Derandomized Graph Product Results Using the Low Degree Long Code

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Abstract

In this paper, we address the question of whether the recent derandomization results obtained by the use of the low-degree long code can be extended to other product settings. We consider two settings: (1) the graph product results of Alon, Dinur, Friedgut and Sudakov [*GAFA*, 2004] and (2) the "majority is stablest" type of result obtained by Dinur, Mossel and Regev [*SICOMP*, 2009] and Dinur and Shinkar [In *Proc. APPROX*, 2010] while studying the hardness of approximate graph coloring.

In our first result, we show that there exists a considerably smaller subgraph of $K_3^{\otimes R}$ which exhibits the following property (shown for $K_3^{\otimes R}$ by Alon *et al.*): independent sets close in size to the maximum independent set are well approximated by dictators.

The "majority is stablest" type of result of Dinur *et al.* and Dinur and Shinkar shows that if there exist two sets of vertices A and B in $K_3^{\otimes R}$ with very few edges with one endpoint in A and another in B, then it must be the case that the two sets A and B share a single influential coordinate. In our second result, we show that a similar "majority is stablest" statement holds good for a considerably smaller subgraph of $K_3^{\otimes R}$. Furthermore using this result, we give a more efficient reduction from Unique Games to the graph coloring problem, leading to improved hardness of approximation results for coloring.

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1   Introduction

The discovery of the low-degree long code (aka short code) by Barak *et al.* [2] has over the last year led to several more efficient inapproximability reductions [2, 5, 9, 13, 15]. The low-degree long code is a derandomization of the long code in the following sense. Given a finite field $\mathbb{F}$, the long code of a string $x \in \mathbb{F}^n$ is the evaluation of every $\mathbb{F}$-valued function on

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$\mathbb{F}^n$ at the point x while the degree d long code of of x is the evaluation of every n-variate polynomial of total degree at most d at the point x. The crucial observation of Barak *et al.* [2] was that the optimal testing results for Reed-Muller codes [3, 10] proved that the low-degree long code could be used as a surrogate for the long code in several inapproximability results. In this paper, we ask if we can extend this application of low-degree long code to other product settings. In particular, we prove the following two results. (1) We show that result due to Alon *et al.* [1] on the size of maximum independent sets in product graphs can be derandomized (Theorem 1.2). (2) We show that the "majority is stablest" type of result obtained by Dinur *et al.* [7] and Dinur and Shinkar [8] can be derandomized (Theorem 1.4).

1.1 Derandomized graph products

As a first application, we consider the following graph product result due to Alon *et al.* [1]. Consider the undirected weighted graph K_3 on the three vertices $V = \{0, 1, 2\}$ and edges weighted as follows: $W(f, f') = 1/2$ iff $f' \neq f \in \{0, 1, 2\}$. Let $K_3^{\otimes R}$ be the graph with vertex set $V^{\otimes R}$ and weights-matrix the R-wise tensor of the matrix W. Clearly, for any $i \in [R]$ and $a \in \{0, 1, 2\}$, the set $V_{i,a} := \{v \in V^{\otimes R} : v_i = a\}$ is an independent set in $K_3^{\otimes R}$ of fractional size $1/3$ since K_3 does not have any self loops. We call such an independent set a *dictator* for obvious reasons. Alon *et al.* [1] showed that these are the maximal independent sets in $K_3^{\otimes R}$ and in fact any independent set of size close to the maximum is close to a dictator.

▶ **Theorem 1.1** ([1])**.** *Let A be an independent set in $K_3^{\otimes R}$ of size $\delta 3^R$. Then,*
1. $\delta \leq 1/3$.
2. $\delta = 1/3$ *iff A is a dictator.*
3. *If $\delta \geq 1/3 - \varepsilon$, then A is $O(\varepsilon)$-close to a dictator. That is, there is a dictator A' such that $|A \Delta A'| = O(\varepsilon 3^R)$.*

Note that the above graph has 3^R vertices. Our first result (Theorem 1.2) shows that there exists a considerably smaller subgraph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ of $K^{\otimes R}$ with only $3^{\text{poly}(\log R)}$ vertices that has the same properties. In order to describe the subgraph, it will be convenient to think of K_3 as having vertex set $\mathbb{F}_3$ and

$$W(f, f') = \Pr_{p \in \mathbb{F}_3, a \in \{1, 2\}}[f' = f + a(p^2 + 1)].$$

Let $\mathsf{P}_{r,d}$ be the set of polynomials on r variables over $\mathbb{F}_3$ of total degree at most d and individual degrees of the variables at most 2. Let r and d be two parameters and let $R = 3^r$. Note that $V^{\otimes R}$ can be identified with $\mathsf{P}_{r,2r}$, since $\mathsf{P}_{r,2r}$ is the set of all functions from $\mathbb{F}_3^r$ to $\mathbb{F}_3$. The subgraph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is as follows : $\mathcal{V} := \mathsf{P}_{r,2d}$ and the edges are given by the weights-matrix defined below

$$\mathcal{W}(f, f') = \Pr_{p \in \mathsf{P}_{r,d}, a \in \{1, 2\}}[f' = f + a(p^2 + 1)].$$

Note that since $\mathsf{P}_{r,2d}$ is a subspace of dimension $r^{O(d)}$, the size of the vertex set is $3^{r^{O(d)}}$, which is considerably smaller than 3^R for constant d.

▶ **Theorem 1.2.** *There is a constant d for which the following holds. If A is an independent set of size $\delta|\mathcal{V}|$ in $\mathcal{G}$ then*
1. $\delta \leq 1/3$.
2. $\delta = 1/3$ *iff A is a dictator.*
3. *If $\delta \geq 1/3 - \varepsilon$ then A is $O(\varepsilon)$-close to a dictator.*

A crucial element in the proof of Theorem 1.1 is a hypercontractivity theorem for functions which do not have any heavy Fourier coefficients. Theorem 1.2 is proved by observing that a similar hypercontractivity theorem also holds good in the low-degree long code setting (see Lemma 3.4).

1.2 Derandomized "majority is stablest" result

While studying the hardness of approximate graph coloring, Dinur, Mossel and Regev [7] proved the following "majority is stablest" type of result: if there is a pair of subsets of vertices in $K_3^{\otimes R}$ of sufficiently large size such that the average weight of edges between them is small, then their indicator functions must have a common influential coordinate. Subsequently, Dinur and Shinkar [8] obtained the following quantitative improvement to the above theorem.

▶ **Theorem 1.3** ([8, Theorem 1.3]). *For all $\mu > 0$ there exists $\delta = \mu^{O(1)}$ and $k = O(\log 1/\mu)$ such that the following holds: For any two functions $A, B : \{0, 1, 2\}^R \to [0, 1]$ if*

$$\mathbb{E}\, A > \mu, \ \ \mathbb{E}\, B > \mu, \ \ and \ \ \underset{f, f'}{\mathbb{E}}\, A(f)B(f') \leq \delta^1$$

where f is chosen randomly from $V^{\otimes R}$ and f' is chosen with probability $W^{\otimes R}(f, f')$ then

$$\exists x \in [R] \ such \ that \ \mathrm{Inf}_x^{\leq k}(A) \geq \delta \ and \ \mathrm{Inf}_x^{\leq k}(B) \geq \delta.$$

Our second result (Theorem 1.4) shows that the above theorem can be derandomized to obtain a similar result for the subgraph $\mathcal{G}$. For defining influence for real valued functions on $\mathsf{P}_{r,2d}$, we note that the characters of $\mathsf{P}_{r,2d}$ are restrictions of characters of $\mathbb{F}_3^R \equiv \mathsf{P}_{r,2r}$. So the definition of influence for functions on $\mathbb{F}_3^R$ also extends naturally to functions on $\mathsf{P}_{r,2d}$.

▶ **Theorem 1.4.** *For all $\mu > 0$ there exists $\delta = \mu^{O(1)}, k = O(\log 1/\mu), d = O(\log 1/\mu)$ such that the following holds: For any two functions $A, B : \mathsf{P}_{r,2d} \to [0, 1]$ if*

$$\mathbb{E}\, A > \mu, \ \ \mathbb{E}\, B > \mu, \ \ and \ \ \underset{f, f'}{\mathbb{E}}\, A(f)B(f') \leq \delta$$

where f is chosen randomly from $\mathsf{P}_{r,2d}$, $f' = f + a(p^2 + 1)$, p are chosen randomly from $\mathsf{P}_{r,d}$ and $a \in_R \{1, 2\}$ then

$$\exists x \in \mathbb{F}_3^r \ such \ that \ \mathrm{Inf}_x^{\leq k}(A) \geq \delta \ and \ \mathrm{Inf}_x^{\leq k}(B) \geq \delta.$$

A similar derandomized "majority is stablest" result in the case of the noisy hypercube was proved by Barak *et al.* [2, Theorem 5.6] and they used the Meka-Zuckerman pseudorandom generators (PRGs) for polynomial threshold functions [14]. Kane and Meka [11] obtained a quantitative improvement over this derandomization by constructing an improved PRG for Lipschitz functions. Our setting is slightly more involved, (1) we have a two function version (ie., A and B) and (2) the underlying graph in K_3 and the corresponding noise operator in the derandomized setting has not necessarily positive eigenvalues. Yet, we manage to show that a derandomization still holds in this case too (using the Kane-Meka PRG). We conjecture that our derandomization can be further improved to obtain $d = O(\log \log 1/\mu)$.

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[1] The hypothesis in the theorem statement of Dinur-Shinkar [8] requires $\mathbb{E}_{f, f'}\, A(f)B(f') = 0$, however it is easy to check that their theorem also holds good under the weaker hypothesis $\mathbb{E}_{f, f'}\, A(f)B(f') \leq \delta$.

1.2.1 Application to graph coloring

Using a version of Theorem 1.3 for another base graph on 4 vertices, Dinur and Shinkar proved a hardness result for graph coloring.

▶ **Definition 1.5** (Label Cover). An instance $G = (U, V, E, L, R, \{\pi_e\}_{e \in E})$ of a *Label Cover* consists of a bipartite graph (U, V, E) that is right regular along with a projection map $\pi_e : R \to L$ for every edge $e \in E$. Label Cover is a constraint satisfaction problem where the vertices in U are the variables taking values in L and vertices in V taking values in R. The instance is a *Unique Games* instance if $R = L$ and π_e is a permutation for all $e \in E$. Given a labeling $\ell : U \cup V \to L \cup R$, an edge $e = (u, v)$ is said to be satisfied if $\pi_e(\ell(v)) = \ell(u)$.

Dinur and Shinkar gave a reduction from an instance of Label Cover with n vertices, 2-to-1 constraints and label set of size R to a graph of size $n4^R$. Perfectly satisfiable instances were mapped to 4-colorable graphs. Instances for which any labeling can satisfy only an $s(n)$ fraction of edges where mapped to graphs which did not have any independent sets of size poly$(s(n))$. Since the size of the graph produced by the reduction is exponential in R, they needed to assume that $R = O(\log n)$, to get hardness results. We give a more efficient reduction using Theorem 1.4 from Label Cover instances for which the projection constraints have special form. Our reduction is simpler to describe for the case 3-colorable graphs and starts with Unique Games instances. Hence for getting hardness result, we need to assume a conjecture similar to the Unique Games Conjecture with specific parameters.

▶ **Conjecture 1.1** $((c(n), s(n), r(n))$-UG Conjecture). *It is NP-Hard to distinguish between unique games instances (U, V, E, R, Π) on n vertices and $R = \mathbb{F}_3^{r(n)}$ from the following cases:*
- *YES Case : There is a labeling and a set $S \subseteq V$ of size $(1 - c(n))|V|$ such that all edges between vertices in S are satisfied.*
- *NO Case : For any labeling, at most $s(n)$ fraction of edges are satisfied.*

Khot and Regev [12] proved that the Unique Games Conjecture implies that for any constants $c, s \in (0, 1/2)$ there is a constant r such that (c, s, r)-UG Conjecture is true. We also require that the constraints of the Unique Games instance are full rank linear maps.

▶ **Definition 1.6** (Linear constraint). A constraint $\pi : R \to L$ is a *linear constraint* of iff $R = L = \mathbb{F}_3^r$, and π is a linear map of rank r.

The theorem below is obtained by replacing the long code by the low degree long code of degree $d = O(\log 1/\mu)$ in the reduction of Dinur and Shinkar. For want of space, the description of this reduction is deferred to the full version [6, Appendix A].

▶ **Theorem 1.7.** *There is a reduction from (c, s, r)-Unique Games instances G with n vertices, label set $\mathbb{F}_3^r$ and linear constraints to graphs $\mathcal{G}$ of size $n3^{r^{O(\log 1/\mu)}}$ where $\mu = \text{poly}(s)$ such that*
- *If G belongs to the YES case of (c, s, r)-UG Conjecture then there is a subgraph of $\mathcal{G}$ with fractional size $1 - c$ that is 3-colorable.*
- *If G belongs to the NO case of (c, s, r)-UG Conjecture then $\mathcal{G}$ does not have any independent sets of fractional size μ.*

Due to the improved efficiency of the reduction, we are able to get hardness results even if the label cover instances have super-polylogarithmic sized label sets of size at most $2^{2^{O(\sqrt{\log \log n})}}$, while the reduction due to Dinur and Shinkar only works if the label set is of size at most $O(\log^c n)$ for some constant c. More precisely, suppose the UG conjecture were true for soundness $s(n)$ and alphabet size $R = 3^r$ that satisfy $\log_3 R = r = s(n)^{O(1)}$. Then,

the result of Dinur and Shinkar rules out polynomial time algorithms that find an independent set of relative size $1/\operatorname{poly}(\log\log N)$. On the other hand, under the same assumption, our reduction rules out polynomial time algorithms that find an independent set of relative size $1/2^{\operatorname{poly}(\log\log N)}$.

▶ **Corollary 1.8.** *Let c, s, r be functions such that $r(n) = \operatorname{poly}(1/s(n))$. Assuming (c, s, r)-UG Conjecture on instances with linear constraints, given a graph on N vertices which has an induced subgraph of relative size $1 - c$ that is 3-colorable, no polynomial time algorithm can find an independent set of fractional size $2^{-\operatorname{poly}(\log\log N)}$.*

We remark that we can improve the conclusion if Theorem 1.4 can be proved even when $d = O(\log\log 1/\mu)$.

2 Preliminaries

2.1 Low degree polynomials

We will be working over the field $\mathbb{F}_3$. Let $\mathsf{P}_{r,d}$ be the set of degree d polynomials on r variables over $\mathbb{F}_3$, with individual variable degrees at most 2. Let $\mathfrak{F}_r := \mathsf{P}_{r,2r}$. Note that $\mathfrak{F}_r$ is the set of all functions from $\mathbb{F}_3^r$ to $\mathbb{F}_3$. $\mathfrak{F}_r$ is a $\mathbb{F}_3$-vector space of dimension 3^r and $\mathsf{P}_{r,d}$ is a subspace of dimension $r^{O(d)}$. The Hamming distance between f and $g \in \mathfrak{F}_r$, denoted by $\Delta(f, g)$, is the number of inputs on which f and g differ. For $S \subseteq \mathfrak{F}_r$, define $\Delta(f, S) := \min_{g \in S} \Delta(f, g)$. We say that f is δ-far from S if $\Delta(f, S) \geq \delta$ and f is δ-close to S otherwise. Given $f, g, \in \mathfrak{F}_r$, the dot product between them is defined as $\langle f, g \rangle := \sum_{x \in \mathbb{F}^r} f(x)g(x)$. For a subspace $S \subseteq \mathfrak{F}_r$, the dual subspace is defined as $S^\perp := \{g \in \mathfrak{F}_r : \forall f \in S, \langle g, f \rangle = 0\}$. The following theorem relating dual spaces is well known.

▶ **Lemma 2.1.** $\mathsf{P}_{r,d}^\perp = \mathsf{P}_{r,2r-d-1}$.

We need the following Schwartz-Zippel-like Lemma for degree d polynomials over $\mathbb{F}_3$.

▶ **Lemma 2.2** (Schwartz-Zippel lemma [10, Lemma 3.2]). *Let $f \in \mathbb{F}_3[x_1, \cdots, x_r]$ be a non-zero polynomial of degree at most d with individual degrees at most 2. Then $\Pr_{a \in \mathbb{F}_3^r}[f(a) \neq 0] \geq 3^{-d/2}$.*

The following lemma is an easy consequence of Lemma 2.2.

▶ **Lemma 2.3.** *If p is a uniformly random polynomial from $\mathsf{P}_{r,d}$ then as a string of length 3^r over the alphabet $\mathbb{F}_3$, p is $3^{\lfloor (d+1)/2 \rfloor}$-wise independent.*

2.2 Fourier analysis of functions on subspace of low degree polynomials

▶ **Definition 2.4** (Characters). A character of $\mathsf{P}_{r,d}$ is a function $\chi : \mathsf{P}_{r,d} \to \mathbb{C}$ such that

$$\chi(0) = 1 \text{ and } \forall f, g \in \mathsf{P}_{r,d}, \ \chi(f + g) = \chi(f)\chi(g).$$

The following lemma lists the basic properties of characters.

▶ **Lemma 2.5.** *Let $\{1, \omega, \omega^2\}$ be the cube roots of unity and for $\beta \in \mathfrak{F}_r, f \in \mathsf{P}_{r,d}$, $\chi_\beta(f) := \omega^{\langle \beta, f \rangle}$, where $\langle \beta, f \rangle := \sum_{x \in \mathbb{F}_3^r} \beta(x)f(x)$.*
- *The characters of $\mathsf{P}_{r,d}$ are $\{\chi_\beta : \beta \in \mathfrak{F}_r\}$.*
- *For $\beta \in \mathsf{P}_{r,d}^\perp$, χ_β is the constant 1 function.*
- *For any $\beta, \beta' \in \mathfrak{F}_r$, $\chi_\beta = \chi'_\beta$ if and only if $\beta - \beta' \in \mathsf{P}_{r,d}^\perp$.*

- *For any β, let $|\beta|$ be the size of the set of inputs on which β is non-zero. For any distinct $\beta, \beta' \in \mathfrak{F}_r$ with $|\beta|, |\beta'| < 3^{\lfloor(d+1)/2\rfloor}/2$, $\chi_\beta \neq \chi'_\beta$ since $\beta + \beta' \notin \mathsf{P}_{r,d}^\perp$.*

- *$\forall\beta, \exists\beta'$ such that $\beta - \beta' \in \mathsf{P}_{r,d}^\perp$ and $|\beta'| = \Delta(\beta, \mathsf{P}_{r,d}^\perp)$ (i.e., the constant 0 function is (one of) the closest function to β' in $\mathsf{P}_{r,d}^\perp$). We call such a β' a minimum support function for the coset $\beta + \mathsf{P}_{r,d}^\perp$.*

- *Characters forms an orthonormal basis for the vector space of functions from $\mathsf{P}_{r,d}$ to $\mathbb{C}$, under the inner product $\langle A, B \rangle := \mathbb{E}_{f \in \mathsf{P}r,d}\left[A(f)\overline{B(f)}\right]$*

- *Any function $A : \mathsf{P}_{r,d} \to \mathbb{C}$ can be uniquely decomposed as*

$$A(f) = \sum_{\beta \in \Lambda_{r,d}} \widehat{A}(\beta)\chi_\beta(f) \; where \; \widehat{A}(\beta) := \mathbb{E}_{g \in \mathsf{P}_{r,d}}\left[A(g)\overline{\chi_\beta(g)}\right], \tag{2.1}$$

and $\Lambda_{r,d}$ is the set of minimum support functions, one for each of the cosets in $\mathfrak{F}_r/\mathsf{P}_{r,d}^\perp$, with ties broken arbitrarily.

- *Parseval's identity: For any function $A : \mathsf{P}_{r,d} \to \mathbb{C}$,*

$$\sum_{\beta \in \Lambda_{r,d}} |\widehat{A}(\beta)|^2 = \mathbb{E}_{f \in \mathsf{P}_{r,d}}[|A(f)|^2]. \tag{2.2}$$

In particular, if $A : \mathsf{P}_{r,d} \to \{1, \omega, \omega^2\}$,

$$\sum_{\beta \in \Lambda_{r,d}} |\widehat{A}(\beta)|^2 = 1. \tag{2.3}$$

▶ **Definition 2.6** (Influence). For a function $A : \mathsf{P}_{r,d} \to \mathbb{C}$ and a number $k < 3^{\lfloor(d+1)/2\rfloor}/2$, the degree k influence of $a \in \mathbb{F}_3^r$ is defined as

$$\mathrm{Inf}_a^{\leq k}(A) = \sum_{\beta \in \Lambda_{r,d} : \beta(a) \neq 0 \text{ and } |\beta| \leq k} |\widehat{A}(\beta)|^2.$$

▶ **Definition 2.7** (Dictator). A function $A : \mathsf{P}_{r,d} \to \mathbb{C}$ is a dictator if there exists $x \in \mathbb{F}_3^r$ and $\widehat{A}_0, \widehat{A}_1, \widehat{A}_2 \in \mathbb{C}$ such that A can be written as $A = \widehat{A}_0 + \widehat{A}_1\chi_{e_x} + \widehat{A}_2\chi_{2e_x}$ where $e_x : \mathbb{F}_3^r \to \mathbb{F}_3$ the indicator function for x.

The following lemma which follows from the results of Guruswami *et al.* [9], will be crucial for our proofs.

▶ **Lemma 2.8.** *If $\alpha : \mathbb{F}_3^r \to \mathbb{F}_3$ such that $\Delta(\alpha, \mathsf{P}_{r,2d}^\perp) > 3^{d/2}$ then*

$$\left| \mathbb{E}_{p \in \mathsf{P}_{r,d}} \chi_\alpha(p^2) \right| \leq 3^{-\Omega(3^{d/9})}.$$

Proof. By definition, $\left| \mathbb{E}_{p \in \mathsf{P}_{r,d}} \chi_\alpha(p^2) \right| = \left| \mathbb{E}_{p \in \mathsf{P}_{r,d}} \omega^{\langle\alpha, p^2\rangle} \right|$. If $\alpha : \mathbb{F}_3^r \to \mathbb{F}_3$ is such that $\Delta(\alpha, \mathsf{P}_{r,2d}^\perp) > 3^{d/2}$ then for a random $p \in \mathsf{P}_{r,d}$, $\langle\alpha, p^2\rangle$ is $3^{-\Omega(3^{d/9})}$-close to the uniform distribution on $\mathbb{F}_3$ according to [9, Lemma 3.1 and 3.4]. ◀

3 Derandomized $K_3^{\otimes R}$

Alon *et al.* [1] proved Theorem 1.1 by using the following lemma.

▶ **Lemma 3.1.** *There is constant K such that the following holds: If $A : \mathbb{F}_3^R \to \{0,1\}$ satisfies*

$$\sum_{|\alpha|>1} |\widehat{A}_\alpha|^2 \le \varepsilon \ \text{and} \ \widehat{A}_0 = \delta$$

then there exists a dictator $B : \mathbb{F}_3^R \to \{0,1\}$ such that

$$\|A - B\|_2 \le \frac{K\varepsilon}{\delta - \delta^2 - \varepsilon}.$$

The above lemma was proved using the following hypercontractive inequality.

▶ **Lemma 3.2.** *There is a constant C such that for any function $A : \mathbb{F}_3^R \to \mathbb{C}$ with $\widehat{A}_\alpha = 0$ when $|\alpha| > t$,*

$$\|A\|_4 \le C^t \|A\|_2.$$

Our proof of Theorem 1.2 will use a similar lemma for functions on the subspace $\mathsf{P}_{r,2d}$.

▶ **Lemma 3.3.** *There is a constant K such that the following holds: If $A : \mathsf{P}_{r,2d} \to \{0,1\}$ satisfies*

$$\sum_{|\alpha|>1} |\widehat{A}_\alpha|^2 \le \varepsilon \ \text{and} \ \widehat{A}_0 = \delta$$

then there exists a dictator $B : \mathsf{P}_{r,2d} \to \{0,1\}$ such that

$$\|A - B\|_2 \le \frac{K\varepsilon}{\delta - \delta^2 - \varepsilon}.$$

The above lemma follows from hypercontractive inequalities over $\mathsf{P}_{r,2d}$ stated below, in exactly the same way as Alon *et al.* proves Lemma 3.1 from Lemma 3.2.

▶ **Lemma 3.4.** *There is a constant C such that for $4t \le 3^{d-1}$ and any function $A : \mathsf{P}_{r,2d} \to \mathbb{C}$ with $\widehat{A}_\alpha = 0$ when $|\alpha| > t$,*

$$\|A\|_4 \le C^t \|A\|_2.$$

Proof. Follows from Lemma 3.6 and Lemma 3.2. ◀

▶ **Definition 3.5** (Lift). For a function $B : \mathsf{P}_{r,2d} \to \mathbb{C}$ with the Fourier decomposition $B = \sum_{\alpha \in \Lambda_{r,d}} \widehat{B}_\alpha \chi_\alpha$, the lift of B denoted by B' is a function $B' : \mathfrak{F}_r \to \mathbb{C}$ with the Fourier decomposition $B' = \sum_{\alpha \in \Lambda_{r,d}} \widehat{B}_\alpha \chi_\alpha$. In the decomposition of B', χ_α's are functions with domain $\mathfrak{F}_r$.

▶ **Lemma 3.6.** *If $2kt \le 3^{d-1}$ and $B : \mathsf{P}_{r,2d} \to \mathbb{C}$ be a function such that $\widehat{B}_\alpha = 0$ when $|\alpha| > t$ then*

$$\|B\|_{2k} = \|B'\|_{2k}.$$

Proof. From the Lemma 2.2 and Lemma 2.1, we have that $\forall \alpha \in \mathsf{P}_{r,2d}^\perp \setminus \{0\}$, $|\alpha| > 3^{d-1}$. So if $\exists \{\alpha_i, \beta_i\}_{i \in [k]}$ with $|\alpha_i|, |\beta_i| \le t$, then

$$\sum_{i \in [k]} \alpha_i - \beta_i \in \mathsf{P}_{r,2d}^\perp \Rightarrow \sum_{i \in [k]} \alpha_i - \beta_i = 0. \tag{3.1}$$

This is because $\sum_{i\in[t]}\alpha_i-\beta_i$ has support size at most $2kt<3^{d-1}$. We use this fact to prove the theorem as follows:

$$\|B\|_{2k}^{2k}=\underset{f\in\mathsf{P}_{r,2d}}{\mathbb{E}}|B(f)|^{2k}=\underset{f\in\mathsf{P}_{r,2d}}{\mathbb{E}}\prod_{i\in[k]}B(f)\overline{B(f)}$$

$$=\sum_{\alpha_1,\beta_1,\cdots,\alpha_k,\beta_k\in\Lambda_{n,2d}}\left(\prod_{i\in[k]}\widehat{B}_{\alpha_i}\overline{\widehat{B}_{\beta_i}}\right)\underset{f\in\mathsf{P}_{r,2d}}{\mathbb{E}}\prod_{i\in[k]}\chi_{\alpha_i}(f)\overline{\chi_{\beta_i}(f)}\quad(\text{ from } (2.1)\text{ })$$

$$=\sum_{\substack{\alpha_1,\beta_1,\cdots,\alpha_k,\beta_k\in\Lambda_{r,2d}\\\sum_i\alpha_i-\beta_i\in\mathsf{P}_{r,2d}^{\perp}}}\prod_{i\in[k]}\widehat{B}_{\alpha_i}\overline{\widehat{B}_{\beta_i}}$$

$$=\sum_{\substack{\alpha_1,\beta_1,\cdots,\alpha_k,\beta_k\in\Lambda_{r,2d}\\\sum_i\alpha_i-\beta_i=0}}\prod_{i\in[k]}\widehat{B}_{\alpha_i}\overline{\widehat{B}_{\beta_i}}\quad(\text{ from } (3.1)\text{ })$$

$$=\sum_{\alpha_1,\beta_1,\cdots,\alpha_k,\beta_k\in\Lambda_{r,2d}}\left(\prod_{i\in[k]}\widehat{B}_{\alpha_i}\overline{\widehat{B}_{\beta_i}}\right)\underset{f\in\mathfrak{F}_r}{\mathbb{E}}\prod_{i\in[k]}\chi_{\alpha_i}(f)\overline{\chi_{\beta_i}(f)}$$

$$=\underset{f\in\mathfrak{F}_r}{\mathbb{E}}\prod_{i\in[k]}B'(f)\overline{B'(f)}=\underset{f\in\mathfrak{F}_r}{\mathbb{E}}|B'(f)|^{2k}=\|B'\|_{2k}^{2k}$$

◀

3.1   Proof of Theorem 1.2

Proof of 1. For $f\in V$, consider the set $\{f,f+1,f+2\}\subseteq V$. These sets form a partition of V and are triangles in the graph. Hence $\delta\le1/3$. ◀

Proof of 2. Let $A:\mathsf{P}_{r,2d}\to\{0,1\}$ be the indicator set of the independent set of size $\delta|V|$. By Parseval's equation (2.2) and the fact that $\widehat{A}_0=\delta$, we have that

$$\sum_{\alpha\in\Lambda_{r,2d}\setminus\{0\}}|\widehat{A}_\alpha|^2=\delta-\delta^2.\tag{3.2}$$

Since A is an independent set,

$$\underset{p\in\mathsf{P}_{r,d},a\in\mathbb{F}_3,f\in\mathsf{P}_{r,2d}}{\mathbb{E}}A(f)A(f+a(p^2+1))=\sum_{\alpha\in\Lambda_{r,2d}}|\widehat{A}_\alpha|^2\underset{p\in\mathsf{P}_{r,d},a\in\mathbb{F}_3}{\mathbb{E}}\chi_\alpha(a(p^2+1))=0.$$

Taking the real parts of the equation on both sides and rearranging, we get

$$\sum_{\alpha\in\Lambda_{r,2d}\setminus\{0\}}|\widehat{A}_\alpha|^2\mathsf{Re}\left(\underset{p\in\mathsf{P}_{r,d}}{\mathbb{E}}\chi_\alpha(p^2+1)\right)=-\delta^2.\tag{3.3}$$

Let T be a random variable such that $\Pr[T=\alpha]=|\widehat{A}_\alpha|^2/(\delta-\delta^2)$ and X be the random variable $X(T)=\mathsf{Re}\left(\mathbb{E}_{p\in\mathsf{P}_{r,d},a\in\mathbb{F}_3}\chi_\alpha(a(p^2+1))\right)$. From (3.2) and (3.3), we have that

$$\mathbb{E}\,X=\frac{-\delta}{1-\delta}.$$

Since p is a random degree d polynomial, it is $3^{d/2}$-wise independent from Lemma 2.3. So if

$|T| \leq 3^{d/2}$ then

$$\left| \mathsf{Re} \left(\mathop{\mathbb{E}}_{p \in \mathsf{P}_{r,d}, a \in \mathbb{F}_3} \chi_\alpha(a(p^2 + 1)) \right) \right|$$

$$= \left| \frac{1}{2} \mathsf{Re} \left(\left(\frac{\omega^2 - 1}{3} \right)^{|\alpha|_1} \left(\frac{\omega - 1}{3} \right)^{|\alpha|_2} + \left(\frac{\omega - 1}{3} \right)^{|\alpha|_1} \left(\frac{\omega^2 - 1}{3} \right)^{|\alpha|_2} \right) \right| \leq \left(\frac{1}{\sqrt{3}} \right)^{|\alpha|}$$

where $|\alpha|_a = \{x \in \mathbb{F}_3^r : \alpha(x) = a\}$.

If $|T| > 3^{d/2}$, we know from Lemma 2.8 that $|X(T)| \leq 3^{-\Omega(3^{d/9})}$.

Note that for T with $|T| = 1$, $X(T) = -1/2$. For T with $|T| = 2$, $X(T) \geq 0$. For T with $|T| \geq 3$, $X(T) \geq \frac{-1}{3\sqrt{3}}$. So if $\mathbb{E} X = -1/2$ then $\Pr[|T| = 1] = 1$. So A is a Boolean valued function with non zero Fourier coefficients of supports only 0 and 1. Using arguments similar to Proof of [1, Lemma 2.3], it can be shown that there is an $x \in \mathbb{F}_3^r$ such that $A(f)$ only depends on $f(x)$.

◀

Proof of 3. Suppose $\delta = 1/3 - \varepsilon$. First we show that most of Fourier weights are concentrated in the first two levels

▶ **Lemma 3.7.**

$$\sum_{\alpha \in \Lambda_{r,2d}: |\alpha| > 1} |\widehat{A}_\alpha|^2 \leq 2\varepsilon$$

Proof. Consider the random variables X and T defined in the Proof of 2. Since $\delta = 1/3 - \varepsilon$ and since $\varepsilon < 1/3$, $\mathbb{E} X = -1/2 + \varepsilon$. Let Y be the random variable $X + 1/2$. Note that $Y \geq 0$ and when $Y > 0$, $Y \geq 1/6$. Therefore by Markov, $\Pr[Y > 0] \leq 6\varepsilon$ and

$$\sum_{\alpha \in \Lambda_{r,2d}: |\alpha| > 1} |\widehat{A}_\alpha|^2 \leq (\delta - \delta^2) \Pr[Y > 0] \leq 2\varepsilon.$$

◀

Then we use Lemma 3.3 to obtain the result.

◀

4   Derandomized Majority is Stablest

In this section, we prove Theorem 1.4. The graphs described in Theorem 1.4 and Theorem 1.3 can be viewed as Cayley graphs on a suitable group. For the proof, we will need bounds on the eigenvalues of these Cayley graphs. For a group G, $\mathbb{R}^G$ denotes the vector space of real valued functions on G.

▶ **Definition 4.1** (Cayley Operator). For a group G with operation $+$, an operator $M : \mathbb{R}^G \to \mathbb{R}^G$ is a *Cayley operator* if there is a distribution μ on G such that for any function $A : G \to \mathbb{R}$,

$$(MA)(f) = \mathop{\mathbb{E}}_{\eta \in \mu} A(f + \eta).$$

It is easy to see that a character $\chi : G \to \mathbb{C}$ is an eigenvector of M with eigenvalue $\mathbb{E}_{\eta \in \mu} \chi(\eta)$.

▶ **Definition 4.2.** We define the following Cayley operators:
1. For the group $\mathbb{F}_3$, let $T : \mathbb{R}^{\mathbb{F}_3} \to \mathbb{R}^{\mathbb{F}_3}$ be the Cayley operator corresponding to the distribution μ that is uniform on $\mathbb{F}_3 \setminus \{0\}$. Let λ be the second largest eigenvalue in absolute value of T.

2. For the group $\mathfrak{F}_r$, let $T_r : \mathbb{R}^{\mathfrak{F}_r} \to \mathbb{R}^{\mathfrak{F}_r}$ be the Cayley operator corresponding to the distribution μ_r that is uniform on $\{f \in \mathfrak{F}_r : f^{-1}(0) = \emptyset\}$. Let $\lambda_r(\alpha)$ be the eigenvalue of T_r corresponding to the eigenvector χ_α, for $\alpha \in \mathfrak{F}_r$.

3. For the group $\mathsf{P}_{r,2d}$, let $T_{r,d} : \mathbb{R}^{\mathsf{P}_{r,2d}} \to \mathbb{R}^{\mathsf{P}_{r,2d}}$ be the Cayley operator corresponding to the distribution $\mu_{r,2d}$ of choosing a uniformly random element in $\{p^2 + 1, -p^2 - 1\}$ where $p \in \mathsf{P}_{r,2d}$ is chosen uniformly at random. Let $\lambda_{r,d}(\alpha)$ be the eigenvalue of $T_{r,d}$ corresponding to χ_α, for $\alpha \in \mathfrak{F}_r$.

4. For the group $\mathsf{P}_{r,2d}$, let $S_{r,d} : \mathbb{R}^{\mathsf{P}_{r,2d}} \to \mathbb{R}^{\mathsf{P}_{r,2d}}$ be the Cayley operator corresponding to the distribution of $a \cdot \prod_{i=1}^{d}(\ell_i - 1)(\ell_i - 2)$ where $\ell_1, \cdots, \ell_d$ are linearly independent degree 1 polynomials chosen uniformly at random and a is randomly chosen from $\mathbb{F}_3$. Let $\rho_{r,d}(\alpha)$ be the eigenvalue of $S_{r,d}$ corresponding to χ_α, for $\alpha \in \mathfrak{F}_r$.

Now we will list some known bounds of the eigenvalues of the above operators. It is easy to see that λ is a constant < 1. Since $\mathbb{F}_3^R$ can be identified with $\mathfrak{F}_r$, $T^{\otimes R}$ can be identified with T_r. Hence we have the following lemma.

▶ **Lemma 4.3.**
$$|\lambda_r(\alpha)| \le |\lambda|^{|\alpha|}.$$

▶ **Lemma 4.4.** *For $\alpha \in \Lambda_{r,2d}$,*

$$|\lambda_{r,d}(\alpha)| \begin{cases} = |\lambda_r(\alpha)| & \text{if } |\alpha| \le 3^{d/2} \\ \le 3^{-3^{C_1 d}} & \text{otherwise.} \end{cases} \tag{4.1}$$

Proof. The first case follows from the fact that a random element η according to $\mu_{r,2d}$ (the distribution that defines $T_{r,d}$) is $3^{d/2}$-wise independent (see Lemma 2.3) as a string of length 3^r over alphabet $\mathbb{F}_3$. The latter case follows from Lemma 2.8. ◀

We will derive bounds on the eigenvalues of $S_{r,d}$ using the results of Haramaty *et al.* [10]. Haramaty *et al.* analyses the following test for checking whether a polynomial is of degree $2r - 2d - 1$: Choose a random affine subspace S of dimension $r - d$ and check if the polynomial is of degree $2r - 2d - 1$ on S. Note that for any $\alpha \in \mathsf{P}_{r,2r-2d-1}$ and subspace S of dimension $r - d$, $\sum_{x \in S} \alpha(x) = 0$. Hence this test is equivalent to choosing $\ell_1, \cdots \ell_d \in \mathsf{P}_{r,1}$ that are linearly independent and checking whether $\langle \alpha, \prod_{i=1}^{d}(\ell_i - 1)(\ell_i - 2) \rangle \ne 0$. Haramaty *et al.* proved the following lemma.

▶ **Lemma 4.5.** *There exists constants C_1, C_2 such that*

$$\Pr_{\ell_i}\left[\langle \alpha, \prod_{i=1}^{d}(\ell_i - 1)(\ell_i - 2) \rangle = 0\right] \le \max\left\{1 - \frac{C_1 \Delta(\alpha, \mathsf{P}_{r,2r-2d-1})}{3^d}, C_2\right\}$$

where $\ell_1, \cdots, \ell_i \in \mathsf{P}_{r,1}$ are random linearly independent polynomials.

▶ **Lemma 4.6.** *There exists constants C_1', C_2' such that, for $\alpha \in \Lambda_{r,2d}$,*

$$1 - \frac{2|\alpha|}{3^d} \le |\rho_{r,d}(\alpha)| \le \max\left\{1 - \frac{C_1' \Delta(\alpha, \mathsf{P}_{r,2r-2d-1})}{3^d}, C_2'\right\} \tag{4.2}$$

Proof. First we will prove the lower bound. By definition

$$\rho_{r,d}(\alpha) = \mathbb{E}_{\ell_i,a}\, \omega^{a \cdot \sum_x \alpha(x) \prod_{i=1}^{d}(\ell_i(x)-1)(\ell_i(x)-2)}.$$

For any x in support of α, the probability that $\prod_{i=1}^{d}(\ell_i(x)-1)(\ell_i(x)-2) \neq 0$ is $1/3^d$. Hence by union bound, $\prod_{i=1}^{d}(\ell_i(x)-1)(\ell_i(x)-2) = 0$ for every x in support of α with probability $1 - |\alpha|/3^d$ and when this happens the expectation is 1. Also note that the quantity inside the expectation has absolute value 1.

For proving the upper bound we will use Lemma 4.5. Let p_{acc} be the probability mentioned in Lemma 4.5. Then

$$\rho_{r,d}(\alpha) = \underset{\ell_i,a}{\mathbb{E}}\, \omega^{a\langle \alpha, \prod_{i=1}^{d}(\ell_i-1)(\ell_i-2)\rangle} = p_{\mathsf{acc}} + \frac{1 - p_{\mathsf{acc}}}{2}(\omega + \omega^2) = \frac{3}{2}p_{\mathsf{acc}} - \frac{1}{2}.$$

From the above equation and Lemma 4.5, the constants C_1', C_2' can be obtained. ◄

▶ **Lemma 4.7.** *For $A, B : \mathsf{P}_{r,2d} \to [0,1]$, let $A' := S_{r,d}^t A$ and similarly define B'. Then*

$$|\langle A, T_{r,d}B\rangle - \langle A', T_{r,d}B'\rangle| \leq 2dt/3^d$$

Proof.

$$
\begin{aligned}
|\langle A, T_{r,d}B\rangle - \langle A', T_{r,d}B'\rangle| &\leq |\langle A, T_{r,d}B\rangle - \langle A, T_{r,d}B'\rangle| \\
&\quad + |\langle A, T_{r,d}B'\rangle - \langle A', T_{r,d}B'\rangle| \\
&= \left|\langle A - \mathbb{E}\,A, T_{r,d}(1 - S_{r,d}^t)(B - \mathbb{E}\,B)\rangle\right| \\
&\quad + \left|\langle T_{r,d}(1 - S_{r,d}^t)(A - \mathbb{E}\,A), B' - \mathbb{E}\,B'\rangle\right| \\
&\leq \|T_{r,d}(1 - S_{r,d}^t)(B - \mathbb{E}\,B)\| + \|T_{r,d}(1 - S_{r,d}^t)(A - \mathbb{E}\,A)\| \\
&\leq 2td/3^d
\end{aligned}
$$

The last step follows from the fact that the operators $T_{r,d}, (1 - S_{r,d}^t)$ have the same set of eigenvectors and the largest eigenvalue in absolute value of $T_{r,d}(1 - S_{r,d}^t)$ is $2td/3^d$ from Lemma 4.4 and Lemma 4.6. ◄

Theorem 1.4 will follow from the following lemma.

▶ **Lemma 4.8.** $\forall \varepsilon > 0, \exists k = O(1/\varepsilon^2), d = O(\log(1/\varepsilon))$ *such that the following holds: if* $A, B : \mathsf{P}_{r,2d} \to [0,1]$ *then* $\exists \mathcal{A}, \mathcal{B} : \mathfrak{F}_r \to [0,1]$ *such that*
1. $|\mathbb{E}\,A - \mathbb{E}\,\mathcal{A}|, |\mathbb{E}\,B - \mathbb{E}\,\mathcal{B}| \leq \varepsilon$,
2. *For all $x \in \mathbb{F}_3^r, k' \leq k$,*
$$\mathrm{Inf}_{\overline{x}}^{\leq k'}(\mathcal{A}) \leq \mathrm{Inf}_{\overline{x}}^{\leq k'}(A) + \varepsilon$$
$$\mathrm{Inf}_{\overline{x}}^{\leq k'}(\mathcal{B}) \leq \mathrm{Inf}_{\overline{x}}^{\leq k'}(B) + \varepsilon$$

3. $|\langle A, T_{r,d}B\rangle - \langle \mathcal{A}, T_r\mathcal{B}\rangle| \leq \varepsilon$.

Proof of Theorem 1.4. We will show that if Theorem 1.4 is false then Theorem 1.3 is also false. First using Lemma 4.8 with parameter $\varepsilon = \mu^{O(1)}$, we obtain functions $\mathcal{A}, \mathcal{B} : \mathfrak{F}_r \to [0,1]$ such that
1. $\mathbb{E}\,\mathcal{A}, \mathbb{E}\,\mathcal{B} \geq \mu - \varepsilon$,
2. For all $x \in \mathbb{F}_2^r, k' \leq k$,

$$\mathrm{Inf}_{\overline{x}}^{\leq k'}(\mathcal{A}) \leq \delta + \varepsilon \quad \text{and} \quad \mathrm{Inf}_{\overline{x}}^{\leq k'}(\mathcal{B}) \leq \delta + \varepsilon$$

3. $|\langle \mathcal{A}, T_r\mathcal{B}\rangle| \leq |\langle A, T_{r,d}B\rangle| + \varepsilon$.

Now applying Theorem 1.3 to the functions $\mathcal{A}, \mathcal{B}$, we obtain that $|\langle \mathcal{A}, T_r\mathcal{B}\rangle| \geq \delta'$, where $\delta' = \mu^{O(1)}$. Hence $|\langle A, T_{r,d}B\rangle| \geq \delta' - \varepsilon$, and we set the parameters $\delta = \delta' - \varepsilon, d = O(\log 1/\mu)$ and $k = O(\log 1/\mu)$.

◄

4.1   Proof of Lemma 4.8

For proving Lemma 4.8, crucially use the following lemma by Kane and Meka [11].

▶ **Lemma 4.9.** *Let* $\xi : \mathbb{R} \to \mathbb{R}_+$ *be the function* $\xi(x) := (\max\{-x, x-1, 0\})^2$. *For any parameters* $k \in \mathbb{N}$ *and* $\varepsilon \in (0, 1)$, *there is a* $d = O(\log(k/\varepsilon))$ *such that the following holds: If the polynomial* $P : \mathfrak{F}_r \to \mathbb{R}$ *satisfies* $\|P\| \leq 1$ *and* $\widehat{P}(\alpha) = 0$ *for* $\alpha \in \Lambda_{r,d}$ *such that* $|\alpha| > k$, *then*

$$\left| \mathop{\mathbb{E}}_{f \in \mathfrak{F}_r} \xi(P(f)) - \mathop{\mathbb{E}}_{f \in \mathsf{P}_{r,d}} \xi(P(f)) \right| \leq \varepsilon.$$

▶ **Remark.** For proving Lemma 4.9, a generalization of [11, Lemma 4.1] to polynomials of the form $P : \{1, \omega, \omega^2\}^R \to \mathbb{R}$ is required. However, we observe that the polynomials we consider are real-valued $P : \mathfrak{F}_r \to \mathbb{R}$ and hence satisfy $\widehat{P}(\alpha) = \overline{P}(-\alpha)$.

Using this observation, the proof of [11, Lemma 4.1] generalizes to our setting (the above property is preserved throughout the proof). The result of [11] also requires an earlier result of Diakonikolas, Gopalana, Jaiswal, Servedio, and Viola [4] on fooling Linear Threshold functions (LTFs) with sample spaces of bounded independence. This proof also goes through for Thresholds of real-valued linear functions defined on variables that are uniformly distributed in $\{1, \omega, \omega^2\}$. [2]

Proof of Lemma 4.8. Let $t = \frac{3^d \log(10/\varepsilon)}{2k}$, and $A_1 = S_{r,d}^t A, B_1 = S_{r,d}^t B$. Then from Lemma 4.7

$$|\langle A, T_{r,d}B \rangle - \langle A_1, T_{r,d}B_1 \rangle| \leq 2dt/3^d \tag{4.3}$$

and similarly for B_1. Let k be a number $< 3^{d/2}$ and $A_2 = \mathsf{Re}(A_1^{\leq k})$. Using the fact that A_1 is real valued,

$$\|A_1 - A_2\| \leq \|A_1 - A_1^{\leq k}\| \leq (1 - 2k/3^d)^t \leq e^{-2tk/3^d} = \varepsilon/10 \tag{4.4}$$

Let $A_3 : \mathfrak{F}_r \to \mathbb{R}$ be defined as $A_3 := \mathsf{Re}((A_1^{\leq k})')$ where $(A_1^{\leq k})'$ is the lift of $A_1^{\leq k}$. Since a random degree d polynomial is $3^{d/2}$-wise independent,

$$\langle A_2, T_{r,d}B_2 \rangle = \langle A_3, T_r B_3 \rangle \tag{4.5}$$

Note that A_3 may not be a $[0, 1]$-valued function. But since A is $[0, 1]$-valued, so is A_1. Let $\xi : \mathbb{R} \to \mathbb{R}_+$ be the function $\xi(x) := (\max\{-x, x-1, 0\})^2$. Notice that $\mathbb{E}_f \xi \circ A(f)$ gives the ℓ_2^2 distance of A from $[0, 1]$-valued functions. Using Lemma 4.9, for $d = O(\log(k/\varepsilon))$,

$$\left| \mathop{\mathbb{E}}_{f \in \mathsf{P}_{r,2d}} \xi(A_2(f)) - \mathop{\mathbb{E}}_{f \in \mathfrak{F}_r} \xi(A_3(f)) \right| \leq \varepsilon/10 \tag{4.6}$$

and similarly for B_2. Hence there exists functions $\mathcal{A}, \mathcal{B} : \mathfrak{F}_r \to [0, 1]$ such that
1. $|\mathbb{E}\, A - \mathbb{E}\, \mathcal{A}| \leq \|A_1' - \mathcal{A}\| \leq \varepsilon$ (similarly for B),
2. For all $x \in \mathbb{F}_3^r, k' \leq k$, $\mathsf{Inf}_x^{\leq k'}(\mathcal{A}) \leq \mathsf{Inf}_x^{\leq k'}(A) + \varepsilon$ (similarly for B),
3. $|\langle A, T_{r,d}B \rangle - \langle \mathcal{A}, T_r \mathcal{B} \rangle| \leq \varepsilon$.

◀

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[2] Such a function is the sign of a "linear polynomial" of the form $\left(\sum_{i=1}^R \alpha_i x_i + \overline{\alpha_i x_i}\right) - \theta$ for $\theta \in \mathbb{R}$.

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