A Fire Fighter's Problem

Rolf Klein[1], Elmar Langetepe[1], and Christos Levcopoulos[2]

1 **Institute of Computer Science I, University of Bonn, Germany**
2 **Department of Computer Science, Lund University, Sweden**

Abstract

Suppose that a circular fire spreads in the plane at unit speed. A fire fighter can build a barrier at speed $v > 1$. How large must v be to ensure that the fire can be contained, and how should the fire fighter proceed? We provide two results. First, we analyze the natural strategy where the fighter keeps building a barrier along the frontier of the expanding fire. We prove that this approach contains the fire if $v > v_c = 2.6144\ldots$ holds. Second, we show that any "spiralling" strategy must have speed $v > 1.618$, the golden ratio, in order to succeed.

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1 Introduction

Fighting wildfires and epidemics has become a serious issue in the last decades. Professional fire fighters need models and simulation tools on which strategic decisions can be based; for example see [5]. Thus, a good understanding of the theoretical foundations seems necessary.

Substantial work has been done on the fire fighting problem in graphs; see, e.g., the survey article [3]. Here, initially one vertex is on fire. Then an immobile firefighter can be placed at one of the other vertices. Next, the fire spreads to each adjacent vertex that is not defended by a fighter, and so on. The game continues until the fire cannot spread anymore. The objective, to save a maximum number of vertices from the fire, is NP-hard to achieve, even for trees.

A more geometric setting has recently been studied in [6]. Suppose that inside a simple polygon P a candidate set of disjoint diagonal barriers has been defined. If a fire starts at some point inside P one wants to build a subset of these barriers in order to save a maximum area from the fire. But each point on a barrier must be built before the fire arrives there. This maximization problem is also NP-hard, even if the candidate barriers are the diagonals of a convex polygon, but there exists an 11.65 approximation algorithm.

In this paper we study a purely geometric version of the fire fighter problem. Suppose there is a circular fire of initial radius A in the plane, centered at the origin. The fire spreads at unit speed. Initially, the plane is empty, except for a single fire fighter who is placed on the boundary of the fire. The fighter can move at speed v, and build a barrier along his path. The fire cannot cross this barrier, and the fighter cannot move into the fire. Will the fighter be able to contain the fire, and how should she proceed to achieve this?

Clearly, the answer depends on speed v. For $v = 1$ the fighter can barely save herself by moving along a straight line away from the fire.

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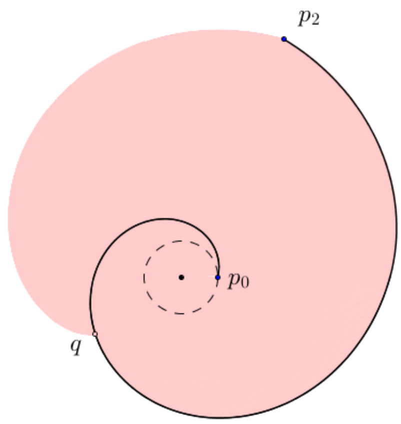

Figure 1 The race between the fire and the fighter for speed $v = 3.738$. The firebreak was constructed from p_0 to p_2 whereas the fire expands along the outer side of the barrier up to point q. Can the fire figther finally catch the fire?

At speed $v > 2\pi + 1$, the fire fighter can move a distance x away from the fire and build a complete circular barrier before the fire can reach it. This requires $(x + 2\pi(x + A))/v \leq x$ or $(2\pi + 1) + 2\pi A/x \leq v$.

What happens in between 1 and $2\pi + 1$? *In this paper we show that a speed* $v > 2.6144$ *is sufficient to contain a fire, and that a speed* $v > 1.618$ *is necessary, at least for a reasonably large class of strategies.*

The first bound is established in the following way. We consider a conscientious fire fighter who tries to contain the fire by building a barrier along its ever expanding frontier, at her maximum speed v. Let us denote this strategy by FF (short for Follow Fire). A spiralling barrier curve results. While the fighter keeps building the barrier, the fire is coming after her along the outside of the barrier, as shown in Figure 1. Intuitively, the fighter can only win this race, and contain the fire, if the last coil of the barrier hits the previous one.

In the hand-drawn example shown in Figure 2 this happens in the second round if $v = 4.1932$; but for smaller values of v, more rounds may be necessary.

We have the following result.

▶ **Theorem 1.**
(i) *Strategy FF contains the fire if* $v > v_c \approx 2.6144$ *holds.*
(ii) *As* v *decreases to* v_c*, the number of rounds to containment tends to infinity.*

Although strategy FF is rather simple, the proof of Theorem 1 is not. First, we establish a recursive system of linear differential equations associated with each round. They can be solved easily by standard methods, but the resulting recursions are complicated. Therefore, we apply techniques from analytic combinatorics. We look at the generating function $F(Z)$ that arises from these recursions, and find a presentation of $F(Z)$ as a ratio of analytic functions. The denominator equals

$$e^{w\,Z} - s\,Z \;=\; 0, \tag{1}$$

where $w = \frac{2\pi+\alpha}{\sin\alpha}$ and $s = e^{(2\pi+\alpha)\cot\alpha}$ are functions of a real variable α which equals $\cos^{-1}(1/v)$ in our setting. Our targets are the coefficients of $F(Z)$; they are linked to the zeroes of equation 1.

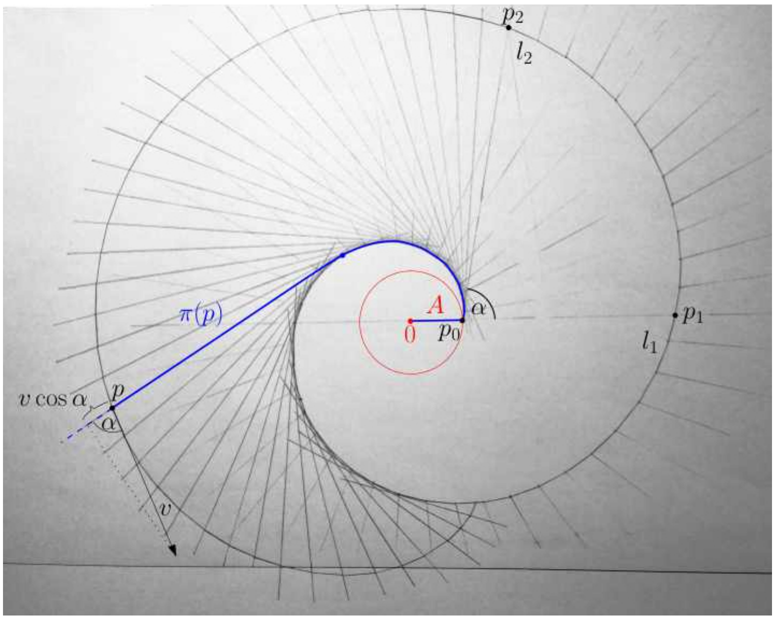

Figure 2 At speed $v = 4.1932$ the fire will be fully contained by the fire figther's barrier in the second round.

Let $\alpha_c \approx 1.1783$ be the smallest positive solution of $s = e\,w$, corresponding to $v_c \approx 2.6144$. For this value of α, equation 1 has a real zero $Z = 1/w$, as direct substitution shows. For $\alpha > \alpha_c$, corresponding to $v > v_c$, this real zero splits into a complex zero $z_0 = \rho(\cos\phi + \sin\phi\, i)$ and its conjugate, where $\phi \in (0, \pi)$, and no real zeroes of equation 1 remain.

At this point, part (i) of Theorem 1 follows from a Theorem of Pringsheim's in complex function theory; see Section 6. To find out how many rounds it takes to contain the fire, we apply Cauchy's residue theorem and find that their number is $\approx \pi/\phi$. Since ϕ, the angle of the complex root z_0, tends to zero as z_0 becomes real for $\alpha \to \alpha_c$, part (ii) of Theorem 1 also follows. How j, the number of rounds, depends on v is shown in Figure 3. For speeds $v \geq 3$ strategy FF needs at most 4 rounds to contain the fire.

In addition to the above upper bound we prove the following lower bound. To this end we restrict ourselves to the class of "spiralling" strategies that visit the four coordinate half-axes in cyclic order, and at increasing distances from the origin. Note that strategy FF is spiralling even though the fighter's distance to the origin may be decreasing: the barrier's intersection points with any ray from 0 are in increasing order since the curve does not self-intersect. Here we have the following.

▶ **Theorem 2.** *In order to enclose the fire, a spiralling strategy must be of speed*

$$v \; > \; \frac{1+\sqrt{5}}{2} \approx 1.618,$$

the golden ratio.

The proof of Theorem 2 is given in Section 7. An (almost) complete proof of Theorem 1 (i) is given in the main text; only for some details we refer to the technical report of this paper; see [7]. Proving part (ii) of Theorem 1 requires considerably more work; we sketch only the essential ideas in the main text. A complete proof of (i) and (ii), which can be read independently of the main text, is given in the Appendix of the technical report [7].

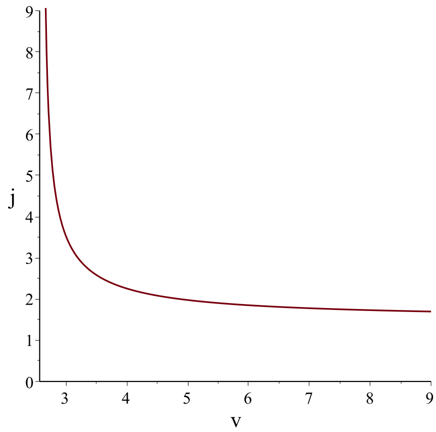

Figure 3 The approximate number of rounds needed by strategy FF, as a function of speed v.

2 The barrier curve generated by strategy FF

We would like to show how the barrier curve shown in Figure 2 has been developed. A more detailed view of the starting situation of Figure 2 from p_0 to p_2 is depicted in Figure 4.

Consider some point p in the first round between p_0 and p_1 as shown in Figure 4. If α denotes the angle between the fighter's velocity vector at p and the ray from 0 through p, the fighter advances at speed $v \cos \alpha$ away fromat 0. This implies $v \cos \alpha = 1$ because the fire expands at unit speed and the fighter stays on its frontier, by definition of strategy FF. Consequently, the barrier curve between p_0 and p_1 is part of a logarithmic spiral centered at 0, whose tangents forms the angle $\alpha = \cos^{-1}(1/v)$ with the extensions of the rays from 0 through p.

In polar coordinates a logarithmic spiral (with excentricity α) is defined by $(\varphi, A \cdot e^{\varphi \cot \alpha})$ and the barrier curve from p_0 to p_1 is represented by the interval $\varphi \in [0, 2\pi]$. The curve length of the logarithmic spiral of excentricity α around origin O between two points C and D appearing on the spiral in this order is given by $\frac{1}{\cos \alpha}(|DO| - |CO|)$, where $|CO|$ and $|DO|$ denote the distances from D and C to the origin 0, respectively. Thus, for example the curve length from p_0 to p_1 is given by $l_1 = \frac{A}{\cos(\alpha)} \cdot (e^{2\pi \cot(\alpha)} - 1)$.

From point p_1 on, the geodesic shortest paths $\pi(p)$ from 0 to p, along which the fire spreads, start with segment $0p_0$, followed by segment p_0p, until the fighter reaches the point p_2 on the barrier's tangent to p_0; see Figure 4. Thus, by the previous argument, between p_1 and p_2 the barrier curve constructed by FF is part of a logarithmic spiral of excentricity α now centered at p_0. This spiral starts at p_1 with distance $A' = A(e^{2\pi \cot(\alpha)} - 1)$ from its origin p_0, and the curve length from p_1 to p_2 is given by $l'_2 = \frac{A'}{\cos(\alpha)}(e^{\alpha \cot(\alpha)} - 1) = \frac{A}{\cos(\alpha)}(e^{2\pi \cot(\alpha)} - 1)(e^{\alpha \cot(\alpha)} - 1)$. This means that the overall curve length from p_0 to p_2 is given by $l_1 + l'_2 = l_2 = \frac{A}{\cos(\alpha)}(e^{2\pi \cot(\alpha)} - 1)e^{\alpha \cot(\alpha)}$.

How does the curve constructed by FF develop from p_2 on? We turn over to Figure 2. From p_2 on, the geodesic shortest path $\pi(p)$ from 0 to fighter's current position p starts wrapping around the existing spiral part of the curve, beginning at p_0. The last edge of $\pi(p)$ ending at p will be called the *free string* in the sequel. The fire will be contained if and only if the free string ever attains length 0.

Thus, after the first round the curve is drawn by endpoint p of the free string. But unlike an involute, the string is not normal to the outer layer. Rather, its extension beyond p forms the angle α with the barrier's tangent at p. This causes the string to grow in length by $\cos \alpha$

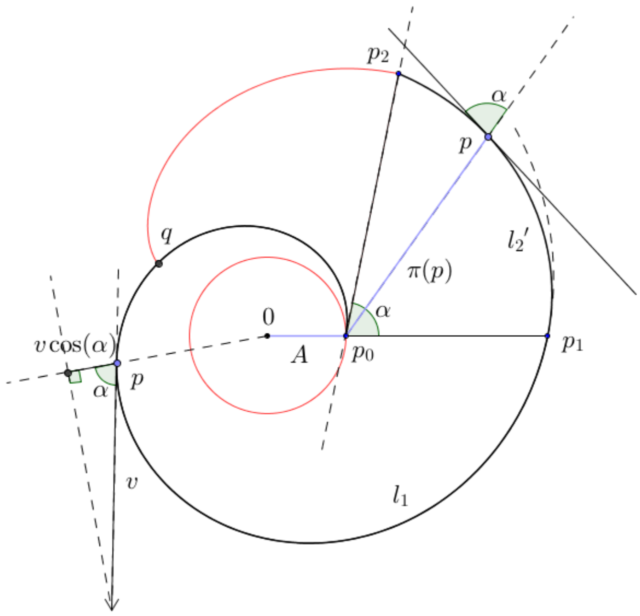

Figure 4 The first part of the barrier curve constructed by FF consists of two different logarithmic spirals of excentricity α where $\alpha = \cos^{-1}(1/v)$ holds. Namely, a logarithmic spiral around the origin 0 from p_0 to p_1 and a logarithmic spiral around p_0 from p_1 to p_2. At p_2 the fire figther's curve starts wrapping around the constructed barrier as show in Figure 2.

for each unit drawn. At the same time, part of the string gets wrapped around the inner layer. It is this interplay between growing and shrinking that we will investigate below. Note that the curve starting at p_2 is no longer a logarithmic spiral.

As the fighter is building the barrier at speed $1/\cos\alpha$, the fire is coming after her at unit speed along the outside of the barrier, as indicated in Figure 1. Thus, each barrier point p is caught by fire twice, once from the inside, when the fighter passes through p, and a second time from the outside, if the fire is not stopped before.

3 Linkages

That the innermost part of the curve consists of two different spiral segments, around 0 and around p_0, carries over to subsequent layers. The structure of the curve can be described as follows. Let

$$l_1 = \frac{A}{\cos(\alpha)} \cdot (e^{2\pi\cot(\alpha)} - 1)$$
$$l_2 = \frac{A}{\cos(\alpha)} \cdot (e^{2\pi\cot(\alpha)} - 1)e^{\alpha\cot(\alpha)}$$

denote the curve lengths from p_0 to p_1 and p_2, respectively, as derived before in Section 2. For $l \in [0, l_1]$ let $F_0(l)$ denote the segment connecting 0 to the point of curve length l; see the sketch given in Figure 5.

At the endpoint of $F_0(l)$ we construct the tangent and extend it until it hits the next layer of the curve, creating a segment $F_1(l)$, and so on. This construction gives rise to a "linkage" connecting adjacent layers of the curve. Each edge of the linkage is turned counterclockwise by α with respect to its predecessor. The outermost edge of a linkage is the free string

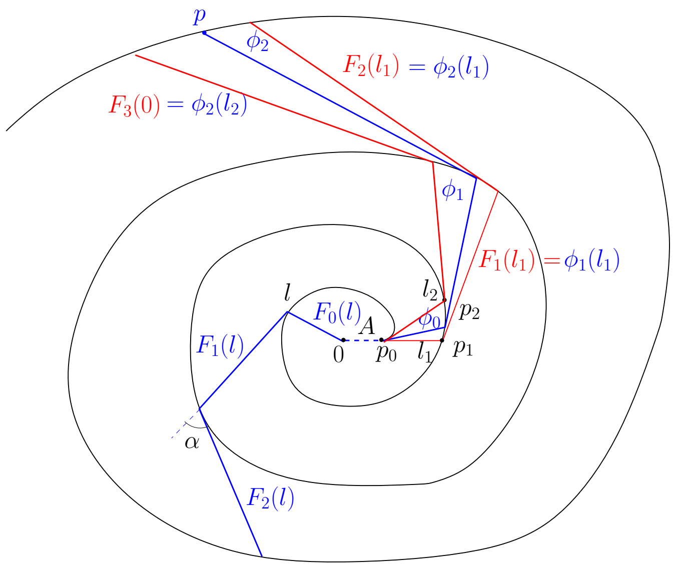

Figure 5 A sketch of the general situation. Two types of linkages defining subsegments of the curve.

mentioned above. As parameter l increases from 0 to l_1, edge $F_0(l)$, and the whole linkage, rotate counterclockwise. While $F_0(0)$ equals the line segment from the center to p_0, edge $F_0(l_1)$ equals segment $0p_1$.

Analogously, let $l \in [l_1, l_2]$, and let $\phi_0(l)$ denote the segment from p_0 to the point at curve length l from p_1. This segment can be extended into a linkage in the same way. We observe that

$$F_{j+1}(l_1) = \phi_{j+1}(l_1) \tag{2}$$
$$F_{j+1}(0) = \phi_j(l_2) \tag{3}$$

hold. But initially, we have $F_0(l) = A + \cos(\alpha)\, l$ and $\phi_0(l) = \cos(\alpha)\, l$, so that $F_0(l_1) \neq \phi_0(l_1)$. Clearly, each point on the curve can be reached by a linkage, as tangents can be constructed backwards. We refer to the two types of linkages by F-type and ϕ-type.

4 Analysis

A detailed proof of the following general facts is given in the Appendix of the technical report [7] in Lemma 7 and 8. We present the intuitive ideas here.

As the endpoint of a taut string of length F, tangent to a smooth curve C at some point p, is moved in direction α, as shown in Figure 6 (i), the length l of the wrapped string grows at rate $r \sin \alpha / F$, where r denotes the curve's radius of curvature at p. (Intuitively, the more perpendicular motion w acts on the string and the larger the osculating circle, the more of the string gets wrapped; but the larger F, the smaller is the effect of the perpendicular motion.)

The center of the osculating circle at p is known to be the limit of the intersections of the normals of all points near p with the normal at p. If, instead of the normals, we consider the lines turned by the angle $\pi/2 - \alpha$, their limit intersection point has distance $r \sin \alpha$ from p; an example is shown in Figure 6 (ii) for the case where curve C itself is a circle.

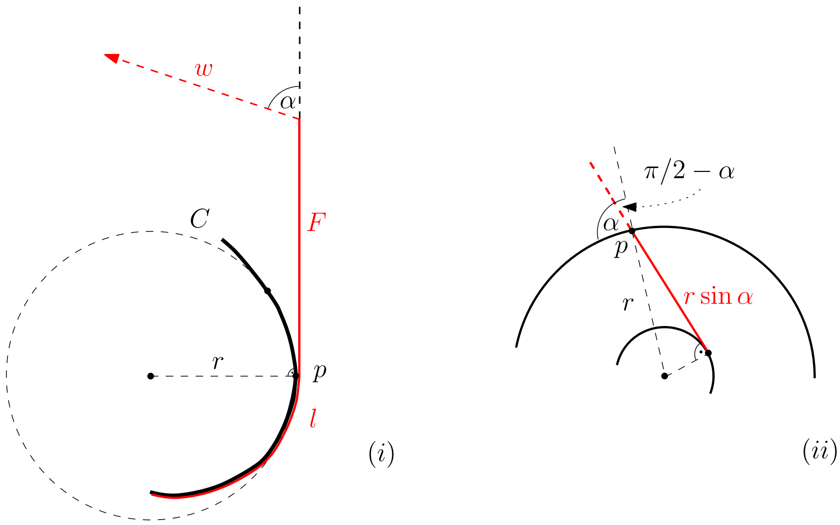

Figure 6 In (i), the wrapped string grows at a rate of $r \sin \alpha / F$. In (ii), the turned normals meet at a point $r \sin \alpha$ away from p.

For the barrier curve, the limit intersection point of the turned normals near p is just the tangent point from p to the previous layer of the curve. If we denote by L_i the length of the barrier curve from p_0 to the outer endpoint of the ith edge of an F-linkage, the above observations imply the following for L_{j-1}, F_j and F_{j-1} as functions of L_j.

$$\frac{L'_{j-1}}{L'_j} = \frac{L'_{j-1}}{1} = \frac{r \sin \alpha}{F_j} = \frac{F_{j-1}}{F_j}.$$

Now we change the former variable L_j to $L_j(l)$ for $l \in [0, l_1]$ introduced in Section 3. Observing that the derivatives of the inner functions cancel out we obtain

▶ **Lemma 3.**

$$\frac{L'_{j-1}(l)}{L'_j(l)} = \frac{F_{j-1}(l)}{F_j(l)}.$$

By multiplication, Lemma 3 generalizes to non-consecutive edges. Thus,

$$\frac{F_j(l)}{F_0(l)} = \frac{L'_j(l)}{l'} = L'_j(l) \tag{4}$$

holds.

On the other hand, a point p on the jth layer of the barrier curve has geodesic distance $L_{j-1}(l) + F_j(l)$ from the initial fire of radius A, and the fire arrives at p (from the inside) simultaneously with the fighter, who has then completed a barrier of length $L_j(l)$ at speed $1/\cos\alpha$. This yields, $F_j(l) + L_{j-1}(l) = \cos\alpha \, L_j(l)$ and after taking derivatives,

$$F'_j(l) + L'_{j-1}(l) = \cos\alpha \, L'_j(l). \tag{5}$$

From 5 and 4 we obtain a linear differential equation for $F_j(l)$,

$$F'_j(l) - \frac{\cos(\alpha)}{F_0(l)} F_j(l) = -\frac{F_{j-1}(l)}{F_0(l)}.$$

The textbook solution for $y'(x) + f(x)y(x) = g(x)$ is

$$y(x) = \exp(-a(x)) \left(\int g(t) \exp(a(t))\, \mathrm{d}t + \kappa \right),$$

where $a = \int f$ and κ denotes a constant that can be chosen arbitrarily. In our case,

$$a(l) = \int -\frac{\cos(\alpha)}{A + \cos(\alpha)\, l} = -\ln(F_0(l))$$

because of $F_0(l) = A + \cos(\alpha)\, l$, and we obtain

$$F_j(l) = F_0(l)\Big(\kappa_j - \int \frac{F_{j-1}(t)}{F_0^2(t)}\,\mathrm{d}t\Big). \tag{6}$$

Next, we consider a linkage of ϕ-type, for parameters $l \in [l_1, l_2]$, and obtain analogously

$$\phi_j(l) = \phi_0(l)\Big(\lambda_j - \int \frac{\phi_{j-1}(t)}{\phi_0^2(t)}\,\mathrm{d}t\Big). \tag{7}$$

Now we determine the constants κ_j, λ_j such that the solutions 6 and 7 describe a contiguous curve. To this end, we must satisfy conditions 2 and 3.

We define $\kappa_0 := 1$ and

$$\kappa_{j+1} := \frac{\phi_j(l_2)}{F_0(0)} + \int \frac{F_j(t)}{F_0^2(t)}\mathrm{d}t|_{l=0}$$

so that 6 becomes

$$F_{j+1}(l) = F_0(l)\left(\frac{\phi_j(l_2)}{F_0(0)} - \int_0^l \frac{F_j(t)}{F_0^2(t)}\,\mathrm{d}t\right),$$

which, for $l = 0$, yields $F_{j+1}(0) = \phi_j(l_2)$ (condition 3).

Similarly, we set $\lambda_0 := 1$ and

$$\lambda_{j+1} := \frac{F_{j+1}(l_1)}{\phi_0(l_1)} + \int \frac{\phi_j(t)}{\phi_0^2(t)}\mathrm{d}t|_{l=l_1}$$

so that 7 becomes

$$\phi_{j+1}(l) = \phi_0(l)\left(\frac{F_{j+1}(l_1)}{\phi_0(l_1)} - \int_{l_1}^l \frac{\phi_j(t)}{\phi_0^2(t)}\,\mathrm{d}t\right),$$

and for $l = l_1$ we get $F_{j+1}(l_1) = \phi_{j+1}(l_1)$ (condition 2).

For simplicity, let us write

$$G_j(l) := \frac{F_j(l)}{F_0(l)} \quad \text{and} \quad \chi_j(l) := \frac{\phi_j(l)}{\phi_0(l)}, \tag{8}$$

which leads to

$$G_{j+1}(l) = \frac{\phi_0(l_2)}{F_0(0)}\chi_j(l_2) - \int_0^l \frac{G_j(t)}{F_0(t)}\,\mathrm{d}t \tag{9}$$

$$\chi_{j+1}(l) = \frac{F_0(l_1)}{\phi_0(l_1)}G_{j+1}(l_1) - \int_{l_1}^l \frac{\chi_j(t)}{\phi_0(t)}\,\mathrm{d}t. \tag{10}$$

In order to find out if the fire fighter is successful we only need to check the values of $F_j(l)$ at the end of each round, as the following lemma shows.

▶ **Lemma 4.** *The curve encloses the fire if and only if there exists an index j such that $F_j(l_1) \leq 0$ holds.*

Proof. The free string shrinks to zero if and only if there exist an index j and argument l such that $F_j(l) \leq 0$ or $\phi_j(l) \leq 0$. Clearly, G_j and F_j have identical signs, as well as χ_j and ϕ_j do. Suppose that $G_j > 0$ and $G_{j+1}(l) = 0$, for some j and some $l \in [0, l_1]$. By 9, function G_{j+1} is decreasing, therefore $G_{j+1}(l_1) \leq 0$. Now assume that $G_i > 0$ holds for all i, and that we have $\chi_{j-1} > 0$ and $\chi_j(l) = 0$ for some j and some $l \in [l_1, l_2]$. By 10 this implies $\chi_j(l_2) \leq 0$, and from 9 we conclude $G_{j+1} \leq 0$, in particular $G_{j+1}(l_1) \leq 0$. ◀

5 Recursions

The integrals in 9 and 10 disappear by iterated substitution. This process is not entirely trivial, and the calculations can be found in Section C in the Appendix of the technical report [7]. After plugging in values, one obtains cross-wise recursions

$$F_j(l_1) = \frac{F_0(l_1)}{F_0(0)} \sum_{\nu=0}^{j} \frac{(-1)^\nu}{\nu!} \left(\frac{2\pi}{\sin\alpha}\right)^\nu \phi_{j-1-\nu}(l_2) \tag{11}$$

$$\phi_j(l_2) = \frac{\phi_0(l_2)}{\phi_0(l_1)} \sum_{\nu=0}^{j} \frac{(-1)^\nu}{\nu!} \left(\frac{\alpha}{\sin\alpha}\right)^\nu \hat{F}_{j-\nu}(l_1) \tag{12}$$

where $\phi_{-1}(l_2) := F_0(0)$, $\hat{F}_0(l_1) := \phi_0(l_1)$, and $\hat{F}_{i+1}(l_1) := F_{i+1}(l_1)$.

In order to solve the cross-wise recursions 11 and 12 for the numbers $F_j(l_1)$ we define the formal power series

$$F(X) := \sum_{j=0}^{\infty} F_j X^j \quad \text{and} \quad \phi(X) := \sum_{j=0}^{\infty} \phi_j X^j$$

where $F_j := F_j(l_1)$ and $\phi_j := \phi_j(l_2)$, for short. From 11 we obtain

$$F(X) = \frac{F_0}{F_0(0)} e^{-\frac{2\pi}{\sin\alpha} X} \big(X\,\phi(X) + F_0(0)\big), \tag{13}$$

and from 12,

$$\phi(X) = \frac{\phi_0}{\phi_0(l_1)} e^{-\frac{\alpha}{\sin\alpha} X} \big(X\,F(X) - F_0 + \phi_0(l_1)\big); \tag{14}$$

both equalities can be easily verified by computing the products and comparing coefficients. Now we substitute 14 into 13, solve for $F(X)$, divide both sides by F_0 and expand by $e^{\frac{2\pi+\alpha}{\sin\alpha}}$ to obtain

$$\frac{F(X)}{F_0} = \frac{e^{vX} - r\,X}{e^{wX} - s\,X}, \tag{15}$$

where v, r, w, s are the following functions of α:

$$v = \frac{\alpha}{\sin\alpha} \quad \text{and} \quad r = e^{\alpha\cot\alpha}$$
$$w = \frac{2\pi+\alpha}{\sin\alpha} \quad \text{and} \quad s = e^{(2\pi+\alpha)\cot\alpha}. \tag{16}$$

Note that here the parameter v does no longer represent the speed parameter, the speed is given by $\frac{1}{\cos\alpha}$.

It is possible to expand the inverse of the denominator in 15 into a power series. This leads to interesting expressions for the F_j; but how to derive their signs seems not obvious.

6 Singularities and Residues

Now we consider the right hand side of (15) as a function

$$f(z) \;:=\; \frac{e^{vz} \;-\; r\,z}{e^{wz} \;-\; s\,z}, \tag{17}$$

of a complex variable, z. Both numerator and denominator of f are analytic on the complex plane. Thus, singularities of f can only arise from zeroes of the denominator $e^{wZ} - sZ$. This equation has received some attention in the area of delay differential equations [2]. As in the Introduction, let $\alpha_c \approx 1.1783$ be the unique solution of $s = ew$ in $(0, \pi/2]$, corresponding to speed $v_c = 1/\cos\alpha_c \approx 2.6144$.

▶ **Lemma 5.** *For $\alpha = \alpha_c$, equation $e^{wZ} - sZ$ has a real root $1/w \approx 0.1238$. For $\alpha > \alpha_c$ (corresponding to speed $v > v_c$), this root splits into a complex conjugate pair z_0 and $\overline{z_0}$, whose absolute values are < 0.31. All other zeroes of numerator and denominator in 15 are strictly complex, and of absolute values ≥ 1. Function $f(z)$ in 17 has only poles as singularities.*

For a proof of Lemma 5 see Lemmata 10 to 13 in the Appendix of the technical report [7].

From now on we assume that $\alpha > \alpha_c$ holds. Now we would like to make use of a general Theorem concerning the sign of coefficients of power series within their convergence radius, in order to prove the first part of Theorem 1.

▶ **Theorem 6** (Pringsheim's Theorem (see for example [4, p. 240]). *Let $h(z) = \sum_{n=0}^{\infty} a_n z^n$ be a power series with finite convergence radius R. If $h(z)$ has non-negative coefficients, a_j, then point $z = R$ is a singularity of $h(z)$.*

Proof of Theorem 1 (i). Let $\alpha > \alpha_c$. Because of the singularities z_0 and $\overline{z_0}$, the power series expansion of $f(z)$ in 17 has a finite radius, R, of convergence. If all coefficients F_i were ≥ 0 then, by Pringsheim's Theorem function $f(z)$ would have a singularity at R. But, by Lemma 5, there can be only complex singularities. Thus, there must be coefficients $F_j < 0$, proving that the fire fighter succeeds. ◀

Now we sketch the proof of Theorem 1(ii). A complete version can be found in the Appendix Sections E and F of the technical report [7]. This will also lead to another, and constructive, proof of part (i) of Theorem 1.

We are using a technique described in [4, p. 258 ff.]. Let Γ denote the circle of radius 0.9 around the origin. By Cauchy's Residue Theorem,

$$\frac{1}{2\pi\, i} \int_\Gamma \frac{f(u)}{u^{j+1}}\, \mathrm{d}u \;=\; \sum_{z \text{ inside } \Gamma} \mathrm{res}(z)$$

holds, where the sum is over all residues of the poles of $\frac{f(z)}{z^{j+1}}$ encircled by Γ. By Lemma 5, these poles are z_0, $\overline{z_0}$, and 0, which has residue F_j/F_0. Computing the residues of z_0, $\overline{z_0}$ yields

$$\frac{F_j}{F_0} \;=\; \sin(j\phi + p)\, \frac{|z_0|^{-j}}{|z_0 - x_0|} \Theta(1) + \; \frac{1}{2\pi\, i} \int_\Gamma \frac{f(u)}{u^{j+1}}\, \mathrm{d}u, \tag{18}$$

where $z_0 = \rho(\cos\phi + \sin\phi\, i)$, with $0 < \phi < \pi$, and $x_0 = (1/w, 0)$ is the limit of z_0 as α_c tends to α. The rightmost term's absolute value is upper bounded by the maximum of $|f(z)|$ on Γ, times 0.9^{-j}; its influence turns out to be negligible.

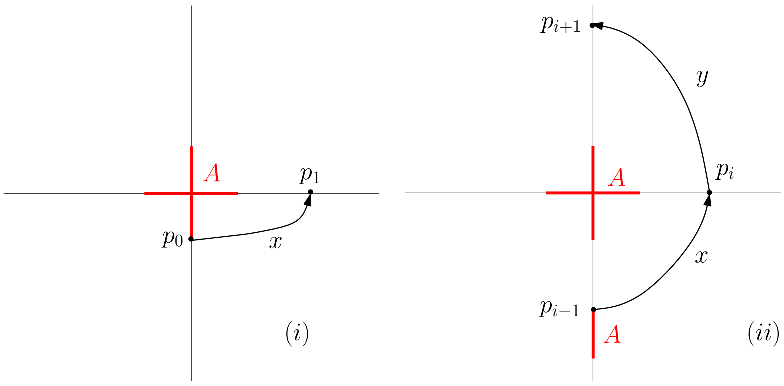

Figure 7 Proof of Lemma 7.

The oscillation $\sin(t\phi + p)$ has wavelength $2\pi/\phi$. For j near its negative minimum, the value of 18 becomes negative. This proves that the fire fighter will succeed in containing the fire in round j, for some $j \leq c \cdot 2\pi/\phi$ (in fact, one can choose $c = 1$). As α decreases towards α_c, both ϕ and phase p tend to zero, but

$$\lim_{\alpha \to \alpha_c} \frac{p}{\phi} \approx 1.315$$

holds. This value denotes how much the graph of $\sin(t\phi+p)$ is shifted to the left, as compared to $\sin t$. We see that j must increase through almost the whole positive halfwave of $\sin(t\phi+p)$ before negative values can occur. Since wavelength $2\pi/\phi$ goes to infinity, so does the number of rounds the fire fighter needs. This completes the proof of Theorem 1. All details are given in the Appendix of the technical report [7].

7 Lower bound

Let us recall that a barrier building strategy S is *spiralling* if it starts on the boundary of a fire of radius A, and visits the four coordinate half-axes in counterclockwise order and at increasing distances from the origin.

Now let S be a spiralling strategy of maximum speed $v \leq (1+\sqrt{5})/2 \approx 1.618$, the golden ratio. We can assume that S proceeds at constant speed v. Let $p_0, p_1, p_2, \ldots$ denote the points on the coordinate axes visited, in this order, by S. The following lemma shows that S cannot succeed because there is still fire burning outside the barrier on the axis previously visited.

▶ **Lemma 7.** *Let A be the initial fire radius. When S visits point p_{i+1}, the interval $[p_i, p_i + sign(p_i)A]$ on the axis visited before is on fire.*

Proof. The proof is by induction on i. Suppose strategy S builds a barrier of length x between p_0 and p_1, as shown in Figure 7 (i). During this time the fire advances x/v along the positive X-axis, so that $A + x/v \leq p_1 \leq x$ must hold, or

$$\frac{x}{v} \geq \frac{1}{v-1}A > A;$$

the last inequality follows from $v < 2$. Thus, the fire has enough time to move a distance of A from p_0 downwards along the negative Y-axis.

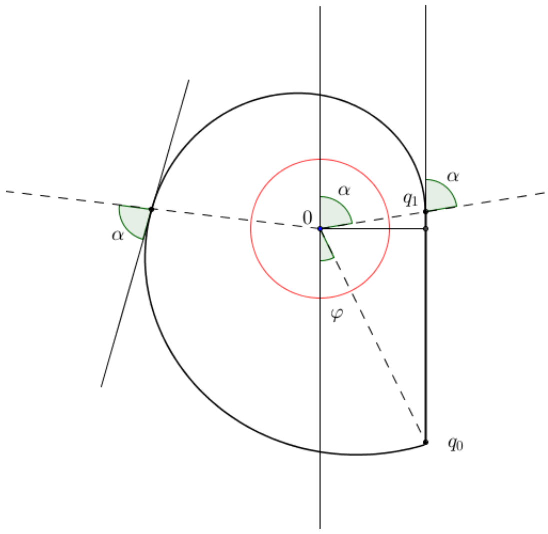

Figure 8 A completion time optimal single closed loop solution for $v \approx 6.25$ starts with a line segment outside the fire and ends with a logarithmic spiral along the boundary of the fire. A single loop solution exists only for $v \geq 3.7788\ldots$

Now let us assume that strategy S builds a barrier of length y between p_i and p_{i+1}, as shown in Figure 7 (ii). By induction, the interval of length A below p_{i-1} is on fire. Also, when the fighter moves on from p_i, there must be a burning interval of length at least $A+x/v$ on the positive Y-axis which is not bounded by a barrier from above. This is clear if p_{i+1} is the first point visited on the positive Y-axis, and it follows by induction, otherwise. Thus, we must have $A + x/v + y/v \leq p_{i+1} \leq y$, hence

$$\frac{y}{v} \geq \frac{1}{v-1}A + \frac{1}{v(v-1)}x > A + x,$$

since the assumption on v implies $v^2 \leq v + 1$. This shows that the fire can crawl along the barrier from p_{i-1} to p_i, and a distance A to the right, as the fighter moves to p_{i+1}, completing the proof of Theorem 2. ◀

8 Conclusions

A number of interesting questions arise. Are there strategies that can contain the fire at a speed $v < v_c$? How about starting points away from the fire? Given a speed $v \geq v_c$, there can be many barrier curves that contain a fire. Which one should the figther choose, to minimize the time to completion, or the area burned? Is it possible to generalize to fires of more realistic shapes, as they result under the influence of wind as for example suggested in [5]? These problems define a new and nice area in the field of path planning in dynamic environments, where obstacle shapes depend on the agent's actions.

For practical purposes, one would wish for a strategy that contains the fire in a single closed round. Also, starting points away from the fire could be allowed. If the fighter is free to pick her starting point she can contain the fire in a single closed round if, and only if, her speed is at least $v \geq 3.7788\ldots$ In this case the shortest possible (i.e., completion time optimal) solution consists of a line segment q_0q_1 followed by a segment of a logarithmic spiral of excentricity α, where $v = \frac{1}{\cos(\alpha)}$. See Figure 8 for an example of the time optimal single closed loop for $\alpha = 1.41$ and $v \approx 6.25$.

A single closed loop solution only exists for

$$\alpha > \arctan\left(\frac{\frac{3}{2}\pi}{W\left(\frac{3}{2}\pi\right)}\right) \approx 74.66^\circ$$

in which W denotes Lambert's W function [1] defined by the functional equation $W(x)\, e^{W(x)} = x$. This gives $\alpha \geq 1.3029\ldots$ or $v \geq 3.7788\ldots$

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