The Complexity of Constraint Satisfaction Problems*

Manuel Bodirsky

Institut für Algebra, Technische Universität Dresden, Germany
Manuel.Bodirsky@tu-dresden.de

Abstract

The *tractability conjecture* for constraint satisfaction problems (CSPs) describes the constraint languages over a finite domain whose CSP can be solved in polynomial-time. The precise formulation of the conjecture uses basic notions from universal algebra. In this talk, we give a short introduction to the universal-algebraic approach to the study of the complexity of CSPs. Finally, we discuss attempts to generalise the tractability conjecture to large classes of constraint languages over infinite domains, in particular for constraint languages that arise in qualitative temporal and spatial reasoning.

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1 The Constraint Satisfaction Problem

Constraint satisfaction problems are computational problems that can be formalised in several equivalent ways. A mathematically convenient way is to view CSPs as structural homomorphism problems, as follows. Fix a structure Γ with a finite relational signature τ. The domain of Γ need not be finite for the following computational problem to be well-defined.

▶ **Definition 1** (CSP(Γ)). The constraint satisfaction problem for Γ, denoted by CSP(Γ), is the computational problem to decide for a given *finite* τ-structure A whether there exists a homomorphism to Γ.

The fixed structure Γ is often referred to as the *constraint language* of the constraint satisfaction problem, since we choose from the relations in Γ to formulate our constraints in the input structure A. We give some concrete examples of CSPs.

1. Graph n-colorability can be formulated as CSP(K_n) where K_n is the complete loopless graph on n vertices.
2. The question whether a given finite digraph is acyclic, i.e., does not contain a directed cycle, can be formulated as CSP($\mathbb{Q}; <$).
3. The question whether a given directed graph has a vertex bipartition such that both parts are acyclic can be formulated as CSP($\mathbb{N}; E$) where

$$E := \{(a, b) \in \mathbb{N}^2 \mid a < b \text{ or } (a - b) \text{ is odd}\}.$$

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4. $\mathrm{CSP}(\mathbb{R}; \leq, A, O)$ for $A := \{(a,b,c) \in \mathbb{R}^3 \mid a + b = c\}$ and $O := \{1\}$ is essentially the feasibility problem for linear programs (see [5]).

The list can be prolonged easily, and contains a variety of problems that appeared in the literature throughout theoretical computer science.

There is a great amount of work about the computational complexity of $\mathrm{CSP}(\Gamma)$ when Γ is a finite structure (i.e., has a finite domain), stimulated by the following dichotomy conjecture.

▶ **Conjecture 1** (Feder and Vardi [18]). *Let Γ be a finite structure with a finite relational signature. Then $\mathrm{CSP}(\Gamma)$ is in P or NP-complete.*

2 The Universal-Algebraic Approach

The central notion for the universal algebraic approach is the notion of a *polymorphism* of a constraint language Γ. A polymorphism of Γ is a homomorphism h from finite powers of Γ into Γ. In other words, when h has arity k, then we require for all relations R of Γ and $(a^1_1, \dots, a^1_n) \in R, \dots, (a^k_1, \dots, a^k_n) \in R$ that $(h(a^1_1, \dots, a^k_1), \dots, h(a^1_n, \dots, a^k_n)) \in R$. Unary polymorphisms are also known as *endomorphisms*. Thus, polymorphisms generalise endomorphisms, and endomorphisms generalise automorphisms of Γ. We write $\mathrm{Pol}(\Gamma)$ for the set of all polymorphisms of Γ, and $\mathrm{Aut}(\Gamma)$ for the set of all polymorphisms of Γ.

The following result for structures with a finite domain, which relies on a fundamental theorem in universal algebra [19, 16], hints at the relevance of polymorphisms for CSPs.

▶ **Theorem 2** ([23]). *Let Γ_1 and Γ_2 be finite structures with the same domain and finite relational signatures such that $\mathrm{Pol}(\Gamma_1) \subseteq \mathrm{Pol}(\Gamma_2)$. Then there is a deterministic linear-time many-one reduction from $\mathrm{CSP}(\Gamma_2)$ to $\mathrm{CSP}(\Gamma_1)$.*

Theorem 2 has an important advancement, Theorem 3 below, which is particularly important when we want to reduce between CSPs where the constraint languages have different domains. Let us first mention that the set $\mathrm{Pol}(\Gamma)$ is a *function clone*. A function clone is a set $\mathcal{S}$ of functions of finite arity that

- is closed under *composition*: for k-ary $g \in \mathcal{S}$ and l-ary $f_1, \dots, f_k \in \mathcal{S}$ the l-ary function $g(f_1, \dots, f_k)$ given by $(x_1, \dots, x_l) \mapsto g(f_1(x_1, \dots, x_l), \dots, f_k(x_1, \dots, x_l))$ is also in $\mathcal{S}$, and
- contains the *projections* π^k_i given by $(x_1, \dots, x_k) \mapsto x_i$.

A map $\xi\colon \mathrm{Pol}(\Gamma_1) \to \mathrm{Pol}(\Gamma_2)$ is called a *clone homomorphism* if for all $g, f_1, \dots, f_k \in \mathrm{Pol}(\Gamma_1)$

$$\xi(g(f_1, \dots, f_k)) = \xi(g)(\xi(f_1), \dots, \xi(f_k))$$

and $\xi(\pi^k_i) = \pi^k_i$ for all $1 \leq i \leq k$. A *clone isomorphism* is a bijective clone homomorphism.

▶ **Theorem 3.** *Suppose that Γ_1 and Γ_2 are finite structures with finite relational signature such that there exists a clone isomorphism between $\mathrm{Pol}(\Gamma_1)$ to $\mathrm{Pol}(\Gamma_2)$. Then $\mathrm{CSP}(\Gamma_1)$ and $\mathrm{CSP}(\Gamma_2)$ are equivalent under deterministic linear-time many-one reductions.*

3 The Finite Domain Tractability Conjecture

Theorem 3 from the previous section tells us that the computational complexity of $\mathrm{CSP}(\Gamma)$ is coded into the equations that hold on the polymorphisms. We even have a candidate equation that might characterise the CSPs in P.

▶ **Theorem 4** ([17, 28, 21]). *Suppose that Γ is a finite structure. Then Γ has a* Taylor[1] *polymorphism f, that is, when f has arity n then it satisfies for every $i \leq n$ an equation of the form*

$$\forall x, y.\, f(x_1, \dots, x_n) = f(y_1, \dots, y_1)\ ,$$

where $x_1, \dots, x_n, y_1, \dots, y_n \in \{x, y\}$ and $x_i \neq y_i$, or there is a structure Γ' obtained from Γ by dropping all but finitely many relations such that $\mathrm{CSP}(\Gamma')$ is NP-complete.

The condition given in Theorem 4 has been improved recently: the existence of a Taylor polymorphism is equivalent to the existence of an operation that satisfies an equation that is must easier to grasp.

▶ **Theorem 5** ([1]). *A finite structure Γ has a Taylor polymorphism if and only if it has a* cyclic *polymorphism f, that is, f has arity $n \geq 2$ and satisfies*

$$\forall x_1, \dots, x_n.\, f(x_1, \dots, x_n) = f(x_2, \dots, x_n, x_1)\ .$$

The following conjecture has been made in different form by Bulatov, Jeavons, and Krokhin [17]; the formulation given below is equivalent by well-known facts. The conjecture complements Theorem 4, and its truth would settle the dichotomy conjecture.

▶ **Conjecture 2** (Tractability Conjecture). *Let Γ be a finite structure with finite relational signature and a Taylor (or, equivalently, cyclic) polymorphism. Then $\mathrm{CSP}(\Gamma)$ is in P.*

4 Infinite Domains

The universal-algebraic approach can be generalised to constraint languages Γ over infinite domains. This generalisation is most straightforward when the automorphism group of Γ is large, in the following sense.

▶ **Definition 6.** A permutation group G on a set X is called *oligomorphic* if the componentwise action of G on X^n has only finitely many orbits, for all $n \in \mathbb{N}$.

An example of an oligomorphic permutation group is the automorphism group of $(\mathbb{Q}; <)$. Countable structures Γ with an oligomorphic permutation group are well-known to model-theorists: by a theorem independently due to Ryll-Nardzewski, Engeler, and Svenonius (see, e.g., [22]), these are precisely the countable structures that are ω-categorical, that is, Γ has the property that all countable models of the first-order theory of Γ are isomorphic to Γ.

A versatile method to construct ω-categorical structures is via Fraïssé-limits, and taking reducts, which we briefly recall here. We need the standard notion of homogeneity (sometimes called ultrahomogeneity) from model theory. A structure Γ is called *homogeneous* if all isomorphisms between finite substructures can be extended to automorphisms of Γ. Homogeneous structures with finite relational signature are ω-categorical [22]. Homogeneous structures are uniquely given by their *age*, which is the class of finite structures that embed into them. The age of a homogeneous structure must have the amalgamation property (we again refer to [22]), and every amalgamation class $\mathcal{C}$ gives rise to a homogeneous structure of age $\mathcal{C}$. The fundamental model theory of homogeneous structures goes back to Fraïssé, and hence the unique homogeneous structure for a given amalgamation class is called the *Fraïssé-limit* of this class.

[1] Note that, contrary to what can often be found in the literature, in our definition of Taylor operations, we do not insist on idempotency of f.

A *reduct* of a structure Δ is a structure Γ on the same domain such that all relations of Γ are first-order definable (without parameters) in Δ. For example, the structure $(\mathbb{Q}; \mathrm{Betw})$ where $\mathrm{Betw} := \{(x, y, z) \mid x < y < z \vee z < y < x\}$ (the so-called *Betweenness relation*) is a reduct of $(\mathbb{Q}; <)$. Reducts of homogeneous structures need not be homogeneous, but reducts of ω-categorical structures remain ω-categorical.

When Γ is ω-categorical, then the complexity of Γ is still coded into the polymorphisms.

▶ **Theorem 7** ([8]). *Let Γ_1 and Γ_2 be ω-categorical structures with the same domain and finite relational signatures such that $\mathrm{Pol}(\Gamma_1) = \mathrm{Pol}(\Gamma_2)$. Then Γ_1 and Γ_2 are equivalent under deterministic linear-time many-one reductions.*

An example of a permutation group which is *not* oligomorphic is the automorphism group of the structure $(\mathbb{N}; E)$ discussed in the introduction: it has infinitely many orbits in its componentwise action on $\mathbb{N}^2$. However, in this case it is easy to come up with a structure that has precisely the same CSP, but whose automorphism group *is* oligomorphic: let Q_1, Q_2 be a partition of $\mathbb{Q}$ such that both Q_1 and Q_2 are dense in $\mathbb{Q}$, and consider the structure $(\mathbb{Q}; E')$ where

$$E' := \{(a, b) \in \mathbb{Q}^2 \mid a < b \text{ or } a \in Q_1 \Leftrightarrow b \in Q_2\} .$$

This is a frequent phenomenon: many computational problems in temporal and spatial reasoning can be formulated as CSPs, but often some extra care is needed to show that they can be formulated with ω-categorical constraint languages. A necessary and sufficient Myhill-Nerode-type condition that characterises the CSPs that can be formulated with an ω-categorical constraint language can be found in [3]. An example of a structure that does not satisfy the mentioned Myhill-Nerode-type condition of Example 4, in the introduction. Hence, $\mathrm{CSP}(\mathbb{R}; A, O)$ (which is essentially the feasibility problem for linear programs) cannot be formulated as $\mathrm{CSP}(\Gamma)$ with an ω-categorical constraint language.

We do not know whether Theorem 3 remains valid for ω-categorical structures Γ, that is, whether the isomorphism type of the polymorphism clone of Γ determines the complexity of $\mathrm{CSP}(\Gamma)$. However, the theorem can be rescued by a slight modification.

▶ **Theorem 8** ([12]). *Suppose that Γ_1 and Γ_2 are ω-categorical structures with finite relational signature such that there exists a clone isomorphism between $\mathrm{Pol}(\Gamma_1)$ and $\mathrm{Pol}(\Gamma_2)$ which is also a* homeomorphism*. Then $\mathrm{CSP}(\Gamma_1)$ and $\mathrm{CSP}(\Gamma_2)$ are equivalent under deterministic linear-time many-one reductions.*

The homeomorphicity requirement in Theorem 8 is with respect to the *topology of pointwise convergence* on the space of all functions of finite arity, which is defined as follows. For elements $a, b_1, \dots, b_k$ of the domain D, define $F_{a,b_1,\dots,b_k} := \{f \mid f(b_1, \dots, b_k) = a\}$. Then the topology of pointwise convergence is the smallest topology where the open sets include $\{F_{a,b_1,\dots,b_k} \mid k \in \mathbb{N}, a, b_1, \dots, b_k \in D\}$. It is a basic fact that a clone $\mathcal{C}$ is closed in this space, $\overline{\mathcal{C}} = \mathcal{C}$, if and only if it is the polymorphism clone of a structure.

5 A general tractability conjecture

Cyclic polymorphisms do not characterise the tractability of the CSP for ω-categorical structures: a simple counterexample is the structure $(\mathbb{N}; \neq, I_4)$ where I_4 is the quaternary relation defined as $I_4 := \{(a, b, c, d) \in \mathbb{N}^4 \mid a = b \Rightarrow c = d\}$. The automorphism group of this structure is the set of all permutations of $\mathbb{N}$, which is clearly oligomorphic. The polymorphisms of this structure are precisely all functions that are composed from injective functions and projections. Hence, the clone does not contain cyclic operations. But $\mathrm{CSP}(\mathbb{N}; \neq, I_4)$ is easily seen to be in P; see [6].

The structure $(\mathbb{N}; \neq, I_4)$ has polymorphisms that are almost as good as cyclic operations: every binary injective operation f will be a polymorphism, and we can always pick an injection i from $\mathbb{N} \to \mathbb{N}$ such that the following holds:

$$\forall x_1, x_2.\, f(x_1, x_2) = i(f(x_2, x_1)) \ .$$

We also have to describe an obstruction to general algorithmic results for the class of all ω-categorical structures. Henson [20] constructed uncountably many homogeneous directed graphs Γ, and all of these directed graphs have distinct CSPs. Since there are only countably many algorithms, there must be directed graphs in this class with an undecidable CSP. There are also CSPs of various intermediate complexities [2]. All of Henson's digraphs have a binary polymorphism f and endomorphisms e_1, e_2 satisfying

$$\forall x_1, x_2.\, e_1(f(x_1, x_2)) = e_2(f(x_2, x_1)) \ ,$$

that is, from a universal-algebraic perspective, they all 'look like easy CSPs', but they are not.

Henson's directed homogeneous graphs are based on forbidding *infinite* families of finite structures. On the other hand, the ω-categorical structures that appear 'in nature' (either in mathematics or to formulate computational problems as CSPs) can typically be described by forbidding only finitely many finite structures. More formally, we say that a homogeneous structure Γ is *finitely bounded* if there exists a finite set $\mathcal{F}$ of finite structures such that the age of Γ is given as the class of all finite structures that do not embed any of the structures from $\mathcal{F}$. We now generalise the tractability conjecture by modifying the idea of Taylor polymorphisms so that it involves outside applications of endomorphisms, as follows.

▶ **Conjecture 3.** *Let Γ be the reduct of a finitely bounded homogeneous structure. If Γ has a polymorphism f of arity $n \geq 2$ such that for every $i \leq n$ there are endomorphisms e_1, e_2 and $x_1, \ldots, x_n, y_1, \ldots, y_n \in \{x, y\}$ with $x_i \neq y_i$ such that f satisfies*

$$e_1(f(x_1, \ldots, x_n)) = e_2(f(y_1, \ldots, y_n))$$

then $\mathrm{CSP}(\Gamma)$ is in P. Otherwise, $\mathrm{CSP}(\Gamma)$ is NP-complete.

The conjecture has been verified for several classes of ω-categorical structures:
- All reducts of $(\mathbb{Q}; <)$ in [7];
- All reducts of the Random graph (the Fraïssé-limit of the class of all finite graphs) in [10];
- All reducts of the homogeneous equivalence relation with infinitely many infinite classes in [15].

The strongest tool we have for attacking this conjecture will be introduced in the next section.

6 Ramsey Theory

The complexity classification results for ω-categorical structures mentioned in Section 5 rely on results from structural Ramsey theory. We say that a homogeneous structure Γ is *Ramsey* if for all finite substructures A and B of Γ and every colouring of the embeddings of A into Γ with finitely many colours, there exists an embedding $e \colon B \to \Gamma$ such that all embeddings of A into $e(B)$ have the same color. Examples of homogeneous Ramsey structures are

- $(\mathbb{Q}; <)$;
- the *ordered* Random graph, and other generically ordered Fraïssé-limits of so-called *free amalgamation classes* (examples are ordered versions of the Henson digraphs) [27];
- the *convexly ordered* homogeneous binary branching C-relation, and other tree-like structures [25, 26];
- the *lexicographically ordered* vector space over a finite field (see [24]);
- the *lexicographically ordered* atomless Boolean algebra (see [24]).

The fact that a structure is Ramsey can be exploited when analysing its automorphism group, endomorphism monoid, or polymorphism clone. Our usage of Ramsey theory is almost exclusively via the concept of *canonical functions*. For simplicity, we explain this concept for unary functions only; however, the ideas generalize straightforwardly to finitary functions; see [9] for an in-depth introduction to the method of canonical functions. A function $f\colon \Gamma \to \Gamma$ is called *canonical* if for all $\beta \in \mathrm{Aut}(\Gamma)$ we have $f \circ \beta \in \overline{\{\alpha f \mid \alpha \in \mathrm{Aut}(\Gamma)\}}$. When Γ is an ordered Ramsey structure, then an arbitrary function 'looks as a canonical function on large parts of the domain': formally, for every function f over the domain of Γ, there exists a canonical function g in $\overline{\{\alpha f \beta \mid \alpha, \beta \in \mathrm{Aut}(\Gamma)\}}$ – the *canonisation lemma*. In practice, we often use a generalisation of canonisation involving constants – we refer to [9] for details. Suppose now that Γ is homogeneous in a finite relational signature. Then there are only finitely many behaviours of canonical functions, and this is essential to break classification arguments dealing with endomorphisms (and polymorphisms) into finitely many cases. We hope that canonical functions and canonization can be used to reduce Conjecture 3 to Theorem 4 and Conjecture 2.

The method of canonical functions has been used extensively in [7, 13, 4, 10, 9, 15, 11], in two contexts: complexity classification of CSPs and classification of reducts of homogeneous structures.

When is it possible to apply this method to analyse the endomorphisms (and polymorphisms) of $\mathfrak{C}$? We do not need $\mathfrak{C}$ to be Ramsey, it suffices that $\mathfrak{C}$ has a homogeneous expansion with finite relational signature which is Ramsey. The following question is therefore of essential importance.

▶ **Question 1** ([14])**.** *Is it true that every homogeneous structure with finite relational signature has a homogeneous Ramsey expansion with finite relational signature?*

Similar in spirit, we ask the following.

▶ **Question 2** ([14])**.** *Can every ω-categorical structure be expanded to an ω-categorical structure which is Ramsey?*

These questions are closely related to recent research in topological dynamics – we refer to a recent survey article for more on this connection [29]. A positive answer to Question 1 would imply that the method of Ramsey theory and canonical functions can be used to approach the tractability conjecture from Section 5 in general.

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