for all i, i.e. there exists $x_i \in B_n^+$ such that $\sigma_i^{-1} = x_i\Delta_n^{-1}$. Given a word w in the σ_i and their inverses, we first replace the leftmost instance of an inverse generator (say it is σ_i^{-1}) with $x_i\Delta_n^{-1}$. We then insert $\Delta_n^{-1}\Delta_n$ in front of x_i, and observe that conjugating a positive braid x by Δ_n results in another positive braid (specifically, the rotation of x by π in the z-plane). In this way, we can push Δ_n^{-1} all the way to the left. We repeat this process for each inverse generator appearing in the word, resulting in a word of the form $\Delta_n^p b$ where $p \in \mathbb{Z}$ and $b \in B_n^+$. Since we can solve the word problem in B_n^+, we can factor out the maximal power of Δ_n appearing as a left divisor of b. We thus have that, as elements of the braid group, $w = \Delta_n^{p'} b'$ with Δ_n not a left divisor of b' and p' unique. This solves the word problem in B_n.

We can make the above algorithm efficient by finding an efficiently computable normal form for a positive braid word b that does not have Δ_n as a left divisor. Recall that the symmetric group S_n has a remarkably similar presentation to B_n. Indeed, starting with (1), letting $\sigma_i = (i\ i+1)$ and adding the relations $\sigma_i^2 = 1$ for all i results in the standard presentation of S_n. In other words, there is a surjective homomorphism $\pi : B_n \to S_n$ with $\sigma_i \mapsto (i\ i+1)$. In terms of the geometric interpretation, a braid is mapped to the permutation on $[n]$ defined by the connections between the top and bottom "pegs," as in Figure 1. For each $\sigma \in S_n$, there is a unique preimage of σ that can be drawn so that any given pair of strands cross only in the positive direction, and at most once. We call such braids *simple braids*, and they form a subset of B_n^+ of size $n!$.

▶ **Definition 3** (p. 4 of [14]).

1. A sequence of simple braids $(s_1, \dots, s_p)$ is said to be *normal* if, for each j, every σ_i that is a left divisor of s_{j+1} is a right divisor of s_j.
2. A sequence of permutations $(f_1, \dots, f_p)$ is said to be *normal* if, for each j, $f_{j+1}^{-1}(i) > f_{j+1}^{-1}(i+1)$ implies $f_j(i) > f_j(i+1)$.

A sequence of simple braids $(s_1, \dots, s_p)$ is normal if and only if the sequence of permutations $(\pi(s_1), \dots, \pi(s_p))$ is normal. Given a permutation $f \in S_n$, let $\hat{f}$ denote the simple braid of B_n satisfying $\pi(\hat{f}) = f$.

▶ **Theorem 4** (p. 4 of [14] and Ch. 9 of [15]).

1. *Every braid z in B_n admits a unique decomposition of the form $\Delta_n^m s_1 \dots s_p$ with $m \in \mathbb{Z}$ and $(s_1, \dots, s_p)$ a normal sequence of simple braids satisfying $s_1 \neq \Delta_n$ and $s_p \neq 1$.*
2. *Every braid z in B_n admits a unique decomposition of the form $\Delta_n^m \hat{f}_1 \dots \hat{f}_p$ with $m \in \mathbb{Z}$ and $(f_1, \dots, f_p)$ a normal sequence of permutations satisfying $f_1 \neq \pi(\Delta_n)$ and $f_p \neq 1$.*

The most efficient algorithms for computing the normal form of a word w in the generators of B_n have complexity $O(|w|^2 n \log n)$ [14].

3 Obfuscation of Classical Reversible Circuits

3.1 Reversible Circuits

In the next section, we will describe a gate R which is universal for classical computation and satisfies Definition 2 when Γ is the set of relations of the braid group. Because group elements are invertible, R must be a reversible gate, that is, it must bijectively map its possible inputs to its possible outputs. We will thus work in the setting of *reversible classical circuits*. These circuits are composed entirely of reversible gates. For more background on reversible computation see [9, 17, 36].

Because reversible circuits cannot erase any information, they operate using ancillary dits ("ancillas") to store unerasable data left over from intermediate steps in the computation. A reversible circuit evaluating a function $f : \{0, \ldots, d-1\}^n \to \{0, \ldots, d-1\}^m$ thus operates on $r \geq \max(n, m)$ dits, where $r - n$ of the input dits are work dits to be initialized to some fixed value independent of the problem instance, and $r - m$ of the output dits contain unerasable leftover data, to be ignored. Efficient procedures are known for compiling arbitrary logic circuits into reversible form, e.g., by using the Toffoli (or CCNOT) gate [9, 17].

In adapting Definitions 1 and 2 to reversible circuits, one is faced with two natural choices for the notion of functional equivalence. One may either demand that the original and obfuscated circuits implement the same function $f : \{0,1\}^n \to \{0,1\}^m$, ignoring the ancilla dits (*weak equivalence*), or demand that they implement the same transformation on the entire set of r dits, including the ancillas (*strong equivalence*). Our constructions will satisfy the latter. Strong equivalence implies weak equivalence, so our construction proves that both possible definitions of partial-indistinguishability are polynomial-time achievable when Γ is the set of relations of the braid group. We remark that, as with ordinary irreversible circuits, determining if two arbitrary reversible circuits are equivalent (weakly or strongly) is coNP-complete [29].

3.2 Classical computation with braids

We now briefly describe a classical reversible gate R which satisfies the braid relations. The complete details of the construction and the proof of universality of R are given in Appendix A. Taken together with Theorem 4, this yields an obfuscator satisfying Definition 2.

Let G be a finite group and set $d = |G|$. Consider the reversible gate R that acts on pairs of dits encoding group elements by

$$R(a, b) = (b, b^{-1}ab). \tag{2}$$

Let R_i denote R acting on the i and $(i+1)^{\text{th}}$ wires of a circuit. By direct calculation, one can check that the set $\{R_1, \ldots, R_{n-1}\}$ satisfies the braid relations, that is,

$$\begin{aligned} R_i R_j &= R_j R_i && \forall\ |i-j| \geq 2 \\ R_i R_{i+1} R_i &= R_{i+1} R_i R_{i+1} && \forall\ i. \end{aligned} \tag{3}$$

In 1997, Kitaev discovered that the gate set $\{R, R^{-1}\}$ is universal for classical reversible computation when G is the symmetric group S_5 [31]. Ogburn and Preskill subsequently showed that the alternating group A_5, which is half as large as S_5, is already sufficient [37]. The universality construction for A_5 was subsequently presented in greater detail and generalized to all non-solvable groups by Mochon [34]. To make our presentation more accessible and self-contained, we give in Appendix A an explicit description of Mochon's universality construction in the the case $G = A_5$. The construction proves computational universality by showing how to efficiently compile Toffoli circuits into R-circuits.

Given any R-circuit, we can apply the algorithm of Theorem 4 by interpreting each R_i as σ_i and each R_i^{-1} as σ_i^{-1}. This leads to partial-indistinguishability obfuscation of R-circuits. A discussion of whether this can also yield meaningful obfuscation for classical circuits constructed from other gate sets is given in Section 5.

4 Quantum Circuits

4.1 Quantum computation with braids

In Section 3.2 and Appendix A, we discuss classical universality of circuits encoded as braids. It turns out that an analogous theory can be developed for quantum circuits, and is well-understood. The family of so-called Fibonacci representations of the braid groups have dense image in the unitary group, and there are efficient classical algorithms for translating any quantum circuit into a braid (and vice-versa) in a way that preserves unitary functionality [18]. A brief synopsis of these facts is given below. We remark that there are in fact many unitary representations of the braid groups that satisfy these properties, and which are physically motivated by the so-called fractional quantum Hall effect. In this setting, the image of these representations consists of unitary operators which describe the braiding of excitations in a 2-dimensional medium [31].

Approachable descriptions of the Fibonacci representation are given in [40, 42]. In [40], what we call the "Fibonacci representation" here, is called the "$\star\star$" irreducible subrepresentation. This is a family of representations $\rho^{(n)}_{\mathrm{Fib}} : B_n \to U(F_{n-4})$, where F_k is the k-th Fibonacci number. For our application, the essential properties of the Fibonacci representation are *locality* and *local density*. These two properties mean that, under a certain qubit encoding, braid generators correspond to local unitaries, and local unitaries correspond to short braid words. Standard arguments from quantum computation tell us that we can achieve the latter to precision ϵ with $O(\log^{2.71}(1/\epsilon))$ braid generators by means of the Solovay-Kitaev algorithm [13].

A natural basis for the space of $\rho^{(n)}_{\mathrm{Fib}}$ can be identified with strings of length n from the alphabet $\{\star, p\}$, which begin with $\star$, end with p, and do not contain "$\star\star$" as a substring[3]. Following [2][4], for n a multiple of four, we identify a particular subspace V_n of $\rho^{(n)}_{\mathrm{Fib}}$ by discarding some basis elements, as follows. Partition a string s into substrings of length four. If each of these substrings is equal to either $\star p \star p$ (this will encode a 0) or $\star ppp$ (this will encode a 1), then the basis element corresponding to s is in V_n; otherwise, it is not. Note that $\dim V_n = 2^{n/4}$. The following theorem follows from [2, 13].

▶ **Theorem 5.** *There is a classical algorithm which, given an $(n/4)$-qubit quantum circuit C and $\epsilon > 0$, outputs a braid $b \in B_n$ of length $O(|C|\log^{2.71}(1/\epsilon))$ satisfying*

$$\left\| C - \rho^{(n)}_{\mathrm{Fib}}(b)\Big|_{V_n} \right\| \leq \epsilon \;;$$

this algorithm has complexity $O(|b|)$.

For the opposite direction, we can identify a subspace $W_n \subset (\mathbb{C}_2)^{\otimes n}$ by discarding all bitstrings except those that start with 0, end with 1 and do not have "00" as a substring. Then $\dim W_n = \dim \rho^{(n)}_{\mathrm{Fib}}$ and we have the following.

▶ **Theorem 6.** *There is a classical algorithm which, given $b \in B_n$ and $\epsilon > 0$, outputs a quantum circuit C on n qubits of length $O(|b|\log^{2.71}(1/\epsilon))$ such that*

$$\left\| C|_{W_n} - \rho^{(n)}_{\mathrm{Fib}}(b) \right\| \leq \epsilon \;;$$

this algorithm has complexity $O(|C|)$.

[3] In [40] the $\star\star$ subrepresentation of B_n acts on strings of length $n+1$ that begin and end with $\star$. One can leave the initial and/or final $\star$ implicit as these are left unchanged by all braiding operations. We omit the final $\star$ leaving us strings of length n that begin with $\star$ and end with p.

[4] Reference [2] describes the basis vectors in terms of "paths". The correspondence between the path notation and the $p\star$ notation is given in appendix C of [40].

The two algorithms in the above theorems are described explicitly in [2].

4.2 Obfuscating quantum computations

While the state of knowledge about classical obfuscation is limited, essentially nothing is known about the quantum case. Here we discuss how to use the facts from the previous section to construct a partial-indistinguishability obfuscator for quantum circuits.

In light of Theorem 5, $\{\rho_{\mathrm{Fib}}(\sigma_1), \ldots, \rho_{\mathrm{Fib}}(\sigma_{n-1})\}$ may be regarded as a universal set of elementary quantum gates. By the homomorphism property of ρ_{Fib}, this set satisfies the braid relations. These gates differ from conventional quantum gates in that they do not possess locality defined in terms of a strict tensor product structure. Nevertheless, as shown above, the power of unitary circuits composed from these gates is equivalent to standard quantum computation. By interpreting each $\rho_{\mathrm{Fib}}(\sigma_j)$ as a braid-group generator σ_j, we can apply the algorithm from Theorem 4 directly to circuits from this gate set, resulting in a partial-indistinguishability obfuscator satisfying Definition 2.

With the algorithms from the previous section in hand, we could also attempt to apply the obfuscation algorithm, Algorithm 1, directly to quantum circuits. For an input circuit C on n qubits, the running times of both of this algorithm is $O(|C|^2 n \cdot \mathrm{polylog}(n, 1/\epsilon))$. The length of the output cannot be longer than the running time. We are not aware of a better upper bound for the length of the output. The security of this algorithms is questionable, and some attacks and possible countermeasures are discussed in Section 5.

Note that reduction of arbitrary quantum circuits to a normal form using a *complete* set of gate relations should not be possible in polynomial time; this would yield a polynomial-time algorithm for deciding whether a quantum circuit is equivalent to the identity, which is a coQMA-complete problem [28].

4.3 Testing claimed quantum computers with a quantum obfuscator

It is natural to consider quantum analogues of the applications of obfuscation from classical computer science. We now consider a potential application of quantum circuit obfuscation that does not fit this mold: testing claimed quantum computers. A similar proposal using a restricted class of quantum circuits has been previously made in [39].

Suppose Bob claims to have access to a universal quantum computer with some fixed finite number of qubits. Alice has access to a classical computer only, as well as a classical communication channel with Bob. Can Alice determine if Bob is telling the truth? Barring tremendous advances in complexity theory, a provably correct test is unlikely;[5] can we still design a test in which we have a high degree of confidence? Given the extensive work on classical algorithms for factoring, a reasonable idea is to simply ask Bob to factor a sufficiently large RSA number. However, Shor's algorithm only begins to outperform the best classical algorithms when thousands of logical qubits can be employed. A much smaller universal quantum computer (e.g., a few dozen qubits) is likely to be a far simpler engineering challenge and could still be quite useful, e.g., for simulating certain quantum systems. A test that works in this case would thus be very valuable. We now outline a new proposal for such a test.

[5] Notice that even a proof that BQP $\neq$ BPP would be insufficient; one would have to find specific problems and instance sizes where some quantum strategy provably beats every classical one. We are thus left with a situation analogous to the practical security guarantees of modern cryptographic systems, which tell us how many bit operations it would take to crack a given instance using the fastest known algorithms.

Simply put, we propose asking questions that are classically easy to answer, but posing them in an obfuscated manner. In this test, Alice would repeatedly generate quantum circuits and ask Bob to run them. At least some of the circuits would in fact be quantumly-obfuscated classical reversible circuits, allowing Alice to easily check the answers. Previous work has yielded tests of quantum computers in the case that the verifier can perform some limited quantum operations [10, 3].

We have considerable freedom when designing an obfuscation-based test of quantum computers. How to choose these parameters in a way that makes the test difficult to fool with a classical computer is an open question. For purposes of illustration, we give one example. Let $\mathcal{O}$ be the obfuscation algorithm for quantum circuits described above.

▶ Algorithm 2.

1. *Select a random bitstring s of length k.*
2. *Let C be the $(k+1)$-bit circuit that, on all-zero input, initializes wires 2 through $k+1$ to s and then computes the parity of s into the first wire.*
3. *Compute $\mathcal{O}(C)$, and let n be the number of qubits needed to run $\mathcal{O}(C)$.*
4. *Ask Bob to run D on the all zeros string and return the first bit of output.*

Clearly, k must be chosen so that n is smaller than the number of logical qubits Bob claims to control. To fool Alice, a purely classical Bob must determine the parity of s. The dictionary attack (*i.e.* Bob repeatedly guesses at k, obfuscates the corresponding circuit, and compares the result to the circuit given by Alice) is of no use provided k is reasonably large, e.g., 80 bits, which can be encoded using a braid of 115 strands using the Zeckendorf encoding described in [40].

We now show that there can be no efficient general-purpose algorithm for breaking our test by detecting whether a given quantum circuit is in fact (almost) classical, and if so, simulating it.

▶ **Definition 7.** Let c be a bit string specifying a quantum circuit via a standard universal set Q of quantum gates, and let U_c be the corresponding unitary operator. Fix some constants $r, d, a \in \mathbb{N}$, and fix a set R of reversible gates. The problem $\mathrm{CLASS}(r, d, a, Q, R)$ is to find a reversible circuit of at most $r|c|^d$ gates from R such that the corresponding permutation matrix P satisfies $\|U_c - P\| \leq 2^{-a|c|}$.

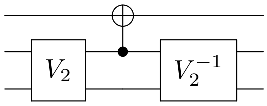
Note that $\mathrm{CLASS}(r, d, a, Q, R)$ is not a decision problem. Thus, to formulate the question of whether this problem can be efficiently solved, we must ask not whether $\mathrm{CLASS}(r, d, a, Q, R)$ is contained in P but whether it is contained in FP. We now provide some formal evidence that this is not the case. Note that the following theorems continue to hold if we change the classicality condition in Definition 7 to $\|U_c - P\| \leq |c|^{-a}$.

▶ **Theorem 8.** *For any fixed $r, d, a \in \mathbb{N}$, any universal reversible gate set R, and any universal quantum gate set Q, if $\mathrm{CLASS}(r, d, a, Q, R) \in \mathrm{FP}$ then $\mathrm{QCMA} \subseteq \mathrm{P}^{\mathrm{NP}}$.*

Note that, $\mathrm{QCMA} \subseteq \mathrm{P}^{\mathrm{NP}}$ would be very surprising because, among other things, it would imply $\mathrm{BQP} \subseteq \mathrm{PH}$, and there is evidence that this is false [1, 16].

Proof. The standard QCMA-complete language $\mathcal{L}$ is as follows. Let $\mathcal{C}$ be the set of all quantum circuits (expressed as a concatenation of bitstrings that index elements of the gate set Q). $\mathcal{C}$ decomposes as the disjoint union of $\mathcal{L}$ and $\bar{\mathcal{L}}$ where $\mathcal{L}$ consists of the quantum circuits that accept at least one classical (*i.e.* computational basis state) input, and $\bar{\mathcal{L}}$ consists of the circuits that reject all inputs. Given a quantum circuit $V_1 \in \mathcal{C}$, (the "verifier") we can amplify it using standard techniques [33, 35] to accept YES instances with probability at least $1 - O(2^{-n})$ and accept NO instances with probability at most $O(2^{-n})$. Let V_2 be such an amplified verifier. Further, let

$V_3 \quad =$



where the second-to-top qubit is the acceptance qubit of V_2. If $V_i \in \bar{\mathcal{L}}$ then $\|V_3 - \mathbb{1}\| = O(2^{-n})$. By assumption, there exists a polynomial time classical algorithm for solving CLASS(r, d, a, Q, R). When presented with V_3, this algorithm will produce a polynomial-size reversible circuit V_4 strongly equivalent to the identity. By querying an oracle for the problem of strong equivalence of reversible circuits, one can decide whether V_4 is equivalent to the circuit of no gates, and hence to the identity operation. If $V_1 \in \bar{\mathcal{L}}$, this oracle will accept. If $V_1 \in \mathcal{L}$ then the algorithm for problem 1 will answer NO or produce a circuit that this oracle rejects. As shown in [29], the problem of deciding strong equivalence of reversible circuits is contained in coNP. Thus, we can decide QCMA in $\mathrm{P}^{\mathrm{coNP}}$, which is equal to the more familiar complexity class P^{NP}. ◀

5 Some Attacks

5.1 Compiler attacks

The security or insecurity of braid-based partial-indistinguishability obfuscation remains an area of current investigation. From a purely information-theoretic point of view, the power of this obfuscation comes from the many-to-one nature of the map N that takes arbitrary braid words to their normal form. If the initial braid words are highly structured because they are obtained by compilation from a different gate set, then this can undermine or destroy the many-to-one feature of N.

In Section 3.2, we describe a reversible gate R on pairs of 60-state dits, corresponding to elements of A_5, that obeys the relations of the braid group and can perform universal classical computation. The gate itself and the proof that it is universal come from the quantum computation literature [31, 37, 34]. Appendix A recounts the universality proof of [34], which can be viewed as a compiler B_R that maps circuits constructed from the well-known universal reversible Toffoli gate into circuits constructed from the R gate. As a cautionary example, we now show that naively obfuscating Toffoli circuits using the composed map $\mathsf{N} \circ \mathsf{B}_R$ is completely insecure.

The construction in Appendix A gives a general mapping from a Toffoli gate to a corresponding braid. We will refer to braids obtained in this way as *Toffoli braids.* Recall that the normal form of a braid in B_n has the form $\Delta_n^m s_1 \ldots s_p$ for a normal sequence of simple braids $(s_1, \ldots, s_p)$. A Toffoli braid obtained from a Toffoli with controls c_1 and c_2 and target t has normal form

$$\Delta_n^0 s_1(c_1, c_2, t) s_2 s_3 s_4 s_5 s_6 s_7 s_8 s_9(c_1, c_2, t) s_{10} s_{11} s_{12} s_{13}(c_1, c_2, t) s_{14}(t). \tag{4}$$

The factors $s_2, \ldots, s_8, s_{10}, s_{11}$ and s_{12} only depend on n, and not on the wires c_1, c_2 or t. Note that this is a positive braid — consisting only of $\sigma_1, \ldots, \sigma_{n-1}$ and none of their inverses. Any product of such braids will thus also be a positive braid, so attempting to obfuscate a circuit in Toffoli gates using this construction will yield only positive braids.

Because Toffoli is a 3-bit gate, there are only $\binom{n}{3}$ ways to apply a Toffoli to n bits. Thus, one may, in polynomial time, test each of these $\binom{n}{3}$ possibilities as a guess for the last gate of the obfuscated circuit. One performs the test by compiling the guessed Toffoli gate into a braid, appending the inverse of this braid to the normal form braid produced as the output the obfuscator, and then reducing the resulting braid to normal form. If the guess is correct, then the resulting braid is still a braid corresponding to a circuit — the original obfuscated circuit with its last Toffoli gate removed — and thus this will result in a positive braid. If the guess is incorrect, then appending the inverse of a positive braid, which consists entirely of $\sigma_1^{-1}, \ldots, \sigma_{n-1}^{-1}$, might result in a braid that is no longer positive — that is, has a negative power of Δ_n, and this seems to be the case with *any* wrong guess, based on some limited tests.

Furthermore, the presence of a negative power of Δ_n is efficiently recognizable, so it is immediately clear whether or not the guess was correct.

This attack is related to so-called length-based attacks. These have been introduced in the cryptanalysis of braid based key-exchanged protocols [25]. In the present context, the natural length-based attack is to guess the final gate, append the inverse of the corresponding braid to the normal-form braid produced by the obfuscator, and the reduce the product braid to normal form. If the result is a shorter word in the braid-group generators than the original normal form, then this can be taken as heuristic evidence that the guess was correct. Intuitively, one expects that the longer the braid words are that implement individual gates from the original gate set, then the better such attacks should work.

One can easily propose modifications to the naive obfuscator $\mathsf{N} \circ \mathsf{B}_R$ that thwart guessing-based attacks such as the two attacks described above. In particular, one finds that the gate R described in Appendix A has order 60. Hence, one can start with the positive Toffoli braid in equation (4) and then each generator σ_i can independently, with probability $\frac{1}{2}$, be replaced with σ_i^{-59}, without altering the functionality of the circuit. The number of generators in a Toffoli braid depends on n, and which wires the Toffoli acts on, but there are always at least 124. Thus, each gate will be compiled into one of 2^{124} braid-words uniformly at random. Thus, guessing-based attacks on the composition of this compiler with N may become impractical. Whether such a scheme is vulnerable to other attacks remains an open question for future research.

5.2 Dictionary attacks

The partial-indistinguishability obfuscator described in the preceding sections deterministically maps input circuits to obfuscated circuits. This creates a potential weakness in the obfuscation. Suppose Alice wishes to run a computation C on Bob's server but does not wish Bob to know what computation she is running. Thus, she sends the obfuscated circuit $\mathcal{O}(C)$ to Bob, who executes it, and returns the result. To improve security, Alice may instead use a circuit C' in which her desired input is hard-coded, and which applies a one-time pad at the end of the computation. If the obfuscation is secure, then Bob is unlikely to learn anything about C, the input, or the output. However, if Bob knows that the circuits Alice is likely to want to execute are drawn from some small set S, then Bob can simply compute $\{\mathcal{O}(s)|s \in S\}$ and identify Alice's computation by finding it in this list. Such attacks are sometimes called "dictionary" attacks after the practice of recovering passwords by feeding all words from a dictionary into the hash function and comparing against the hashed password.

Dictionary attacks may or may not be a serious threat to our obfuscation scheme, depending on the the size of the set of likely circuits to be obfuscated. In cryptographic applications where dictionary attacks are a concern, the standard way to protect against them is to append random bits prior to encryption. (In the context of hashing passwords, this practice is called "salting".) Such a strategy can be applied to our obfuscator, but some care must be taken in doing so. The most obvious strategy is to append a random circuit on the output ancillas prior to obfuscation. However, attackers can defeat this countermeasure by using the polynomial-time algorithms for computing left-greatest-common-divisors in the braid group [15]. However, prior to obfuscation, one may introduce extra dits, and apply random circuits before, after, and simultaneously with the computation, in a way so as not to disrupt it. The problem of optimizing the details of this procedure so as to maximize security and efficiency is left to future work.

6 Future Work

6.1 Classical and quantum universality

It is of interest to consider other computationally universal representations of the braid group, which might provide more efficient translations from circuits to braids. One avenue for obtaining such representations is by finding other solutions to the Yang-Baxter equation, besides the operator R from Appendix A. Our investigations so far prove that no permutation matrix solution of dimension up to 16×16 is a universal gate and suggest that no permutation matrix solution of dimension 25×25 is a universal gate. In the quantum case, it has been shown that no 4×4 unitary solution is universal [4].

More generally, one may look for other finitely-generated groups with computationally universal representations and efficiently computable normal forms. One potential candidate family are the mapping class groups $\mathrm{MCG}(\Sigma_g)$ of unpunctured surfaces of genus g. These groups also have quantumly universal representations [5] and an efficiently solvable word problem [23]. It is not known if there are also classically universal permutation representations, or if there are efficiently computable normal forms.

6.2 Expanding the set of indistinguishability relations

By [29], achieving efficient indistinguishability obfuscation for the complete set of relations of a universal gate set is unlikely. However, it is possible that partial-indistinguishability obfuscation on R gates could be achieved with a larger set of relations than the braid relations. For example, the universal reversible gate described in Appendix A has order 60. If we add the relations $\sigma_i^{60} = \mathbb{1}$ for $i = 1, 2, \ldots, n-1$ to B_n, we obtain a "truncated" (but still infinite for large n [12]) factor of the braid group. If a normal form can still be computed in polynomial time for this group then one could construct an efficient obfuscator using the relations of this truncated group, which would be strictly stronger than our braid group obfuscator. This approach also provides motivation for finding a complete set of relations for the gate R.

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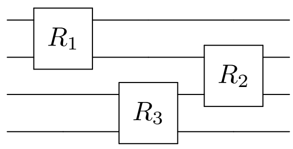


Figure 2 An example of a reversible circuit constructed from a single gate R. As a product of matrices, we write this $R_2R_3R_1$, in keeping with the convention [36] that circuit diagrams are to be read left-to-right, whereas the matrix product acts right-to-left. Note that in subsequent circuit diagrams we drop the subscripts from the R gates as these can be read off from the "wires" the gates act on.

A Classical Computation with Braids

In this section, we present a reversible gate R on pairs of 60-state dits that can perform universal computation and obeys the relations of the braid group. The universality construction for this gate comes from the quantum computation literature [31, 37, 34], but we present it here in purely classical language to make it accessible to a broader audience.

Suppose we arrange n dits on a line, and allow R to act only on neighboring dits. Further, we do not allow R to be applied "upside-down". Then, there are $n-1$ choices for how to apply R. We label these $R_1, R_2, \ldots, R_{n-1}$, as illustrated in Figure 2. Each of $R_1, \ldots, R_{n-1}$ corresponds to a $d^n \times d^n$ permutation matrix. Specifically, R_j is obtained by taking the tensor product of R with identity matrices according to $R_j = \mathbb{1}_{d\times d}^{\otimes(j-1)} \otimes R \otimes \mathbb{1}_{d\times d}^{\otimes(n-j-1)}$.

$R_1, \ldots, R_{n-1}$ generate a subgroup of S_{d^n}. Among others, these generators obey the relations

$$R_i R_j = R_j R_i \quad \forall |i-j| \geq 2. \tag{5}$$

If R satisfies

$$R_1 R_2 R_1 = R_2 R_1 R_2 \tag{6}$$

then

$$R_i R_{i+1} R_i = R_{i+1} R_i R_{i+1} \quad \forall i \tag{7}$$

and in this case the gates $R_1, \ldots, R_{n-1}$ satisfy all the relations of the braid group B_n. In other words, the map defined by $\sigma_i \mapsto R_i$ and $\sigma_i^{-1} \mapsto R_i^{-1}$ is a homomorphism from B_n to S_{d^n}, *i.e.* a representation of the braid group. Note that this representation is never faithful as B_n is infinite.

The condition 6 is known as the Yang-Baxter equation[6]. Finding all the matrices satisfying the Yang-Baxter equation at a given dimension has only been achieved at $d = 2$ [24]. However, certain systematic constructions coming from mathematical physics can produce infinite families of solutions. In particular, let G be any finite group, and let R be the permutation on the set $G \times G$ defined by

$$R(a, b) = (b, b^{-1}ab). \tag{8}$$

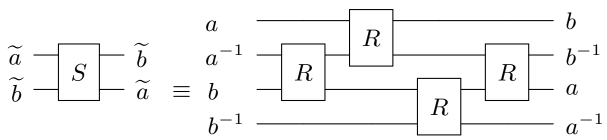
[6] Actually, two slightly different equations go by the name Yang-Baxter in the literature. Careful sources distinguish these as the algebraic Yang-Baxter equation and the braided Yang-Baxter relation (which is sometimes called the quantum Yang-Baxter equation). Equation 6 is the latter. Furthermore, some sources treat a more complicated version of the Yang-Baxter equation in which R depends on a continuous parameter. In such works equation 6 is often referred to as the constant Yang-Baxter equation.

By direct calculation one sees that any such an R satisfies the Yang-Baxter equation. (In physics language, R comes from the braiding statistics of the magnetic fluxes in the quantum double of G.)

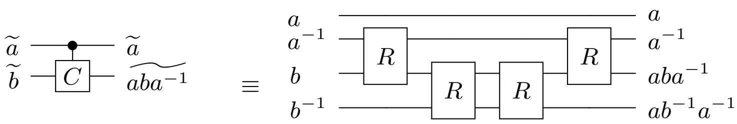
In 1997, Kitaev discovered that choosing G to be the symmetric group S_5 yields an R gate sufficient to perform universal reversible computation [31]. Ogburn and Preskill subsequently showed that the alternating group A_5, which is half as large as S_5, is already sufficient. The universality construction for A_5 was subsequently presented in greater detail and generalized to all non-solvable groups by Mochon [34]. In the remainder of this section we give a self-contained exposition of the universality construction from [34], shorn of physics language.

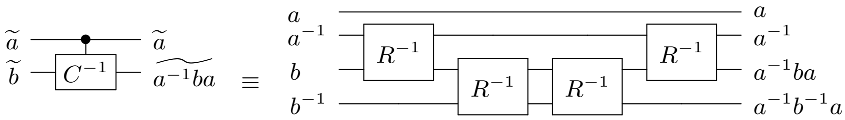
To obtain a representation of the braid group, we must strictly enforce the requirement that application of R to neighboring dits on a line is the only allowed operation. In particular, we are not given as elementary operations the ability to apply R upside-down, or to non-neighboring dits, or to move dits around. Thus, to prove computational universality, it is helpful to first construct a SWAP gate from R gates, which exchanges neighboring dits. As is well-known, the $n-1$ swaps of nearest neighbors on a line generate the full group S_n of permutations, and thus a SWAP gate enables application of R to any pair of dits.

For R gates of the form (2), two pairs of inverse group elements in the order a, a^{-1}, b, b^{-1} can be swapped by applying the product $R_2R_3R_1R_2$. Thus, in the construction of [37, 34], elements of A_5 are always paired with their inverses. This can be regarded as a form of encoding; $|A_5| = 60$, so each 60-state dit is encoded by a corresponding pair of elements of A_5. We introduce the notation $\widetilde{g} \equiv (g, g^{-1})$ for this encoding, and similarly, abbreviate the encoded swap operation as follows.

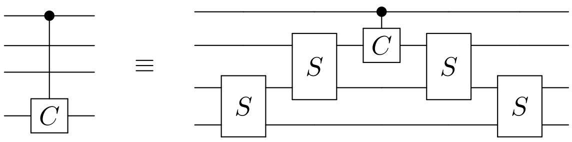


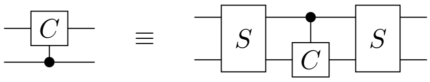
Similarly, the sequence $R_2R_3R_3R_2$ performs the transformation $(\widetilde{a}, \widetilde{b}) \mapsto (\widetilde{a}, \widetilde{aba^{-1}})$ on a pair of encoded dits. We abbreviate this in circuit diagrams as follows.





This notation can easily be extended to provide a shorthand for the sequence of gates needed to implement a C gate between non-neighboring pairs of bits, as illustrated by the following examples.





Next, consider the following product of elements of A_5 (which should be read right-to-left).

$$f(g_1, g_2) = (521)g_1(14352)g_2(124)g_1^{-1}(15342)g_2^{-1}(521) \tag{9}$$

One sees that

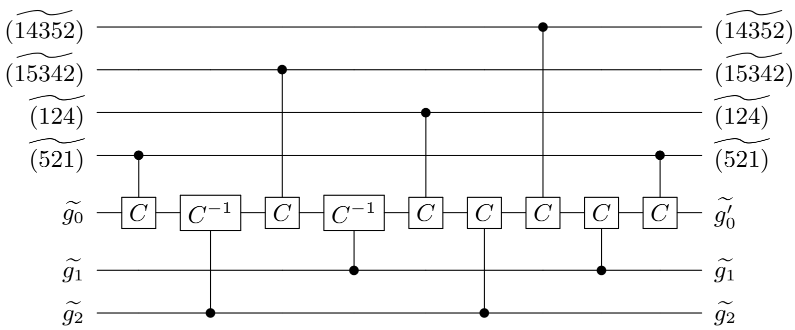
$$\begin{aligned} f((345),(345)) &= \mathbb{1} \\ f((345),(435)) &= \mathbb{1} \\ f((435),(345)) &= \mathbb{1} \\ f((435),(435)) &= (12)(34) \end{aligned}$$

where $\mathbb{1}$ denotes the identity permutation. Furthermore, conjugating (345) by (12)(34) yields (435), and conversely, conjugating (435) by (12)(34) yields (345). Thus, we may think of (345) as an encoded zero and (435) as an encoded one, and we see that

$$f(g_1, g_2)g_0 f(g_1, g_2)^{-1} \tag{10}$$

toggles g_0 between one and zero if g_1 and g_2 are both encoded ones and leaves g_0 unchanged otherwise. Such a doubly-controlled toggling operation is known as a Toffoli gate, which is well-known to be a computationally universal reversible gate [17].

As a circuit diagram, this construction can be expressed as follows.



Here, if g_0, g_1, g_2 encode bits b_0, b_1, b_2 then g_0' encodes $b_0 \oplus b_1 \wedge b_2$. The four ancillary dits $\widetilde{(14352)}$, $\widetilde{(15342)}$, $\widetilde{(124)}$, and $\widetilde{(521)}$, are used to "catalytically" facilitate the construction of a Toffoli gate, and thus computations built from arbitrarily many Toffoli gates can be performed with only one copy of these four dits.

Unpacking the various shorthand notations, one sees that the above circuit represents the following braid of 132 crossings on 14 strands, which encodes a Toffoli gate with the first wire as target, and the second and third wires as controls.

$$T = \begin{array}{llll} \sigma_8\sigma_9\sigma_9\sigma_8 & \sigma_{10}\sigma_{11}\sigma_9\sigma_{10} & \sigma_{10}\sigma_{11}\sigma_{11}\sigma_{10} & \sigma_{10}\sigma_{11}\sigma_9\sigma_{10} \\ \sigma_2\sigma_3\sigma_1\sigma_2 & \sigma_4\sigma_5\sigma_3\sigma_4 & \sigma_6\sigma_7\sigma_5\sigma_6 & \sigma_8\sigma_9\sigma_9\sigma_8 \\ \sigma_6\sigma_7\sigma_5\sigma_6 & \sigma_4\sigma_5\sigma_3\sigma_4 & \sigma_2\sigma_3\sigma_1\sigma_2 & \sigma_{12}\sigma_{13}\sigma_{11}\sigma_{12} \\ \sigma_{10}\sigma_{11}\sigma_9\sigma_{10} & \sigma_{10}\sigma_{11}\sigma_{11}\sigma_{10} & \sigma_{10}\sigma_{11}\sigma_9\sigma_{10} & \sigma_{12}\sigma_{13}\sigma_{11}\sigma_{12} \\ \sigma_6\sigma_7\sigma_5\sigma_6 & \sigma_8\sigma_9\sigma_9\sigma_8 & \sigma_6\sigma_7\sigma_5\sigma_6 & \sigma_{10}\sigma_{11}\sigma_9\sigma_{10} \\ \sigma_{10}^{-1}\sigma_{11}^{-1}\sigma_{11}^{-1}\sigma_{10}^{-1} & \sigma_{10}\sigma_{11}\sigma_9\sigma_{10} & \sigma_4\sigma_5\sigma_3\sigma_4 & \sigma_6\sigma_7\sigma_5\sigma_6 \\ \sigma_8\sigma_9\sigma_9\sigma_8 & \sigma_6\sigma_7\sigma_5\sigma_6 & \sigma_4\sigma_5\sigma_3\sigma_4 & \sigma_{12}\sigma_{13}\sigma_{11}\sigma_{12} \\ \sigma_{10}\sigma_{11}\sigma_9\sigma_{10} & \sigma_{10}^{-1}\sigma_{11}^{-1}\sigma_{11}^{-1}\sigma_{10}^{-1} & \sigma_{10}\sigma_{11}\sigma_9\sigma_{10} & \sigma_{12}\sigma_{13}\sigma_{11}\sigma_{12} \\ \sigma_8\sigma_9\sigma_9\sigma_8 & & & \end{array} \tag{11}$$

Note that we take the convention that this should be read backwards compared to the way one reads English text. This is in keeping with the conventional notation for the composition of functions and our right-to-left multiplication of R matrices. We have used whitespace to divide crossings into groups of four as these correspond to elementary S and R gates.

Given this construction of the Toffoli gate by braid crossings, it is a simple matter to "compile" any given logic circuit into a corresponding braid. A_5 has 60 elements. Thus, encoding a single bit into a a pair of A_5 elements appears somewhat wasteful. It is natural to try to find Yang-Baxter solutions acting on d-state dits for smaller d that achieve universal classical computation. In appendix B, we improve upon the A_5-based construction to show that $d = 44$ suffices. We have also used exhaustive computer search to find all permutation solutions satisfying the Yang-Baxter equation up to $d = 5$ (i.e. up to 25×25 permutation matrices). Our examination of these solutions suggests that none are computationally universal. Where between 5 and 44 lies the minimal d remains an interesting open question.

B Optimizing Classical Braid Gates

In appendix A we have recounted the construction of [34], which shows that the reversible gate R, which acts on pairs of 60-state dits and satisfies the Yang-Baxter equation, can perform universal classical computation. In this section, based on a suggestion of Robert König, we show that R can be modified to obtain a gate acting on pairs of 44-state dits that satisfies the Yang-Baxter equation and can perform universal classical computation. Our computational evidence suggests that no reversible gate on d-state dits satisfying the Yang-Baxter equation can perform universal computation for $d \leq 5$. Where between 5 and 44 the minimal d lies for which computationally universal reversible Yang-Baxter gates acting on d-state qudits exist remains an open question.

The universality construction of [34], recounted in appendix A, starts with all dits initialized to states from the following set.

$$\begin{aligned} S &= \{g, g^{-1} | g \in S_0\} \\ S_0 &= \{(14352), (15342), (124), (521), (345), (435)\} \end{aligned}$$

Here we show that the orbit of S under the action of the gate R is not all of A_5, rather the orbit has only 44 elements. Thus the restriction of the matrix R onto this 44-dimensional subspace is a permutation-matrix that satisfies the Yang-Baxter equation and is capable of universal classical computation.

Recalling (2), one sees that the orbit O_R of S under R is

$$O_R = \{b^{-1}ab | a \in S, b \in \langle S \rangle\} \tag{12}$$

where $\langle S\rangle$ is the subgroup of A_5 generated by S. A simple computer algebra calculation shows that $\langle S\rangle = A_5$, thus O_R consists of exactly those elements of A_5 conjugate to S.

It is well known that the conjugacy classes of A_5 are as follows.

1) the identity	(1 element)
2) 3-cycles	(20 elements)
3) conjugates of (12)(34)	(15 elements)
4) conjugates of (12345)	(12 elements)
5) conjugates of (21345)	(12 elements)

One sees that O_R contains 2), and does not contain 1) or 3). The only remaining question is whether O_R contains both 4) and 5) or just one of them. A simple computer algebra calculation shows that (14352) and (15342) are non-conjugate elements of A_5. Hence O_R must contain both 4) and 5). Therefore, $|O_R| = 44$.