Non-Constructivity in Kan Simplicial Sets

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─────── **Abstract** ───────

We give an analysis of the non-constructivity of the following basic result: if X and Y are simplicial sets and Y has the Kan extension property, then Y^X also has the Kan extension property. By means of Kripke countermodels we show that even simple consequences of this basic result, such as edge reversal and edge composition, are not constructively provable. We also show that our unprovability argument will have to be refined if one strengthens the usual formulation of the Kan extension property to one with explicit horn-filler operations.

1   Introduction

Brouwer's Programme is the constructive reformulation of (as much as possible of) classical mathematics. In [2] it has been shown that the following theorem, though classically true (cf. [10, Corollary 7.11]), cannot be proved constructively.

▶ **Theorem 1** (classical). *The fibers of 0 and 1 of a Kan fibration $p : E \to \Delta^1$ are homotopy equivalent.*

In this paper we show that the following basic theorems cannot be proved constructively.

▶ **Theorem 2** (classical). *If X and Y are Kan simplicial sets, then any edge in Y^X can be reversed.*

▶ **Theorem 3** (classical). *If X and Y are Kan simplicial sets, then compatible edges in Y^X can be composed.*

The above two theorems follow immediately and constructively from the following.

▶ **Theorem 4** (classical). *If X and Y are Kan simplicial sets, then also Y^X is so.*

Hence we obtain that also Theorem 4, though classically true even without requiring that X is Kan (cf. [10, Theorem 6.9]), cannot be proved constructively.

The importance of these results is twofold. First, it is of evident importance for Brouwer's Programme to understand which results of classical mathematics already are constructive and which results are not. Second, Theorem 4 plays a crucial role in the construction of models of type theory with the Univalence Axiom, see [7]. The use of classical logic in proving this crucial property implies in particular that the model construction cannot be used to give a computational interpretation of univalence. Actually, Theorem 4 is a necessary step in the

semantics of the simply typed λ-calculus based on Kan simplicial sets. In what follows we expand on these points; for more motivation we refer to [2].

We would like to use the occasion to say a few not-too-technical words on the role of the Kan extension property of simplicial sets in relation to univalence. Let MLTT be Martin-Löf type theory with universe U and inductive equality $=_U$ on U. Assume we have two distinct copies of the natural numbers, inductively defined by constructors $0 : N$ $(0' : N')$ and $S : N \to N$ $(S' : N' \to N')$. MLTT proves $N =_U N$ and $N' =_U N'$, but not $N =_U N'$. The Univalence Axiom (UA) implies that (homotopy) equivalent types are equal, and in particular $N =_U N'$. By the Leibniz property of inductive equality, this implies that N and N' have the same properties and that all structure on N can be transported to N' and vice versa. This holds even uniformly, for example, $\Pi P : U \to U. (PN \to PN')$ is inhabited under UA. On the other hand, without UA, $\Pi P : U \to U. (PN \to PN')$ is not inhabited in MLTT. (One reason is that MLTT has models in which $N \neq N'$, so one can take $P \equiv \lambda X : U. (N =_U X)$ and get PN but not PN').

The above observation concerns not only the rather artificial type N' but also any other type that is equivalent to N, such as the type of lists over a unit type with one object. In fact the observation concerns all equivalent types. A less artificial example is perhaps the equivalence of the unit type to $\Sigma x : A. (a =_A x)$ for given $a : A : U$. The upshot is that validating UA requires an interpretation of $=_U$ that carries much more information than in MLTT without UA, since the elimination rule for $=_U$ (roughly, the Leibniz property, or substitutivity of equals for equals) has to be much stronger. In our simple example, the interpretation of $=_U$ must be leveraged to give an inhabitant of $\Pi P : U \to U. (PN \to PN')$.

Simplicial sets can be used to build a presheaf-style [6] model of MLTT. In this model the interpretation of $=_U$ does not validate UA. It turns out that if one builds a model of MLTT based on Kan simplicial sets, then it is possible to validate UA. The crucial notion here is that of a Kan fibration. A Kan *fibration* $p : E \to B$ is a map of simplicial sets with a specific lifting property. This lifting property *lifts* a path from b_0 to b_1 in B to a transport function from the fiber $p^{-1}(b_0)$ to the fiber $p^{-1}(b_1)$. In the model based on Kan simplicial sets, an inhabitant of $N =_U N'$ is interpreted as a path from N to N' in U. (Here and below we omit the correct but tedious phrase *the interpretation of* $N, N', U, \ldots$). Any $P : U \to U$ is interpreted as a Kan fibration with fibers PT for any $T : U$. (NB the fibration, being a projection on the base type, has a direction opposite to the arrow in $P : U \to U$). Then the transport function obtained from the lifting property is the desired function $PN \to PN'$. In short, one can say that the transport functions interpret substitutivity of equals by equals.

Finally, to come back to the topic of this paper: if all types are to have Kan structure, one has to prove this inductively following the rules of type formation. One of the induction steps is Theorem 4. The unprovablility of Theorem 4 shows that, from the constructive point of view, there is a problem with using the exponent Y^X in the category of Kan simplicial sets to interpret function types $X \to Y$.

The type theoretic (synthetic) formulation of homotopy equivalence and the Univalence Axiom, as well as the model of MLTT plus UA using Kan simplicial sets are all due to Voevodsky [14, 7]. This model confirms the homotopical interpretation proposed by Awodey and Warren [1].

Theorem 4 (without requiring that X is Kan) has an interesting history. The first appearance seems to be [12, Appendix A, p. 1A-8, Theorem 3]. Moore credits A. Heller for the definition of the function space Y^X on page 1A-4. Moore's proof is combinatorial, using the excluded middle in distinguishing the cases a non/degenerate on page 1A-9, l. 17ff. (Typo: on page 1A-7, l. 12 and 15, the map F is missing on the rhs; evidently $F_{(\mu,\nu)}$ was

intended to depend on F.) The proof in [10, Theorem 6.9] is much the same as the one by Moore (with the F's in place). Several variations of this argument can be found in the literature.

An essentially more abstract proof using anodyne extensions is given by Gabriel and Zisman in [4, Chapter Four, 3.1.2] (take $B = \Delta^0$). Here the classical reasoning shows up when in 2.1.2 amalgamated sums over sets of non-degenerate simplices are taken.

The results of Moore and Heller imply that Kan simplicial sets form a cartesian closed category, which can be seen as a germ of the fact that they model dependent type theory.

The rest of the paper is structured as follows. In Section 2 we give an introduction to simplicial sets, and in Section 3 we provide several examples of simplicial sets which will be in use in the rest of the article. In Section 4 we take a closer look at Theorem 2, and provide a Kripke model showing that a constructive consequence, Lemma 14 cannot be proven constructively. Section 5 deals with edge compostion, much in the same way as Section 4 deals with edge reversal. A summary and evaluation of the results obtained so far is given in Section 6. In Section 7 we strengthen the Kan condition and prove constructively a weak version of Lemma 14. This shows that our unprovability argument will have to be refined for the stronger Kan condition. We sum up our findings and discuss further research in Section 8.

2   Preliminaries

▶ **Definition 5** (Simplicial set). A *simplicial set* A is a collection of sets $A[i]$ for $i \in \mathbb{N}$ such that for every $0 < n$ and $j \leq n$ we have a function (*face map*) $d_j^n : A[n] \to A[n-1]$, and for every $0 \leq n$ and $j \leq n$ we have a function (*degeneracy map*) $s_j^n : A[n] \to A[n+1]$, satisfying the following *simplicial identities* for all suitable superscripts, which we happily omit:

$$d_i d_j = d_{j-1} d_i \qquad\qquad\qquad \text{if } i < j \qquad\qquad (1)$$
$$d_i s_j = s_{j-1} d_i \qquad\qquad\qquad \text{if } i < j \qquad\qquad (2)$$
$$d_i s_j = id \qquad\qquad\qquad \text{for } i = j, j+1 \qquad\qquad (3)$$
$$d_i s_j = s_j d_{i-1} \qquad\qquad\qquad \text{if } i > j+1 \qquad\qquad (4)$$
$$s_i s_j = s_j s_{i-1} \qquad\qquad\qquad \text{if } i > j \qquad\qquad (5)$$

An element of $A[i]$ is called an *i-simplex*, or just simplex when we don't wish to stipulate the dimension. A *degenerate* element is any element $a \in A[i+1]$ in the image of a degeneracy map.

Note that a simplicial identity like, e.g., $d_i^n d_j^{n+1} = d_{j-1}^n d_i^{n+1}$ actually means

$$\forall x \in A[n+1].\ d_i^n(d_j^{n+1}(x)) = d_{j-1}^n(d_i^{n+1}(x)).$$

With a countably infinite signature, the above definition can be expressed completely in many-sorted first-order logic. That means that we can see first-order models which satisfy the above requirement as simplicial sets, and instead of simplicial sets we could talk about first-order models satisfying the above requirements.

Simplicial sets form a category. For two simplicial sets A and B, $Hom_S(A, B)$ is the set of all natural transformations from A to B. A natural transformation is a collection of maps $g[n] : A[n] \to B[n]$ commuting with the face and degeneracy maps of A and B: $g[n]s_i = s_i g[n-1]$ for all $0 \leq i < n$ and $g[n+1]d_i = d_i g[n]$ for all $0 \leq i \leq n+1$. We freely omit the dimension $[n]$ when it can be inferred from the other arguments. For more information on simplicial sets we refer to, for example, [10, 5, 3].

▶ **Definition 6** (Kan simplicial set). A simplicial set Y satisfies the Kan condition if for any collection of simplices $y_0, \ldots, y_{k-1}, y_{k+1}, \ldots, y_n$ in $Y[n-1]$ such that $d_i y_j = d_{j-1} y_i$ for any $i < j$ with $i \neq k$ and $j \neq k$, there is an n-simplex y in Y such that $d_i y = y_i$ for all $i \neq k$. The Kan condition is also called the Kan extension property, and a simplicial set is called a *Kan simplicial set* if it satisfies the Kan condition.

▶ **Definition 7** (Kan graph). A *reflexive multigraph* consists of C_1, C_0, d_0, d_1, s where C_0 is a set of points, C_1 a set of edges, $d_i : C_1 \to C_0$, d_1 the *source* and d_0 the *target* function, and $s : C_0 \to C_1$ the function mapping each $c \in C_0$ to a selfloop of c. We write $e : a \to b$ if e is in C_1 such that $d_1(e) = a$ and $d_0(e) = b$ (note the direction!). In particular we have $d_i(s(c)) = c$ for all $c \in C_0$. A *Kan graph* is a reflexive multigraph having the property that for all a, b, c in C_0, if $e : a \to b$ and $f : a \to c$, then there exists an edge $g : b \to c$ in C_1.

Kan graphs can be viewed as truncated Kan simplicial sets, modelling a truncated proof-relevant equality relation. Note that we don't require the Kan graph to have explicit functions giving the required edges like in [2], we merely require that the edges exists. We discuss this distinction further in Section 7. The special requirement of the edges for the Kan graph is in the literature often called *Euclidean*. Euclidean combined with reflexivity gives both transitivity and symmetry.

3 Examples of simplicial sets

We give some examples of simplicial sets that are used in the sequel.

3.1 Standard simplicial k-simplex Δ^k

Δ^k is the simplicial set with $\Delta^k[j]$ consisting of all non-decreasing sequences of numbers $0, \ldots, k$ of length $j + 1$. Equivalently, $\Delta^k[j]$ is the set of order-preserving functions $[j] \to [k]$, where $[i]$ denotes $0, \ldots, i$ with the natural ordering. Examples are $\Delta^1[0] = \{0, 1\}$, $\Delta^1[1] = \{00, 01, 11\}$, $\Delta^2[1] = \{00, 01, 02, 11, 12, 22\}$ and

$$\Delta^2[2] = \{000, 001, 002, 011, 012, 022, 111, 112, 122, 222\}.$$

The degeneracy map $s_k^j : \Delta^i[j] \to \Delta^i[j+1]$ duplicates the k-th element in its input. So, $s_k^j(x_0 \ldots x_k \ldots x_{j+1}) = x_0 \ldots x_k x_k \ldots x_{j+1}$. The face map $d_k^j : \Delta^i[j] \to \Delta^i[j-1]$ deletes the k-th element. So, $d_k^j(x_0 \ldots x_j) = x_0 \ldots x_{k-1} x_{k+1} \ldots x_j$.

3.2 The k-horns Λ_j^k

Λ_j^k is the j'th horn of the standard k-simplex Δ^k, and defined by $\Lambda_j^k[n] = \{f \in \Delta^k[n] \mid [k] - \{j\} \not\subseteq Im(f)\}$. Alternatively, it is $\Delta^k[n]$ except every element must avoid some element not equal to j. For example, $\Lambda_0^2[1] = \{00, 01, 02, 11, \cancel{12}, 22\} = \Delta^2[1] - \{12\}$ (excluding 12, since 12 does not avoid any element not equal to 0). We also have:

$$\Lambda_0^2[2] = \{000, 001, 002, 011, \cancel{012}, 022, 111, \cancel{112}, \cancel{122}, 222\}.$$

The Kan extension condition for a simplicial set Y can also be formulated as: every map $F : \Lambda_j^k \to Y$ can be extended to a map $F' : \Delta^k \to Y$. This is equivalent to Definition 6.

3.3   Cartesian products

For two simplicial sets A and B, $A \times B$ is the simplicial set given by $(A \times B)[i] = A[i] \times B[i]$, and the structural maps d and s use d^A and d^B component-wise (and likewise for s^A and s^B). So if $a \in A[i]$ and $b \in B[i]$ then $(a, b) \in (A \times B)[i]$, and $d_i((a, b)) = (d_i^A(a), d_i^B(b))$. In particular, the degenerate simplices of $A \times B$ are pairs $(s_j^A(a), s_j^B(b)) \in (A \times B)[i + 1]$. (Caveat: this is stronger than both components being degenerate.)

3.4   Function spaces

We give the standard definition [12, p. 1A-4]: Y^X is the simplicial set given by $Y^X[i] = Hom_S(\Delta^i \times X, Y)$, where Hom_S denotes morphisms (natural transformations) of simplicial sets, and structural maps as follows. The face maps $d_k[i] : Y^X[i] \to Y^X[i - 1]$ need to map elements of $Hom_S(\Delta^i \times X, Y)$ to $Hom_S(\Delta^{i-1} \times X, Y)$ and the degeneracy maps vice versa. For their definition it is convenient to view a k-simplex in Δ^i as an order-preserving function $a : [k] \to [i]$. Let d_k^* be the strictly increasing function on natural numbers such that $d_k^*(n) = n$ if $n < k$ and $d_k^*(n) = n + 1$ otherwise (d_k^* 'jumps' over k). Given $F \in Hom_S(\Delta^i \times X, Y)$, define $(d_k F)[i](a, x) = F[i](d_k^* a, x)$. For the degeneracy maps, let s_k^* be the weakly increasing function on natural numbers such that $s_k^*(n) = n$ if $n \leq k$ and $s_k^*(n) = n - 1$ otherwise (s_k^* 'duplicates' k). Then define $(s_k F)[i](a, x) = F[i](s_k^* a, x)$.

3.5   The simplicial set defined by a reflexive multigraph

The following definition from [2] gives the general construction of a simplicial set from a reflexive multigraph. It is important to note that, even if the reflexive multigraph is transitive, its simplicial set is not the same as the nerve [5, Example 1.4] of the category defined by the multigraph. The difference is subtle: if we have edges $f : x \to y$, $g : y \to z$, $h, h' : x \to z$, where the composition $gf = h$, then the nerve does not contain the 2-simplex with f, g, h', in contrast to below.

▶ **Definition 8.** Given a reflexive multigraph C we define the simplicial set $S(C)$ as follows. $S(C)[0] = C_0$, $S(C)[1] = C_1$ and $S(C)[n]$, for $n \geq 2$, consisting of all tuples of the form $(u_0, \ldots, u_n; \ldots, e_{ij}, \ldots)$ such that

$$e_{ij} : u_i \to u_j \text{ in } C_1 \text{ for all } 0 \leq i < j \leq n.$$

The maps d_k in $S(C)$ are defined by removing from $(u_0, \ldots, u_n; \ldots, e_{ij}, \ldots)$ the point u_k and all edges e_{ik} and e_{kj}. The maps s_k in $S(C)$ are defined by duplicating the point u_k in $(u_0, \ldots, u_n; \ldots, e_{ij}, \ldots)$, adding an edge $e_{k(k+1)} = s(u_k)$, and duplicating edges and incrementing indices of edges as appropriate. This completes the construction of the simplicial set $S(C)$.

We now see why Kan graphs are named as they are: the S construction above turns them into Kan simplicial sets.

▶ **Lemma 9.** $S(Y)$ *is a Kan simplicial set whenever Y is a Kan graph.*

Proof. Consider Λ_k^n for some $n \geq 1$ and $0 \leq k \leq n$ and let $f : \Lambda_k^n \to S(Y)$. We have to define a lifting $h : \Delta^n \to S(Y)$. Δ^n consists of elements in every dimension, but we only need to specify h for every element in $\Delta^n[n]$. This since both the higher and lower dimensional objects are the (possibly repeated) s_i or d_j images of objects in $\Delta^n[n]$, and h must commute with both s_i and d_j, which determines h.

If $n = 1$, note that that Λ_k^1 only consists of one point, and degenerations of that point in the higher dimensions. E.g., if $k = 0$ then $\Lambda_0^1[0] = \{0\}$, $\Lambda_0^1[1] = \{00\}$ etc. In that case we extend f to $h : \Delta^1 \to S(Y)$ by mapping $h(1) = h(0) = f(0)$, which determines h in higher dimensions.

If $n = 2$ we use the fact that Y is a Kan graph, so for any two edges $f : a \to b$ and $g : a \to c$ there is an edge from b to c. The 2-horn gives two edges in the graph with at least one common point, and the fact that the graph is both reflexive, symmetric and transitive (because of the Kan property) enables us to find a third edge with compatible endpoints. The procedure depends on the value of k. We will here give the procedure for $k = 2$; $k = 0, 1$ are just simple adaptations.

Given $f : \Lambda_2^2 \to S(Y)$ we have edges $f(02) : f(0) \to f(2)$ and edges $f(12) : f(1) \to f(2)$, and we need to find a value for $h(01) : f(0) \to f(1)$ such that $d_1 h(01) = d_1 f(02)$ and $d_0 h(01) = d_1 f(12)$. In other words, we need to find that the dotted edge in the diagram actually exists (self-loops are not displayed).
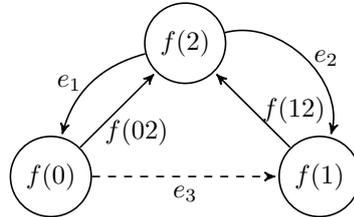
Recall that $S(Y)[1] = Y[1]$, so both $f(01)$ and $f(12)$ are actual edges in Y. By applying the Kan property on $s(f(0))$ and $f(02)$ we get an edge $e_1 : f(2) \to f(0)$. Similarly we get an edge $e_2 : f(2) \to f(1)$. Now, by using the Kan property on e_1 and e_2 we get an edge $e_3 : f(0) \to f(1)$, and we put $h(01) = e_3$.

Finally, if $n \geq 3$ we observe that the horn Λ_k^n contains all points and edges of Δ^n, and we define the lifting by

$$ h(q) = (f_{[0]}(q(0)), \ldots, f_{[0]}(q(m)); \ldots, f_{[1]}(e_{ij}), \ldots). $$

Here $q : [m] \to [n]$ is order-preserving and e_{ij} is the edge from $q(i)$ to $q(j)$ in $\Delta^n[1] = \Lambda_k^n[1]$. ◀

4 Edge reversal

In this section we give the classical proof of Theorem 2 and show that there is no constructive proof.

4.1 Edge reversal, definition and classical proof

▶ **Definition 10** (Edge reversal). A simplicial set Y is said to have *edge reversal* when for every edge $e \in Y[1]$ there exists an edge $f \in Y[1]$ with $d_1(f) = d_0(e)$ and $d_0(f) = d_1(e)$.

▶ **Lemma 11.** *Kan simplicial sets have edge reversal.*

Proof. Given an arbitrary Kan simplicial set Y and an edge $e \in Y[1]$ we can make a map $G : \Lambda_0^2 \to Y$ by letting $G(0) = G(2) = d_1(e)$, $G(1) = d_0(e)$, $G(01) = e$ and $G(02) = s(d_1(e))$. Since Y is Kan we can extend G to $G : \Delta^2 \to Y$, giving us a value for $G(12) \in Y[1]$, which must be an edge between $G(1)$ and $G(2) = G(0)$, giving the reverse edge. ◀

We introduce some convenient ad-hoc terminology for later use.

▶ **Definition 12** (Y^X-*good*)**.** Let X and Y are reflexive multigraphs and $F_{01} : X[1] \to Y[1]$. Define $F_0 = d_1 F_{01} s : X[0] \to Y[0]$ and $F_1 = d_0 F_{01} s : X[0] \to Y[0]$. We say that F_{01} is Y^X-*good* when the following two requirements hold for $i = 0, 1$:

- For all $e, e' \in X[1]$, if $d_i(e) = d_i(e')$ then $d_i F_{01}(e) = d_i F_{01}(e')$;
- For all $e \in X[1]$, $F_i d_0(e) = F_i d_1(e)$.

The first requirement expresses that F_{01}, F_0, F_1 respect endpoints, that is, if $e : a \to b$ in $X[1]$, then $F_{01}(e) : F_0(a) \to F_1(b)$ in $Y[1]$. The second requirement ensures that F_0 and F_1 are constant on each weakly connected component of X. (Notice that $F_{01}s(y)$ for $y \in Y[0]$ does not need to map to a degenerate edge, so F_0 and F_1 are not necessarily identical.)

▶ **Lemma 13.** *If X and Y are reflexive multigraphs and $F_{01} : X[1] \to Y[1]$ is Y^X-good, then we can extend F_{01} to a 1-simplex in $S(Y)^{S(X)}$.*

Proof. To be a map in $S(Y)^{S(X)}$ we need to extend $F_{01} : X[1] \to Y[1]$ to a family of maps $F'_{01}[n] : (\Delta^1 \times S(X))[n] \to S(Y)[n]$ which commute with d_i and s_j. Recall the definitions $F_0 = d_1 F_{01} s$ and $F_1 = d_0 F_{01} s$. We define $F'_{01}[n]$ depending on n. If $n = 0$ then the input will have the form (i, x) where $0 \le i \le 1$ and $x \in X[0]$, and we put $F'_{01}[0](i, x) = F_i(x)$. If $n = 1$ the input will have the form (ij, e) where $0 \le i \le j \le 1$ and $e \in X[1]$. If $i = j$ we put $F'_{01}(ij, e) = s F_i(d_0(e))$. Note that since F_{01} is Y^X-good, we know that $F_i(d_0(e)) = F_i(d_1(e))$, justifying our choice of the degenerate edge as the output. If $i < j$ we let $F'_{01}(01, e) = F_{01}(e)$. If $n > 1$ any input to $F'_{01}[n]$ will have the form $(0^a 1^b, (x_0, \ldots, x_n; \ldots e_{ij}, \ldots))$ such that $a + b = n + 1$. We let $F'_{01}[n]$ map this element to the tuple

$$(F_0(x_0), \ldots, F_0(x_{a-1}), F_1(x_a) \ldots, F_1(x_{a+b-1}); \ldots e'_{ij}, \ldots),$$

where $e'_{ij} = s(F_0(x_a))$ if $i < j < a$, $e'_{ij} = F_{01}(e_{ij})$ if $i < a \le j$, and $e'_{ij} = s(F_1(x_a))$ if $a \le i < j$. That is, the $F'_{01}[n]$ images are sequences of a number of F_0 images followed by b number of F_1 images, with all edges being degenerate, except the bridges between the two nodes. Since each of the derived F_i functions are constant on each connected component, and the input consists exactly of sequences of nodes in the same connected component, all of the elements $F_0(x_0), \ldots, F_0(x_{a-1})$ are the same element in $Y[0]$, and likewise for $F_1(x_a) \ldots, F_1(x_{a+b-1})$. This justifies our choice of e'_{ij} as the degenerate edges.

It should be clear that this map does indeed commute with d_i and s_j, completing the proof.   ◀

▶ **Lemma 14** (classical)**.** *For all Kan graphs Y and X, if $F_{01} : X[1] \to Y[1]$ is Y^X-good, then there is an $F_{10} : X[1] \to Y[1]$ such that $d_0 F_{01} = d_1 F_{10}$ and $d_1 F_{01} = d_0 F_{10}$.*

Proof. Let X and Y be Kan graphs. The $S(Y)$ and $S(X)$ are Kan simplicial sets by Lemma 9. By applying the classical Theorem 2 we get that $S(Y)^{S(X)}$ has edge reversal. Since F_{01} is Y^X-good we extend F_{01} to an edge $F'_{01} \in S(Y)^{S(X)}[1]$ as defined in the proof of Lemma 13. By edge reversal in $S(Y)^{S(X)}$ we get an $F'_{10} \in S(Y)^{S(X)}[1]$ satisfying $d_1(F'_{10}) = d_0(F'_{01})$ and $d_0(F'_{10}) = d_1(F'_{01})$. We put $F_{10}(x) = F'_{10}(01, x)$. By expanding the definition of d_k from Section 3.4, we get the following properties: $F'_{10}(00, e) = F'_{01}(11, e)$ and $F'_{10}(11, e) = F'_{01}(00, e)$, giving $F'_{10}(0, d_i(e)) = F'_{01}(1, d_i(e))$ and $F'_{10}(1, d_i(e)) = F'_{01}(0, d_i(e))$. We calculate $d_0 F_{01}(e) = d_0 F'_{01}(01, e) = F'_{01}(1, d_0(e)) = F_1 d_0(e)$. Since F_{01} is Y^X-good (2nd requirement) we have $F_1 d_0(e) = F_1 d_1(e)$. We continue the calculation: $F_1 d_1(e) = F'_{01}(1, d_1(e)) = F'_{10}(0, d_1(e))$ where the last step is justified above. We continue: $F'_{10}(0, d_1(e)) = d_1 F'_{10}(01, e) = d_1 F_{10}(e)$. In total we have proved $d_0 F_{01}(e) = d_1 F_{10}(e)$ for all $e \in X[1]$. Hence $d_0 F_{01} = d_1 F_{10}$. The other equation is proved symmetrically.   ◀

■ **Table 1** Kripke (counter)model for edge reversal.

	Day 1
X_0	$\{x, x'\}$
X_1	$\{s(x), s(x')\}$
Y_0	$\{y_0, y_1, y_0', y_1'\}$
Y_1	$\{s(y_0), s(y_1), s(y_0'), s(y_1'), y_0 y_1 : y_0 {\to} y_1, y_0' y_1' : y_0' {\to} y_1', a : y_1 {\to} y_0, b : y_1' {\to} y_0'\}$

	Day 2
X_0	$\{x{=}x'\}$
X_1	$\{s(x){=}s(x')\}$
Y_0	$\{y_0 = y_0', y_1 = y_1'\}$
Y_1	$\{s(y_0) = s(y_0'), s(y_1) = s(y_1'), y_0 y_1 = y_0' y_1', a, b\}$

	Input	Output
F_0	x	y_0
F_0	x'	y_0'
F_1	x	y_1
F_1	x'	y_1'
F_{01}	$s(x)$	$y_0 y_1$
F_{01}	$s(x')$	$y_0' y_1'$

Kripke [8] showed that constructive logic is sound for Kripke models, so the existence of a Kripke countermodel of a statement gives the non-existence of a constructive proof of that statement. We will now, by the means of a Kripke model, see that Lemma 14 does not hold constructively.

4.2 Edge reversal, the Kripke countermodel

We describe a Kripke model containing a Y^X-good F_{01} such that there cannot be a function $F_{10} : X[1] \to Y[1]$ with $d_0 F_{01} = d_1 F_{10}$ and $d_1 F_{01} = d_0 F_{10}$, even though X and Y are Kan graphs.

For clarity, the functions $F_0 = d_1 F_{01} s$ and $F_1 = d_0 F_{01} s$ as defined in Definition 12 are also made explicit in this model. Face maps are part of the model, but not made explicit.

The model consists of two days, with an X and a Y part each. On day 1 both X and Y consist of two separate components, which get merged on day 2. We give the model both in Table 1 and, graphically, in Figure 1 and 2.

It is easy to see that both X and Y are Kan graphs by simply observing that each of their two components are strongly connected. It is also clear that we cannot define a consistent F_{10}. In day 1 we would have to set $F_{10}(s(x)) = a$ and $F_{10}(s(x')) = b$ to satisfy the requirement that $d_0 F_{01} = d_1 F_{10}$ and $d_1 F_{01} = d_0 F_{10}$. The problem occurs in day 2, where we have that $s(x) = s(x')$, but $a \neq b$, making it impossible for F_{10} to respect equality. Note that all other functions, F_0, F_1, F_{01}, s, d_0, and d_1 remain consistent after collapsing, that is, they still map equal elements to equal elements.

5 Edge composition

In this section we give the classical proof of Theorem 3 and show that there is no constructive proof.

▶ **Definition 15** (Edge composition). A simplicial set Y is said to have *edge composition* when for every edge $e_1, e_2 \in Y[1]$, if $d_0(e_1) = d_1(e_2)$ then there exists an edge $f \in Y[1]$ with $d_1(f) = d_1(e_1)$ and $d_0(f) = d_0(e_2)$.

▶ **Lemma 16.** *Kan simplicial sets have edge composition.*
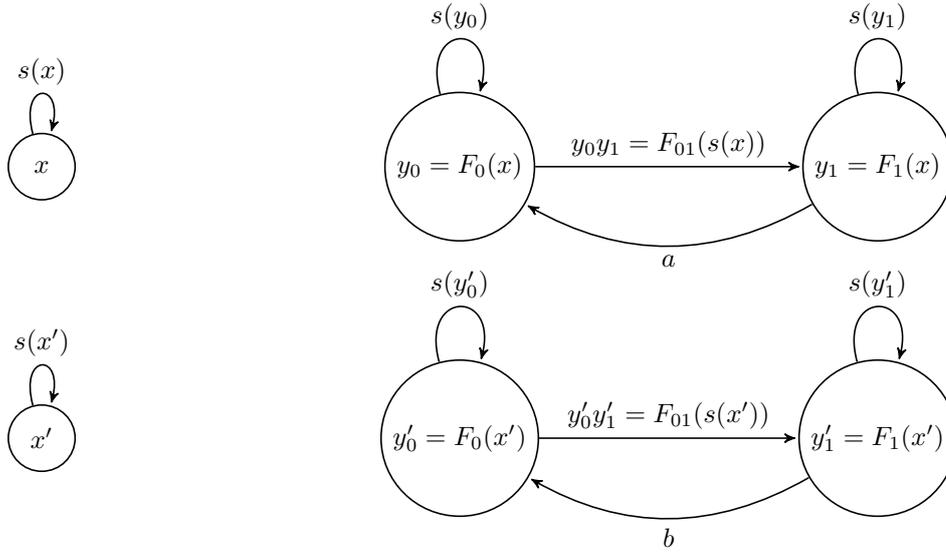
Figure 1 Kripke (counter)model for edge reversal, day 1.
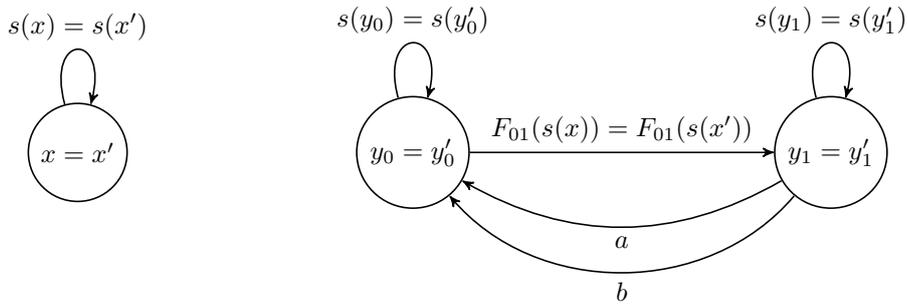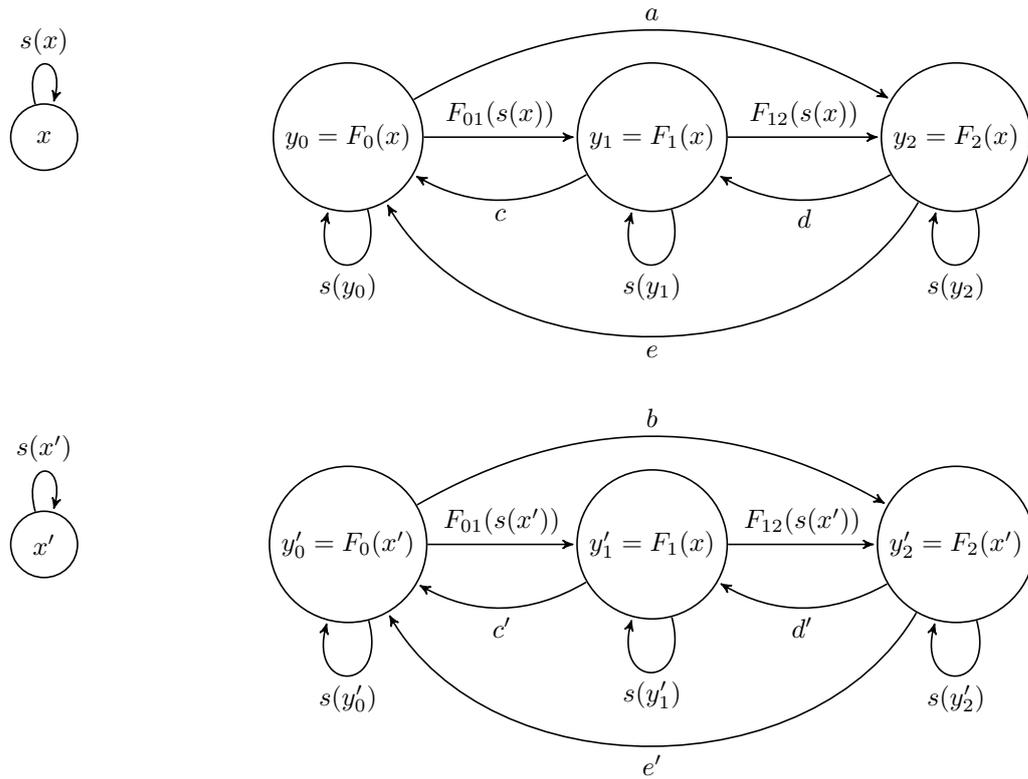

Figure 2 Kripke (counter)model for edge reversal, day 2.

Proof. Given an arbitrary Kan simplicial set Y and edges $e_1, e_2 \in Y[1]$ with $d_0(e_1) = d_1(e_2)$, we can make a map $G : \Lambda_1^2 \to Y$ by putting $G(0) = d_1(e_1)$, $G(1) = d_0(e_1)$, $G(2) = d_0(e_2)$ $G(01) = e_1$ and $G(12) = e_2$. Since Y is Kan we can extend G to $G : \Delta^2 \to Y$, giving us a simplex $G(02) : G(0) \to G(2)$ in $Y[1]$, the composition of e_1 and e_2. ◀

By a proof essentially identical to the proof of Lemma 14 we get the following lemma.

▶ **Lemma 17** (classical). *For all Kan graphs Y and X, if $F_{01} : X[1] \to Y[1]$ and $F_{12} : X[1] \to Y[1]$ are Y^X-good maps satisfying $d_0 F_{01} = d_1 F_{12}$, then there is an $F_{02} : X[1] \to Y[1]$ such that $d_0 F_{01} = d_0 F_{02}$ and $d_1 F_{12} = d_1 F_{02}$.*

In Figure 3 and 4 we see that Lemma 17 is not constructively provable. We have two Y^X-good functions F_{01} and F_{12}, satisfying the requirement, and both X and Y are Kan graphs. If $S(Y)^{S(X)}$ had edge composition we would get a function F_{02} that $d_1 F_{01} = d_1 F_{02}$ and $d_0 F_{12} = d_0 F_{02}$. However, such a function is not definable in the Kripke model. The reason is analogous to the case of edge-reversal: from day 1 to day 2 we have equated objects in the domain of F_{02} while keeping the images distinct. Specifically, on day 1 we are forced to set $F_{02}(s(x)) = a$ and $F_{02}(s(x')) = b$, but on day 2 we have $s(x) = s(x')$, but $a \neq b$.

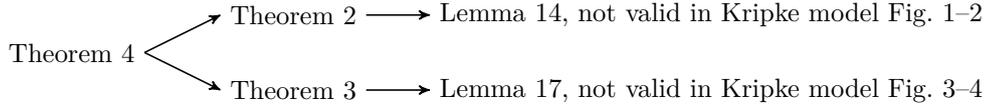**Figure 3** Kripke (counter)model for edge composition, day 1.
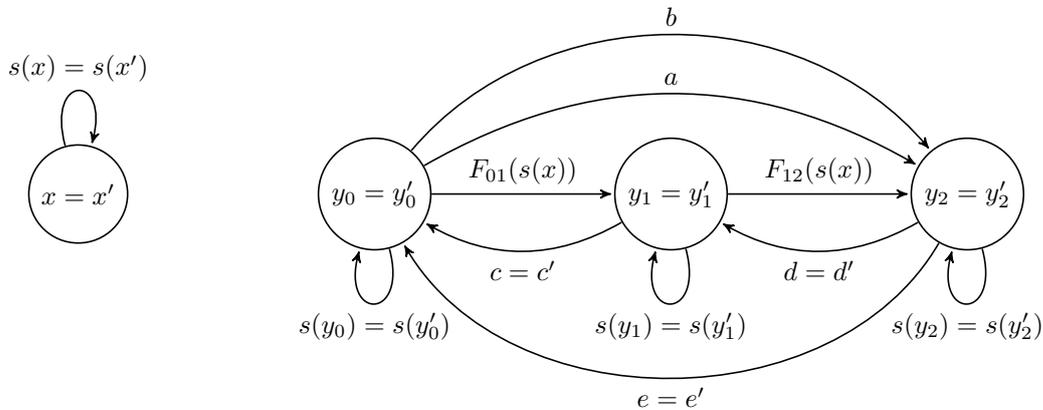
Figure 4 Kripke (counter)model for edge composition, day 2.

6 Evaluation of the results

The results up to now are summarized in Figure 5.

Having concrete, finite Kripke countermodels against Lemma 14 and 17 allows for a further simplification: everything remains valid under the condition that X has at most two points. Likewise, explicit bounds read off from the Kripke models can be imposed on the number of points of Y and on the number of edges in X and in Y. The simplified results are denoted by postfixing the number of the result by a 'b' for bounded, so Lemma 14b is the bounded version of Lemma 14.

Theorem 4 $\nearrow$ Theorem 2 $\longrightarrow$ Lemma 14, not valid in Kripke model Fig. 1–2

$\searrow$ Theorem 3 $\longrightarrow$ Lemma 17, not valid in Kripke model Fig. 3–4

Figure 5 Summary of results, all implications constructive.

With explicit bounds on the size of the domain, functions are completely determined by a finite number of function values. For example, if we have $\forall z \in X.\ (z = x \vee z = x')$ for $x, x' \in X$, then the binary predicate $\mathrm{fun}(y, y') \equiv (x = x' \rightarrow y = y')$ on Y completely describes all functions $X \rightarrow Y$, in evidence $x \mapsto y,\ x' \mapsto y'$. With this in mind it is not difficult to express Lemma 14b as a first-order classical tautology Φ that is not true in all Kripke models.

Now fix a constructive framework that is sufficiently expressive for the results in Figure 5. For example, IZF (Zermelo-Fraenkel set theory in IPL, intuitionistic predicate logic) will do. Let $[\![\Phi]\!]$ be the Tarski interpretation of Φ expressed in IZF. The following fundamental property of IZF could be called the semantic conservativity of IZF over IPL:

If $[\![\Phi]\!]$ is provable in IZF, then Φ is true in all Kripke models.

Lubarsky [9] and McCarty [11] independently provided constructive proofs of the above conservativity property of IZF. We gratefully acknowledge their prompt answers to our question.[1]

Empowered by the proofs of Lubarsky and McCarty we can now conclude that Lemma 14b cannot be proved in IZF. The same is true for Lemma 17b, and for all other results in Figure 5, as well as for their bounded versions.

7   Kan graphs with explicit filler functions

Let us first give an intuitive explanation of our countermodels. They actually exploit the undecidability of equality: on day 1 we don't know what will be equal on day 2. (This is different from the decidability of degeneracy, but the two are related: for example, an edge e is degenerate iff $e = s_0(d_1(e))$.) In Figure 1 and 2, the point is that $y_0 \neq y'_0$ on day 1, so one cannot put $F_{10}(s(x)) = F_{10}(s(x')) = a$ since this conflicts with $d_0 F_{10} = d_1 F_{01}$. One is thus forced to a choice that turns out to be wrong on day 2.

One attempt to deal with this lack of information is to give Kan simplicial sets more structure. One could for example change Definition 6 of a Kan simplicial set into one where we not only know that the required n-simplex exists, but actually have *functions* producing them. In the formulation using horns as in Section 3.2 this would amount to a dependent function $\mathrm{fill}(k, j, F)$ such that $\mathrm{fill}(k, j, F) : \Delta^k \rightarrow Y$ extends $F : \Lambda^k_j \rightarrow Y$, for any k, j, F. This form of Kan simplicial set has been introduced by Nikolaus in [13] under the name of *algebraic Kan complex*. The definition with explicit fill-functions has certain advantages, both classically and constructively, as we will see below. However, one should be careful in defining Y^X: morphisms in the category of algebraic Kan complexes are required to map chosen fillers in X to chosen fillers in Y. As a consequence, there are less maps from X to Y

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[1] Strengthening the semantic conservativity to syntactic conservativity, that is, concluding that Φ is provable in intuitionistic predicate logic, by using the completeness of the Kripke semantics implicates some classical logic. Although not needed for this paper, we think there is some general interest in a constructive proof that IPL $\vdash \Psi$ whenever IZF $\vdash [\![\Psi]\!]$, for any first-order sentence Ψ.

as algebraic Kan complexes than as just simplicial sets. What we propose could be called a *functional* Kan simplicial set, with explicit fill-functions but with maps as for ordinary simplicial sets. As a consequence the exponential Y^X of simplicial sets can be used.

To be able to prove an analogue of Lemma 9 we have to strengthen the notion of Kan graph to also include such *filler functions*, cf. [2].

▶ **Definition 18** (Kan fill-graph). A *Kan fill-graph* is a reflexive multigraph with a partial function fill : $Y[1] \times Y[1] \to Y[1]$ such that for all $e_1, e_2 \in Y[1]$, if $e_1 : a \to b$ and $e_2 : a \to c$, then $\text{fill}(e_1, e_2) : b \to c$.

As noted earlier, the Kan property together with reflexivity implies symmetry and transitivity. We can now define the corresponding functions.

▶ **Definition 19** (Edge reversal). For all $e \in Y[1]$ where Y is a Kan fill-graph let

$$e^{-1} = \text{fill}(e, sd_1(e)).$$

If $e : a \to b$, then $sd_1(e) : a \to a$, and $\text{fill}(e, sd_1(e)) : b \to a$.

Note that we in general don't have $(e^{-1})^{-1} = e$, but we do have that $d_i((e^{-1})^{-1}) = d_i(e)$.

▶ **Definition 20** (Edge composition). Using the inverse for edges in Y we define the composition of two edges $e_1 : a \to b$ and $e_2 : b \to c$ as

$$\text{trans}(e_1, e_2) = \text{fill}(e_1^{-1}, e_2).$$

Again we are in no way guaranteed that $\text{trans}(e_1, s(b)) = e_1$ or $\text{trans}(s(x), s(x)) = s(x)$.

We immediately see that the addition of explicit functions adds power, as we can now prove constructively and trivially an analogue of Lemma 14.

▶ **Lemma 21.** *For all Kan fill-graphs Y, X and for every $F : X[1] \to Y[1]$, the function $F^{-1} : X[1] \to Y[1]$ defined by $F^{-1}(e) = F(e)^{-1}$ satisfies $d_0F = d_1F^{-1}$ and $d_1F = d_0F^{-1}$.*

Note how using explicit functions rules out the Kripke counter-example we gave of Lemma 14. If $s(x) = s(x')$ on day 2, then we immediately get $a = F_{01}^{-1}(s(x)) = F_{01}^{-1}(s(x')) = b$ since equality has to be preserved.

We can even use the above fact to show that:

▶ **Lemma 22.** *For any reflexive multigraph X and Kan fill-graph Y, $S(Y)^{S(X)}$ has edge reversal.*

Proof. Assume an edge $F \in S(Y)^{S(X)}[1]$, we proceed to define F^{-1} such that $d_0(F) = d_1(F^{-1})$ and $d_1(F) = d_0(F^{-1})$. As $F \in S(Y)^{S(X)}[1]$ we have $F[n] : \Delta^1[n] \times X[n] \to Y[n]$. We start with $n = 0$, defining $F^{-1}[0] : (\Delta^1 \times X)[0] \to Y[0]$ by letting $F^{-1}[0](0, x) = F[0](1, x)$ and $F^{-1}[0](1, x) = F[0](0, x)$. Likewise for $n = 1$ we define $F^{-1}(00, e) = F(11, e)$ and $F^{-1}(11, e) = F(00, e)$, these are directly enforced by $d_0(F) = d_1(F^{-1})$ and $d_1(F) = d_0(F^{-1})$. For the case of $F^{-1}(01, e)$ we need to find an edge $F^{-1}(01, e) : F^{-1}(0, d_1e) \to F^{-1}(1, d_0e)$, which from the way we defined $F^{-1}[0]$ is the same as an edge

$$F^{-1}(01, e) : F(1, d_1e) \to F(0, d_0e).$$

The diagram in Figure 6 shows $e \in S(X)[1]$ with its endpoints on the left, and the nodes and edges we have directly reachable in $S(Y)$ using only F on the right.
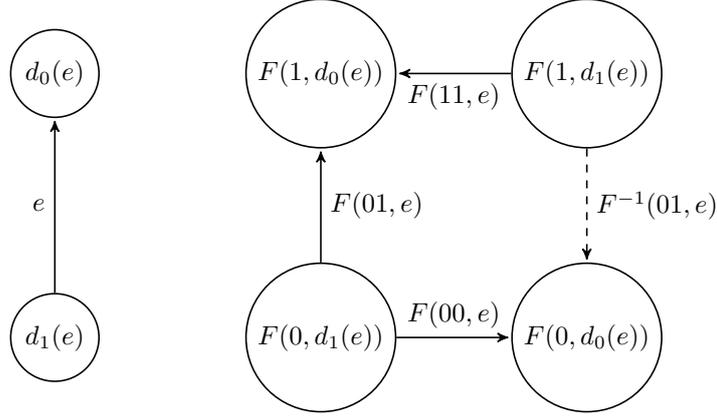
Figure 6 Reversing F.

Reading off the figure we can define $F^{-1}(01, e)$ as follows:

$$F^{-1}(01, e) = \operatorname{trans}(F(11, e)), \operatorname{fill}(F(01, e), F(00, e)))$$

Note that F^{-1} is well-defined since the functions involved in the definition are. Moreover, F^{-1} commutes with s_0, d_0, d_1 by construction.

Having defined F^{-1} for dimension 0 and 1, F^{-1} is also determined in higher dimensions, because of the truncation in $S(X), S(Y)$. In the case of $n > 1$ any input to $F^{-1}[n]$ will have the form

$$F^{-1}(0^a 1^b, (x_0, \ldots, x_n; \ldots e_{ij}, \ldots))$$

where $a + b = n + 1$. We let $F^{-1}[n]$ map this element to the tuple

$$(F^{-1}(0, x_0), \ldots, F^{-1}(0, x_{a-1}), F^{-1}(1, x_a) \ldots, F^{-1}(1, x_{a+b-1}); \ldots e'_{ij}, \ldots),$$

where $e'_{ij} = F^{-1}(00, e_{ij})$ if $i < j < a$, $e'_{ij} = F^{-1}(01, e_{ij})$ if $i < a \le j$, and $e'_{ij} = F^{-1}(11, e_{ij})$ if $a \le i < j$. This commutes with face and degeneracy maps. ◀

Using the same techniques we can constructively prove the following variant of Lemma 17.

▶ **Lemma 23.** *For any Kan graph X and Kan fill-graph Y, if $F_{01} : X[1] \to Y[1]$ and $F_{12} : X[1] \to Y[1]$ satisfy $d_0 F_{01} = d_1 F_{12}$, then there is a $F_{02} : X[1] \to Y[1]$ such that $d_1 F_{01} = d_1 F_{02}$ and $d_0 F_{12} = d_0 F_{02}$.*

▶ **Lemma 24.** *For any reflexive multigraph X and Kan fill-graph Y, $S(Y)^{S(X)}$ has edge composition.*

Proof. Assume edges $F_{01} \in S(Y)^{S(X)}$, $F_{12} \in S(Y)^{S(X)}$ such that $d_0(F_{01}) = d_1(F_{12})$, and we proceed to define $F_{02} \in S(Y)^{S(X)}$ such that $d_1(F_{02}) = d_1(F_{01})$ and $d_0(F_{02}) = d_0(F_{12})$.

As was the case in the proof of Lemma 22, we are forced on $F_{02}(0, x) = F_{01}(0, x)$, $F_{02}(1, x) = F_{12}(1, x)$, $F_{02}(00, e) = F_{01}(00, e)$, and $F_{02}(11, e) = F_{12}(11, e)$.

For the case of $F_{02}(01, e)$ we need to find an edge $F_{02}(01, e) : F_{02}(0, d_1 e) \to F_{02}(1, d_0 e)$, which from the way we defined $F_{02}[0]$ is the same as an edge

$$F_{02}(01, e) : F_{01}(0, d_1 e) \to F_{12}(1, d_0 e).$$

We note that $d_0(F_{01}) = d_1(F_{12})$ enforces $F_{01}(11, e) = F_{12}(00, e)$, which again enforces $F_{01}(1, d_i(e)) = F_{12}(0, d_i(e))$. This gives the diagram in Figure 7, enabling us to read off:

$$F_{02}(01, e) = \operatorname{fill}(\operatorname{trans}(F_{01}(11, e), F_{01}(01, e)^{-1}), F_{12}(01, e)). \quad ◀$$
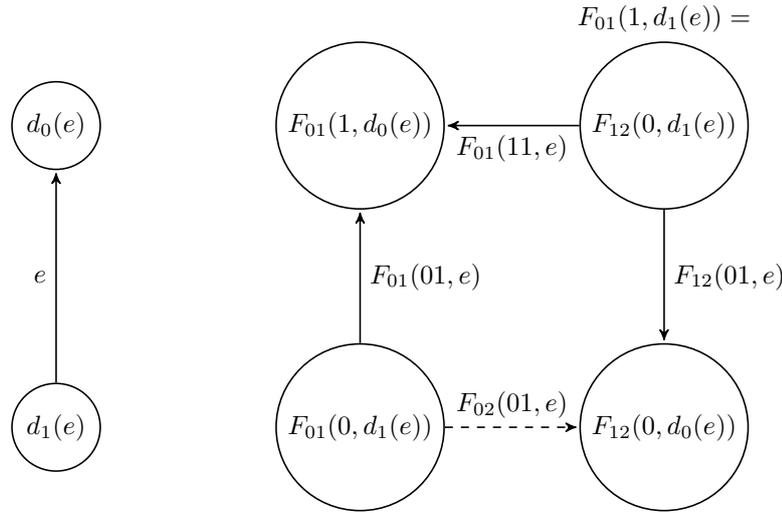
Figure 7 Filling the horn Λ^2_1.

8 Conclusions and Future Research

We have given a thorough analysis of the non-constructivity of the basic result that the Kan extension property is preserved under the usual operation of exponentiation of simplicial sets. An important step in this analysis, also employed in [2], is the truncation of simplicial sets to dimension 1. This allows us to study the basic result in the simplified situation of Kan graphs. Once one has shown the constructive unprovability of the basic result in the situation of Kan graphs, one obtains *a fortiori* its unprovability for Kan simplicial sets.

The much simpler notion of Kan graph (as compared to Kan simplicial set) invites to further thought experiments. One of those is the study of simple, constructive consequences of the Kan extension property, such as edge reversal and edge composition. It turns out that already these consequences cannot be proven constructively.

Another experiment is to strengthen the Kan extension property from existence of an n-simplex as in Definition 6 to having a function, called a *filler*, yielding these n-simplices. This makes quite a difference. None of the Kripke models we have introduced is able to deal with such fillers, since equating objects in X and Y implies that filler-values such as a and b in Figure 1 also have to be equal. The question arises whether this is necessary so, or just coincidental in the particular Kripke model. This question is answered in Section 7, where we prove constructively that, if X is a graph and Y a Kan-fill graph, then $S(Y)^{S(X)}$ has edge reversal and edge composition. This result may be of independent interest. It suggests that showing the (expected) constructive unprovability of Theorem 4 for *algebraic Kan complexes* as in [13] will require more complicated structures than graphs. The above expectation is based on an analysis of filling a 2-horn in Y^X, which requires defining $F(001, t)$. As F has to commute with s_0, one must know whether the 2-simplex t is an s_0-image or not. This can in general only be decided by an appeal to classical logic. We have to leave this to future research.

References

1   Steve Awodey and Michael Warren. Homotopy theoretic models of identity types. *Mathematical Proceedings of the Cambridge Philosophical Society*, 146:45–55, 2009.

2   Marc Bezem and Thierry Coquand. A Kripke model for simplicial sets. *TCS*, 2015. doi:10.1016/j.tcs.2015.01.035.

3   Greg Friedman. An elementary illustrated introduction to simplicial sets. Preprint, `http://arxiv.org/abs/0809.4221`, 2008.

4   Peter Gabriel and Michel Zisman. *Calculus of fractions and homotopy theory*. Springer, 1967.

5   Paul Goerss and J.F. Jardine. *Simplicial Homotopy Theory*. Modern Birkhäuser Classics. Birkhauser Verlag GmbH, 2009. Reprint of Vol. 174 of Progress in Mathematics, 1999.

6   Martin Hofmann. Syntax and semantics of dependent types. In A.M. Pitts and P. Dybjer, editors, *Semantics and logics of computation*, volume 14 of *Publ. Newton Inst.*, pages 79–130. Cambridge University Press, Cambridge, 1997.

7   Chris Kapulkin, Peter LeFanu Lumsdaine, and Vladimir Voevodsky. The simplicial model of univalent foundations. Preprint, `http://arxiv.org/abs/1211.2851`, 2012.

8   Saul Kripke. Semantical analysis of intuitionistic logic I. In M. Dummett and J.N. Crossley, editors, *Formal Systems and Recursive Functions*, pages 92–130. North–Holland, Amsterdam, 1965.

9   Bob Lubarsky, March 2015. Personal communication.

10 Jon Peter May. *Simplicial Objects in Algebraic Topology*. Chicago Lectures in Mathematics. University of Chicago Press, 2nd edition, 1993.

11 Charles McCarty. Two questions about IZF and intuitionistic validity. Pdf file, personal communication, March 2015.

12 John C. Moore. Algebraic homotopy theory. Lectures at Princeton, `http://faculty.tcu.edu/gfriedman/notes/aht1.pdf`, 1956.

13 Thomas Nikolaus. Algebraic models for higher categories. Preprint, `http://arxiv.org/abs/1003.1342`, 2010.

14 Vladimir Voevodsky. Notes on type systems. `http://www.math.ias.edu/~vladimir/Site3/Univalent_Foundations_files/expressions_current.pdf`, 2009.