On-line Coloring between Two Lines*

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Abstract

We study on-line colorings of certain graphs given as intersection graphs of objects "between two lines", i.e., there is a pair of horizontal lines such that each object of the representation is a connected set contained in the strip between the lines and touches both. Some of the graph classes admitting such a representation are permutation graphs (segments), interval graphs (axis-aligned rectangles), trapezoid graphs (trapezoids) and cocomparability graphs (simple curves). We present an on-line algorithm coloring graphs given by convex sets between two lines that uses $O(\omega^3)$ colors on graphs with maximum clique size ω.

In contrast intersection graphs of segments attached to a single line may force any on-line coloring algorithm to use an arbitrary number of colors even when $\omega = 2$.

The *left-of* relation makes the complement of intersection graphs of objects between two lines into a poset. As an aside we discuss the relation of the class $\mathcal{C}$ of posets obtained from convex sets between two lines with some other classes of posets: all 2-dimensional posets and all posets of height 2 are in $\mathcal{C}$ but there is a 3-dimensional poset of height 3 that does not belong to $\mathcal{C}$.

We also show that the on-line coloring problem for curves between two lines is as hard as the on-line chain partition problem for arbitrary posets.

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1 Introduction

In this paper we deal with on-line proper vertex coloring of graphs. In this setting a graph is created vertex by vertex where each new vertex is created with all adjacencies to previously created vertices. An *on-line coloring algorithm* colors each vertex when it is created, immediately and irrevocably, such that adjacent vertices receive distinct colors. In particular, when coloring a vertex an algorithm has no information about future vertices. This means that the color of a vertex depends only on the graph induced by vertices created before. It is convenient to imagine that vertices are created by some adaptive adversary so that the coloring process becomes a game between that adversary and an on-line algorithm.

We are interested in on-line algorithms using a number of colors that is bounded by a function of the chromatic number of the input graph. For general graphs this is too much to

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ask for. Indeed, it is a popular exercise to devise a strategy for adversary forcing any on-line algorithm to use arbitrarily many colors on a forest. However, some restricted graph classes admit competitive on-line coloring algorithms. Examples are P_5-free graphs [13], interval graphs [15] and cocomparability graphs [12]. All of these classes are covered by the main result of Penrice, Kierstead and Trotter in [12] that says that for any tree T with radius 2, the class of graphs that do not contain an induced copy of T can be colored on-line with the number of colors depending only on T and the clique number of the input graph.

We are interested in situations where the on-line graph is presented with a geometric intersection representation. A graph G is an *intersection graph of a family $\mathcal{F}$ of sets* if the vertices of G and the elements of $\mathcal{F}$ are in bijection such that two vertices are adjacent in G if and only if the corresponding sets intersect. For convenience, we identify the intersection graph of the family $\mathcal{F}$ with $\mathcal{F}$ itself. The most important geometric intersection graphs arise from considering compact, arc-connected sets in the Euclidean plane $\mathbb{R}^2$. In the corresponding on-line coloring problem such objects are created one at a time and an on-line coloring algorithm colors each set when it is created in such a way that intersecting sets receive distinct colors. For many geometric objects the on-line coloring problem is still hopeless, e.g., disks and axis-aligned squares (Erlebach and Fiala [8]). Since any intersection graph G of translates of a fixed compact convex set in the plane has a maximum degree bounded by $6\omega(G) - 7$ (Kim, Kostochka and Nakprasit [16]), any on-line algorithm that uses a new color only when it is forced to, colors G with at most $6\omega(G) - 6$ colors.

In this paper we consider the on-line coloring problem of geometric objects *spanned* between two horizontal lines, that is, arc-connected sets that are completely contained in the strip $\mathbf{S}$ between the two lines and have non-empty intersection with each of the lines. Clearly, such a family imposes a partial order on its elements where $x < y$ if x and y are disjoint and x is contained in the left component of $\mathbf{S} \setminus y$. Hence, two sets intersect if and only if they are incomparable in the partial order, i.e., the intersection graph is a cocomparability graph. In particular, $\chi(G) = \omega(G)$ for all such graphs G. Conversely every cocomparability graph has a representation as intersection graph of y-monotone curves between two lines. The usual way to state this result is by saying that cocomparability graphs are function graphs, see [10] or [17]. If the representation is given the cocomparability graph comes with a transitive orientation of the complement. In this setting there is an on-line algorithm that uses $\omega^{O(\log \omega)}$ colors when ω is the clique number of the graph (Bosek and Krawczyk [3], see also [2]). This subexponential function in ω is way smaller than the superexponential function arising from the on-line algorithm for cocomparability graphs from [12]. The best known lower bound for on-line coloring of cocomparability graphs is of order $\Omega(\omega^2)$, see [1]. We present an on-line algorithm that uses only $O(\omega^3)$ colors on convex objects spanned between two lines.

Intersection graphs of convex sets spanned between two lines generalize several well-known graph classes.

- *Permutation graphs* are intersection graphs of segments spanned between two lines and posets admitting such cocomparability graphs are the 2-dimensional posets.
- *Interval graphs* are intersection graphs of axis-aligned rectangles spanned between two horizontal lines.
- *Bounded tolerance graphs* are intersection graphs of parallelograms with two horizontal edges spanned between two horizontal lines. (Bounded tolerance graphs were introduced in [9])
- *Triangle graphs* (a.k.a. PI-graphs) are intersection graphs of triangles with a horizontal side spanned between two horizontal lines. (Triangle graphs were introduced in [5])

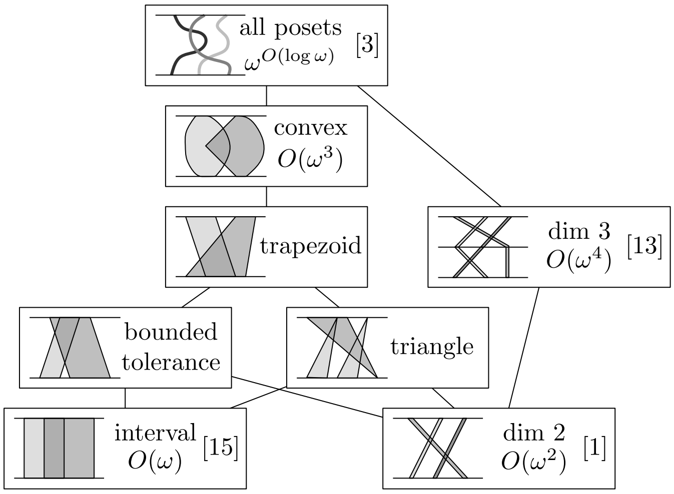

Figure 1 The containment order of some classes of graphs given by objects between two lines together with the performance guarantee for the best known on-line coloring algorithm (given the intersection representation as input).

- *Trapezoid graphs* are intersection graphs of trapezoids with two horizontal edges spanned between two horizontal lines. Posets admitting such cocomparability graphs are the posets of interval-dimension at most 2. (Trapezoid graphs were independently introduced in [5, 6]).

Effective on-line coloring algorithms have been known for some of these classes:
- *Permutation graphs* can be colored on-line with $\binom{\omega+1}{2}$ colors (Schmerl 1979 unpublished, see [1]). Kierstead, McNulty and Trotter [11] generalized Schmerl's idea and gave an on-line algorithm chain partitioning d-dimensional posets, presented with d linear extensions witnessing the dimension, and using $\binom{\omega+1}{2}^{d-1}$ chains, here ω is the width of the poset.
- *Interval graphs* can be colored on-line with $3\omega - 2$ colors (Kierstead and Trotter [15]).

An easy strategy for on-line coloring is given by *First-Fit*, which is the strategy that colors each incoming vertex with the least admissible natural number. While First-Fit uses $O(\omega)$ colors on interval graphs (see [19]) it is easy to trick this strategy and force arbitrary number of colors on permutation graphs of clique-size 2 (see survey [1]). The behavior of the First-Fit algorithm on p-tolerance graphs $(0 < p < 1)$, a subclass of bounded tolerance graphs, was studied in [14]. First-Fit uses there $O(\frac{\omega}{1-p})$ colors.

▶ **Theorem 1.** *There is an on-line algorithm coloring convex sets spanned between two lines with $O(\omega^3)$ colors when ω is the clique number of the intersection graph.*

Note that our on-line coloring algorithm is best known for bounded tolerance graphs, trapezoid and triangle graphs. The best known lower bound $\binom{\omega+1}{2}$ (see [1]) holds already for permutation graphs (segments). Proofs are deferred to later sections.
A poset is called *convex* if its cocomparability graph is an intersection graph of convex sets spanned between two lines. We give a short proof that all height 2 posets are convex. All 2-dimensional posets are convex but not all 3-dimensional.

▶ **Proposition 2.**
1. *Every height 2 poset is convex;*
2. *There is a 3-dimensional height 3 poset that is not convex.*

Rok and Walczak [20] have looked at intersection graphs of connected objects that are attached to a horizontal line and contained in the upper halfplane defined by this line. They show that there is a function f such that $\chi(G) \leqslant f(\omega(G))$ for all G admitting such a representation. However, there is no effective on-line coloring algorithm for graphs in this class, even if we restrict the objects to be segments.

▶ **Proposition 3.** *Any on-line algorithm can be forced to use arbitrarily many colors on a family of segments attached to a line, even if the family contains no three pairwise intersecting segments* ($\omega = 2$).

Recall that it may make a difference for an on-line coloring algorithm whether the input is an abstract cocomparability graph, or the corresponding poset, or a geometric representation. Kierstead, Penrice and Trotter [12] gave an on-line coloring algorithm for cocomparability graphs using a number of colors that is superexponential in ω. Bosek and Krawczyk [3] introduced an on-line coloring algorithm for posets using $\omega^{O(\log \omega)}$ colors where ω is the width of the poset. We show that having a poset represented by y-monotone curves between two lines does not help on-line algorithms. Indeed, such a representation can be constructed on-line if the poset is given.

▶ **Theorem 4.** *There is an on-line algorithm that for any poset draws y-monotone curves spanned between two lines such that $x < y$ in the poset if and only if the curves x and y are disjoint and x lies left of y. That means, for every element of the poset when it is created a curve is drawn in such a way that throughout the set of already drawn curves forms a representation of the current poset.*

Theorem 1 and Proposition 2 are proven in Section 2. Actually we define the class of quasi-convex posets and show the $O(\omega^3)$ bound for this class. Since every convex poset is quasi-convex this implies Theorem 1. The section is concluded with a proposition showing that the class of quasi-convex posets is a proper superclass of convex posets. In Section 3 we discuss general connected sets between two lines. In this context we prove Theorem 4. We conclude the paper with a proof of Proposition 3 in Section 3 and a list of four open problems related to these topics that we would very much like to see answered.

2 Quasi-Convex Sets Between Two Lines

A connected set v spanned between two parallel lines is *quasi-convex* if it contains a segment s_v that has its endpoints on the two lines. When working with a family of quasi-convex sets it is convenient to fix such a segment s_v for each v and call it the *base segment* of the set v. Clearly, every convex set spanned between two lines is also quasi-convex.

Below we show that there is an on-line algorithm coloring a family of quasi-convex sets between two parallel lines with $O(\omega^3)$ colors, when ω is the clique number of the family. This implies Theorem 1

Proof of Theorem 1. We describe an on-line coloring algorithm using at most $\binom{\omega+1}{2} \cdot 24\omega$ colors on quasi-convex sets spanned between two parallel lines with clique number at most ω. The algorithm colors incoming sets with triples (α, β, γ) of positive integers with $\alpha + \beta \leqslant \omega + 1$ and $\gamma \leqslant 24\omega$ in such a way that intersecting sets receive different triples.

Let ℓ^1, ℓ^2 be the two horizontal lines such that the quasi-convex sets of the input are spanned between ℓ^1 and ℓ^2. With a quasi-convex set v we consider a fixed base segment s_v and the points (x-coordinates) $v^i = s_v \cap \ell^i$ for $i = 1, 2$.

A sequence $(v_1, \ldots, v_k)$ of already presented quasi-convex sets is *i-increasing* for $i = 1, 2$ if we have $v_1^i \leqslant v_2^i \leqslant \ldots \leqslant v_k^i$. The reverse of an i-increasing sequence is called *i-decreasing* for $i = 1, 2$. Let α_v be the size of a maximum sequence $S_\alpha(v)$ of already presented sets that is 1-increasing and 2-decreasing and starts with v. Let β_v be the size of a maximum sequence $S_\beta(v)$ of already presented sets that is 1-decreasing and 2-increasing and starts with v.

The algorithm is going to color v with a triple $(\alpha_v, \beta_v, \gamma_v)$ where α_v and β_v are defined as above. The definition of α_v and β_v is as in Schmerl's on-line algorithm for chain partitions of 2-dimensional orders or equivalently on-line coloring of permutation graphs. Indeed, if the input consists of a set of segments, then any two segments with the same α- and β-values are disjoint.

For a fixed pair (α, β) consider the set $X = X(\alpha, \beta)$ of all quasi-convex sets u presented so far that have been colored colored with $(\alpha, \beta, *)$, where $*$ an arbitrary value of the third coordinate. Since $S_\alpha(v) \cup S_\beta(v)$ is a collection of sets with pairwise intersecting base segments we can conclude that $\alpha_v + \beta_v = |S_\alpha(v)| + |S_\beta(v)| = 1 + |S_\alpha(v) \cup S_\beta(v)| \leqslant 1 + \omega$.

To determine γ_v the algorithm uses First-Fit on the set $X(\alpha, \beta)$. Bosek et al. [4] have shown that First-Fit is efficient on cocomparability graphs with no induced $K_{t,t}$. The best bound is due to Dujmović, Joret and Wood [7]: First-Fit uses at most $8(2t-3)\omega$ colors on cocomparability graphs with no induced $K_{t,t}$.

To make the result applicable we show that the intersection graph of each class $X(\alpha, \beta)$ is a cocomparability graph with no induced $K_{3,3}$. As the number of these sets is at most $\binom{\omega+1}{2}$, this will conclude the proof.

▶ **Claim.** *The bases of sets in $X(\alpha, \beta)$ are pairwise disjoint.*

Proof of Claim. Consider any two sets $u_1, u_2 \in X$ with the endpoints $u_j^i \in u_j \cap \ell^i$ for $i = 1, 2$ and $j = 1, 2$ of their bases. It suffices to show that we have $u_1^i < u_2^i$ for $i = 1, 2$ or $u_1^i > u_2^i$ for $i = 1, 2$.

Assume that $u_1^1 \leqslant u_2^1$ and $u_1^2 \geqslant u_2^2$ and that u_1 was presented before u_2. Since $u_1 \in X(\alpha, \beta)$ it is part of a 1-decreasing and 2-increasing sequence $(u_1, v_2, \ldots, v_\beta)$. The sequence $(u_2, u_1, v_2 \ldots, v_\beta)$ is a longer 1-decreasing and 2-increasing sequence starting with u_2. This contradicts the fact that $u_2 \in X(\alpha, \beta)$.

A similar argument applies when u_2 was presented before u_1. In this case we compare the 1-increasing and 2-decreasing sequences $(u_2, v_2, \ldots, v_\alpha)$ and $(u_1, u_2, v_2, \ldots, v_\alpha)$ to arrive at a contradiction. ◀

▶ **Claim.** *The intersection graph of $X(\alpha, \beta)$ contains no induced $K_{3,3}$.*

Proof of Claim. Let U and V be any two disjoint triples of sets in X. We shall show that if U and V are independent, then there is a set in U which is disjoint from a set in V, i.e., that the intersection graph of these six sets is not an induced $K_{3,3}$ with bipartition classes U, V.

By the previous claim the bases of these six sets in $U \cup V$ are disjoint and hence are naturally ordered from left to right within the strip. Without loss of generality amongst the leftmost three bases at least two belong to sets in U and thus amongst the rightmost three bases at least two belong to sets in V. In particular, there are four sets $u_1, u_2 \in U$, $v_1, v_2 \in V$ whose left to right order of bases is u_1, u_2, v_1, v_2.

By assumption u_1, u_2 and v_1, v_2 are non-intersecting. Since the base of each set is contained in the corresponding set (quasi-convexity) we know that u_1 lies completely to the left of the base of u_2 and v_2 lies completely to the right of the base of v_1. Together with the order of the bases of u_2 and v_1 this makes u_1 and v_2 disjoint. ◀

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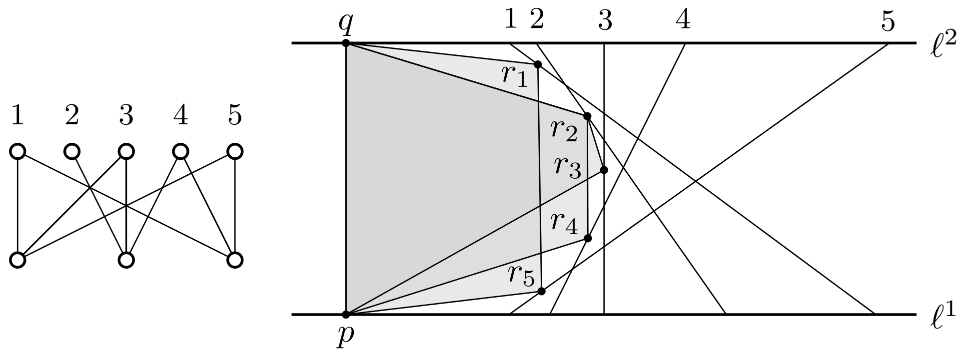

Figure 2 A poset of height 2 and its representation with convex sets spanned between two lines.

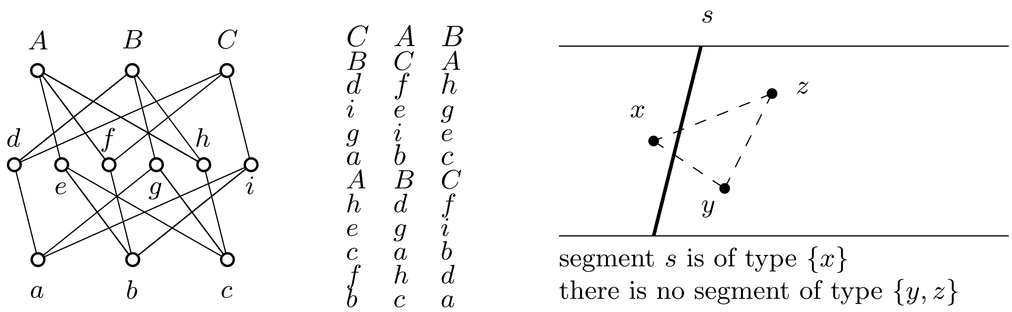

Figure 3 A height 3 and 3-dimensional poset that is not quasi-convex. Provided with its 3 linear extensions witnessing the dimension.

It is possible to decrease the number of colors used by the algorithm from $\binom{\omega+1}{2} \cdot 24\omega$ to $\binom{\omega+1}{2} \cdot 16\omega$ by showing that the pathwidth of the intersection graph of $X(\alpha, \beta)$ is at most $2\omega - 1$ and applying another result from [7]: First-Fit on cocomparability graph of pathwidth at most t uses at most $8(t+1)$ colors.

Proof of Proposition 2. Let P be any poset of height 2, and let X and Y be the sets of minimal and maximal elements in P, respectively. We represent the elements in Y as pairwise intersecting segments so that every segment appears on the left envelope, that is, on every segment $y \in Y$ there is a point r_y such that the horizontal ray emanating from r_y to the left has no further intersection with segments from Y. Choose $p \in \ell^1$ and $q \in \ell^2$ to the left of all segments for Y and define for each $x \in X$ the convex set C_x as the convex hull of p, q and the set of all r_y for which y and x are incomparable in P. It is easy to check that in the resulting representation two sets intersect if and only if the corresponding elements in P are incomparable. See Figure 2 for an illustration.

We claim that the poset Q depicted in Figure 3 is not quasi-convex. Suppose that there is a quasi-convex realization of Q. Fix three points within the strip $x \in a \cap A$, $y \in b \cap B$, and $z \in c \cap C$. The *type* of a segment s spanned in the strip and avoiding x, y and z is the subset of $\{x, y, z\}$ consisting of the points that are to the left of s. How many different types of segments can exist for given x, y and z? We claim that among 8 possible subsets only 7 are realizable. Indeed, consider the point $p \in \{x, y, z\}$ with the middle value with respect to the vertical axis. Then either $\{p\}$ or $\{x, y, z\} \setminus \{p\}$ is not realizable (see Figure 3). A collection of quasi-convex sets representing the elements d, e, f, g, h, i of Q must have base segments of pairwise distinct types. Moreover the types $\emptyset$ and $\{x, y, z\}$ do not occur. This leaves 5 possible types for 6 elements, contradiction. ◀

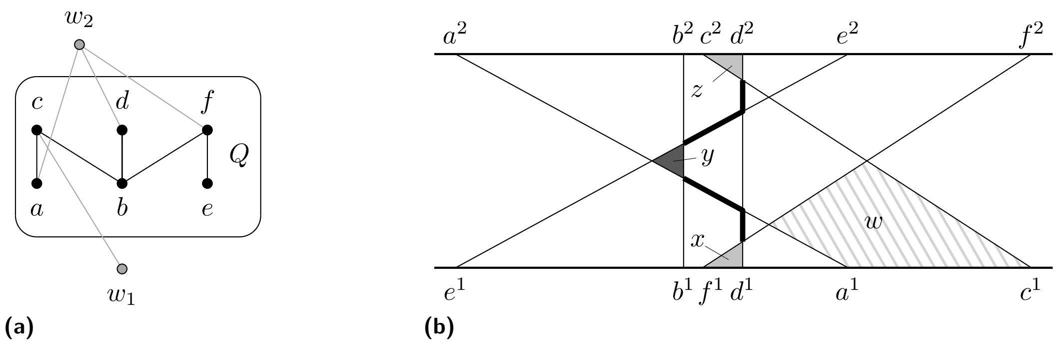

Figure 4 (a) The 6-element poset Q and the elements w_1, w_2 of P for the downset $D = \{a, b, d, e, f\}$ in Q. (b) The segment representation $\mathcal{R}$ of Q with the cells w, x, y and z corresponding to the downsets $\{a, b, d, e, f\}$, $\{b, e, f\}$, $\{a, e\}$ and $\{a, b, c\}$ in Q, respectively.

▶ **Proposition 5.** *There is a quasi-convex poset that is not convex.*

Proof. Consider the poset Q on the set $E = \{a, b, c, d, e, f\}$ as shown in Figure 4a. Moreover, consider the representation $\mathcal{R}$ of Q with segments spanned between two lines given in Figure 4b. Each cell w in $\mathcal{R}$ naturally corresponds to the downset D_w in Q (downwards closed subset of E) formed by those segments in $\mathcal{R}$ that lie to the left of w.

We shall construct a quasi-convex poset $\bar{Q}$ that has Q as an induced subposet. Later we extend $\bar{Q}$ by one point to a quasi-convex poset P and we prove that P is not convex.

Define $\bar{Q} \supset Q$ as follows. For each cell w in $\mathcal{R}$ corresponding to a downset $D_w \subseteq E$ of Q there are two incomparable elements w_1, w_2 in $\bar{Q}$, where w_2 is above all elements in D_w and w_1 is below all elements in $E \setminus D_w$. There are no further comparabilities between w_1, w_2 and elements of $\bar{Q}$, except for those implied by transitivity. We refer again to Figure 4 for an illustration.

We extend $\bar{Q}$ by adding an element g below d and y_2, but incomparable to x_2 and z_2, where y, x and z are the cells in $\mathcal{R}$ corresponding to downsets $\{a, e\}$, $\{b, e, f\}$, $\{a, b, c\}$ in Q, respectively (Figure 4b). Let P be the poset after adding g.

To see that P is quasi-convex take the representation $\mathcal{R}$ of Q, select a point p_w in each cell w, let s and t be two segments such that s is on the left and t is on the right of all segments in $\mathcal{R}$. For each cell w of $\mathcal{R}$ define T-shaped sets for w_1 and w_2 consisting of s and t, respectively, together with a horizontal segment ending at p_w. Finally let g be the union of s and two horizontal segments, one ending at p_x and one at p_z.

Fix any quasi-convex representation $\bar{\mathcal{R}}$ of $\bar{Q}$. By definition each quasi-convex set in E comes with a base segment spanned between the two lines. With $\mathcal{R}'$ we denote the configuration of the base segments corresponding to elements of Q.

▶ **Claim.** *The segment representations $\mathcal{R}$ and $\mathcal{R}'$ are equivalent in the sense that the segments together with ℓ^1 and ℓ^2 induce (up to reflection) the same plane graph where vertices are attachment points and crossings of segments and edges are pieces of segments/lines between consecutive vertices).*

Proof of Claim. Consider any cell w in $\mathcal{R}$ and the corresponding downset $D_w \subseteq E$ of Q. By the definition of w_1, w_2 in $\bar{Q}$ (in particular, the fact that the corresponding sets intersect in $\bar{\mathcal{R}}$), there is a cell $\bar{w}$ in $\mathcal{R}'$ that lies to the right of all sets in D_w and to the left of all sets in $E \setminus D_w$. Since there can be only one such cell in $\mathcal{R}'$, we have an injection φ from the cells of $\mathcal{R}$ into the cells of $\mathcal{R}'$.

Next note that if s, t are two intersecting segments in $\mathcal{R}$, then there is a cell in $\mathcal{R}$ with s to the left and t to the right, as well as another cell with s to the right and t to the left. With φ we have such cells also in $\mathcal{R}'$ and hence the segments for s and t in $\mathcal{R}'$ intersect as well. Disjoint segments in $\mathcal{R}$ represent a comparability in Q, hence, the corresponding segments in $\mathcal{R}'$ have to be disjoint as well. It follows that the number of intersections and, hence, also the number of cells, is the same in $\mathcal{R}$ and $\mathcal{R}'$, proving that φ is a bijection.

To show that $\mathcal{R}$ and $\mathcal{R}'$ are equivalent we now consider the dual graphs. That is we take the cells between the lines as vertices and make them adjacent if and only if the corresponding downsets differ in exactly one element. These dual graphs come with a plane embedding. All the inner faces of these embeddings correspond to crossings and are therefore of degree 4. Moreover, every 4-cycle of these graphs has to be an inner face. This uniquely determines (up to reflection) the embeddings of these dual graphs and hence also of the primal graphs. For the last conclusion we have used that the union of all segments in $\mathcal{R}$ and $\mathcal{R}'$ is connected. ◀

▶ **Claim.** *P is not convex.*

Proof of Claim. By the previous claim every quasi-convex representation of P induces a segment representation $\mathcal{R}'$ of Q equivalent to $\mathcal{R}$. We denote the segments in $\mathcal{R}'$ for elements a, b, c, d, e, f by $a^*, b^*, c^*, d^*, e^*, f^*$, respectively, and the cells in $\mathcal{R}'$ corresponding to x, y, z in $\mathcal{R}$ by x^*, y^*, z^*, respectively. We claim that x^* lies strictly below y^*, which lies strictly below z^*. Indeed, we can construct a y-monotone curve as follows (Figure 4b): Start with the highest point of x^*, i.e., the crossing of f^* and d^*, follow d^* to its crossing with a^*, follow a^* to its crossing with b^*, i.e., the lowest point of the cell y^*. And symmetrically, we go from the lowest point of y^* (the crossing of b^* and e^*) along e^* to its crossing with d^* and along d^* to its crossing with c^*, i.e., the highest point of z^*.

Now, as g is below d, but incomparable to x_2, the set for g contains a point p right of f^* and left of d^*, i.e., $p \in x$. Similarly, the set for g contains a point $q \in z$. Moreover the segment between p and q lies between the segments b^* and d^* as it starts and ends there. However, the base segment for y_2 lies to the right of d^* as y_2 is to the right of e and a. Hence, if g were a convex set, then the sets g and y_2 would intersect, contradicting that g is below y_2 in P. ◀

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3 On-line Curve Representation

In this section we prove Theorem 4, i.e., we show that there is an on-line algorithm that produces a curve representation of any poset that is given on-line. The curves used for the representation are y-monotone.

Recall that a *linear extension* L of a poset P is a total ordering of its elements such that if $x < y$ in P this implies $x < y$ in L. Our construction maintains the invariant that at all times the curve representation $\mathcal{C}$ of the current poset P satisfies the following property $(*)$:

> there is a set $\mathcal{L}$ of horizontal lines such that for every linear extension L of P there is a horizontal line $\ell \in \mathcal{L}$ such that the curves in $\mathcal{C}$ intersect ℓ from left to right in distinct points in the order given by L. $(*)$

For the first element of the poset use any vertical segment in the strip and property $(*)$ is satisfied. Assume that for the current poset P we have a curve representation $\mathcal{C}$ with y-monotone curves respecting $(*)$.

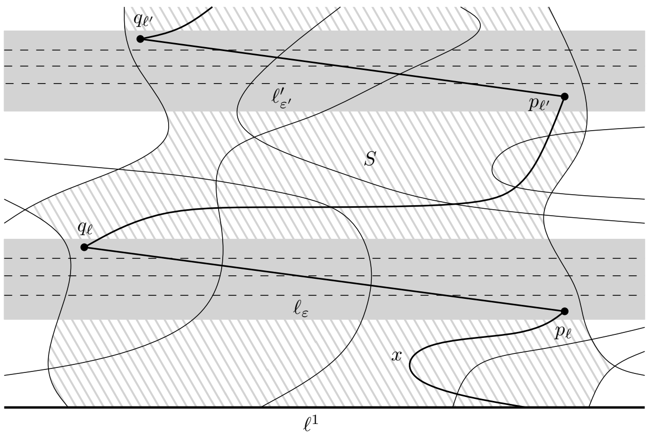

Figure 5 Constructing the curve for a new element x by using segments within ε-tubes for each $\ell \in \mathcal{L}$. Dashed horizontal lines correspond to the lines in $\mathcal{L}_{P\cup\{x\}}$.

Let x be a new element extending P. The elements of P are partitioned into the upset $U(x) = \{y : x < y\}$, the downset $D(x) = \{y : y < x\}$, and the set $I(x) = \{y : x||y\}$ of incomparable elements. Let S be the union of all points in the strip between ℓ^1 and ℓ^2 that lie strictly to the left of all curves in $U(x)$ and strictly to the right of all curves in $D(x)$. Note that S is y-monotone (its intersection with any horizontal line is connected), $S \cap \ell^i \neq \emptyset$ for $i = 1, 2$, and that S is connected since each curve in $U(x)$ lies completely to the right of each curve in $D(x)$. This implies that for any two points $p, q \in S$ with distinct y-coordinates there is a y-monotone curve connecting p and q inside of S.

We use the set $\mathcal{L}$ to draw the curve for x as follows:

- Choose $\varepsilon > 0$ small enough so that within the ε-tube ℓ_ε of any line $\ell \in \mathcal{L}$ no two curves get closer than ε.
- For each line $\ell \in \mathcal{L}$ choose two points $q_\ell, p_\ell \in \ell_\varepsilon \cap S$ such that q_ℓ is above ℓ and has distance at most ε to the left boundary of S while p_ℓ is below ℓ and has distance at most ε to the right boundary of S. Draw a segment from p_ℓ to q_ℓ.
- If ℓ and ℓ' are consecutive in $\mathcal{L}$ with ℓ below ℓ', then we connect q_ℓ and $p_{\ell'}$ by a y-monotone curve in S. We also connect the lowest p and the highest q by y-monotone curves in S to ℓ^1 and ℓ^2 respectively.
- The curve of x is the union of the segments $\overline{p_\ell q_\ell}$ and the connecting curves.

Figure 5 illustrates the construction.

We claim that the curve representation of P together with the curve of x has property $(*)$. Let $L = (\ldots, a, x, b, \ldots)$ be an arbitrary linear extension of $P \cup \{x\}$ and let $L^x = (\ldots, a, b, \ldots)$ be the linear extension of P obtained from L by omitting x. Let $\ell^x \in \mathcal{L}$ be the horizontal line corresponding to L^x. Within the ε-tube of ℓ^x the segment $\overline{p_{\ell^x} q_{\ell^x}}$ contains a subsegment $\overline{\rho_a \rho_b}$ where ρ_a is a point ε to the right of the curve of a and ρ_b is a point ε to the left of the curve of b. The horizontal line ℓ containing the point $\frac{\rho_a+\rho_b}{2}$ is a line representing L in $P \cup \{x\}$. This proves property $(*)$ for the extended collection of curves.

The comparabilities in the intersection of all linear extensions of $P \cup \{x\}$ are exactly the comparabilities of $P \cup \{x\}$. Therefore, property $(*)$ implies that the curve of x is intersecting the curves of all elements of $I(x)$. Since the curve of x is in the region S it is to the right of all curves in $D(x)$ and to the left of all curves in $U(x)$. Hence, the extended family of curves represents $P \cup \{x\}$.

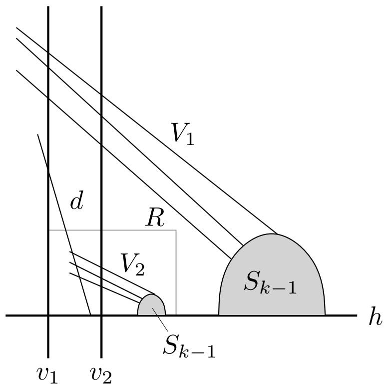

Figure 6 Strategy S_k consists of two calls of strategy S_{k-1} and an addition of an extra segment d. Algorithm A unavoidably uses k colors on the segments intersecting v_1 or on the segments intersecting v_2.

4 Connected Sets Attached to a Line

In this section we give the proof of Proposition 3. Actually, we prove a stronger statement by induction:

▶ **Claim.** *The adversary has a strategy S_k to create a family of segments attached to a horizontal line h with clique number at most 2 against any on-line coloring algorithm A such that there is a vertical line v with the properties:*
1. *any two segments pierced by v are disjoint,*
2. *every segment pierced by v is attached to h to the right of v,*
3. *A uses at least k distinct colors on segments pierced by v.*

Proof of Claim. The strategy S_1 only requires a single segment with negative slope. Now consider $k \geqslant 2$. Fix any on-line algorithm A. The strategy S_k goes as follows. First the adversary uses S_{k-1} to create a family of segments $\mathcal{F}_1$ and a vertical line v_1 piercing a set $V_1 \subseteq \mathcal{F}_1$ of pairwise disjoint segments on which A uses at least $k-1$ colors. Define a rectangle R with bottom-side on h, the left-side in v_1 and small enough such that the vertical line supported by the right-side is piercing the same subset V_1 of $\mathcal{F}_1$, moreover R is disjoint from all the segments in $\mathcal{F}_1$. The adversary uses strategy S_{k-1} again, this time with the restriction that all the segments are contained in R. This creates a family $\mathcal{F}_2$ and a vertical line v_2 piercing a set $V_2 \subseteq \mathcal{F}_2$ of pairwise disjoint segments on which A uses at least $k-1$ colors. By construction segments from $\mathcal{F}_1$ and $\mathcal{F}_2$ are pairwise disjoint. From the definition of R it follows that line v_2 intersects all the segments in V_1 and no other segments from $\mathcal{F}_1$. Strategy S_k is completed with the creation of one additional segment d such that d is attached between v_1 and v_2, d is intersecting all the segments in V_2 and the vertical line v_1 but it intersects none of the segments in V_1 (see Figure 6).

If A uses at least k distinct colors on $V_1 \cup V_2$ then v_2 is the vertical line witnessing the invariant. Otherwise A uses exactly the same set of $k-1$ colors on V_1 and V_2 and since segment d intersects all segments in V_1 it must be colored with a different color. Thus, the vertical line v_1 intersecting $V_1 \cup \{d\}$ intersects segments of at least k distinct colors. ◀

5 Open problems

In this concluding section we collect some open problems related to the results of this paper.

In Figure 1 there are some classes of posets that contain interval orders and 2-dimensional orders and are contained in the class of convex orders. For on-line coloring of the cocomparability graphs of these classes (given with a representation) we have the algorithm from Theorem 1 that uses $O(\omega^3)$ colors.

(1) Find an on-line algorithm that only needs $O(\omega^\tau)$ $(\tau < 3)$ colors for coloring graphs in a class $\mathcal{G}$ between 2-dimensional and convex. Interesting choices for $\mathcal{G}$ would be trapezoid graphs, bounded tolerance graphs, triangle graphs (or simple triangle graphs; for the definition cf. [18]).

By restricting the curves or the intersection pattern of curves spanned between two lines we obtain further classes of orders which are nested between 2-dimensional orders and the class of all orders. We define *k-bend orders* by restricting the number of bends of the polygonal curves representing the elements to k. Clearly, every $k + 2$ dimensional order is a k-bend order. We define *k-simple orders* by restricting the number of intersections of pairs of curves representing elements of the order to k.

(2) Find on-line algorithms that only need polynomially many colors for coloring cocomparability graphs of 2-simple or 1-bend orders when a representation is given.

Another direction would be the study of recognition complexity. Meanwhile the recognition complexity for all classes shown in Figure 1, except convex orders, has been determined (see [18]).

(3) Determine the recognition complexity for convex orders.

We think that the determination of the recognition complexity of 2-simple orders and 1-bend orders are also interesting problems.

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