Head reduction and normalization in a call-by-value lambda-calculus

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Abstract

Recently, a standardization theorem has been proven for a variant of Plotkin's call-by-value lambda-calculus extended by means of two commutation rules (sigma-reductions): this result was based on a partitioning between head and internal reductions. We study the head normalization for this call-by-value calculus with sigma-reductions and we relate it to the weak evaluation of original Plotkin's call-by-value lambda-calculus. We give also a (non-deterministic) normalization strategy for the call-by-value lambda-calculus with sigma-reductions.

1998 ACM Subject Classification D.3.1 Formal Definitions and Theory, F.3.2 Semantics of Programming Language, F.4.1 Mathematical Logic, F.4.2 Grammars and Other Rewriting Systems.

Keywords and phrases sequentialization, lambda-calculus, sigma-reduction, call-by-value, head reduction, internal reduction, (strong) normalization, evaluation, confluence, reduction strategy.

Digital Object Identifier 10.4230/OASIcs.WPTE.2015.3

1 Introduction

The call-by-value λ-calculus (λ_v-calculus or λ_v for short) and the operational machine for its evaluation has been introduced by Plotkin [15] inspired by Landin's seminal work [9] on the programming language ISWIM and the SECD machine. The λ_v-calculus is a paradigmatic language able to capture two features of many functional programming languages: call-by-value parameter passing policy (parameters are evaluated before being passed) and weak evaluation (the body of a function is evaluated only when parameters are supplied).

The syntax of λ_v is the same as that of the ordinary (i.e. call-by-name) λ-calculus (λ for short), but the reduction rule for λ_v, called β_v, is a restriction of the β-rule for λ: β_v allows the contraction of a redex $(\lambda x.M)N$ only in case the argument N is a value, i.e. a variable or an abstraction. Unfortunately, the semantic analysis of the λ_v-calculus has turned out to be more elaborate than that of ordinary λ-calculus. This is due essentially to the "weakness" of (full) β_v-reduction, a fact widely recognized: indeed, there are many proposals of alternative call-by-value λ-calculi extending Plotkin's one [11, 10, 8, 2, 1]. To have an example of the "weakness" of the rewriting rules of λ_v, it is sufficient to consider that it is impossible to have an internal operational characterization (i.e. one that uses the β_v-reduction) of the semantically meaningful notions of call-by-value solvability and potential valuability, as shown in [13, 14, 2].

In this paper we will study the λ_v^σ-calculus (λ_v^σ for short), a call-by-value extension of λ_v recently proposed in [4]: it keeps the λ_v (and λ) syntax and it adds to the β_v-reduction two commutation rules, called σ_1 and σ_3, which unblock "hidden" β_v-redexes that are concealed by the "hyper-sequential structure" of terms. The λ_v^σ-calculus enjoy some basic properties we expect from a calculus, namely confluence (see [4]) and standardization (see [7]). Moreover, λ_v^σ provides elegant characterizations of many semantic properties, e.g. solvability and potential

2nd International Workshop on Rewriting Techniques for Program Transformations and Evaluation (WPTE'15).
Editors: Yuki Chiba, Santiago Escobar, Naoki Nishida, David Sabel, and Manfred Schmidt-Schauß; pp. 3–17
OpenAccess Series in Informatics
OASICS Schloss Dagstuhl – Leibniz-Zentrum für Informatik, Dagstuhl Publishing, Germany

valuability (see [4]), and it is conservative with respect to Plotkin's λ_v: in particular, [7] shows that the notions of solvability and potential valuability for λ_v^σ coincide with those for λ_v.

The v-reduction (i.e. the reduction for λ_v^σ) can be partitioned into head v-reduction and internal v-reduction; the head v-reduction is in turn decomposed into head β_v- and head σ-reduction. The head β_v-reduction is just the deterministic weak evaluation strategy for Plotkin's λ_v-calculus. According to a sequentialization theorem proven in [7, Theorem 22], any v-reduction sequence can be sequentialized in an initial head β_v-reduction sequence followed by a head σ-reduction sequence followed by an internal v-reduction sequence. Similar well-known results hold for λ and λ_v, and starting from them one can define a normalization strategy for λ and λ_v, i.e. a deterministic reduction strategy that reaches a normal form if and only if one exists: for example the leftmost reduction, see [19, Theorem 2.8] and [3, Theorem 13.2.2].

Is there a normalization strategy for λ_v^σ? Theorem 24, one of the main results of this paper, proves that, starting from the sequentialization theorem mentioned above, a normalization strategy can be defined for λ_v^σ, based on the notions of head β_v- and head σ-reductions.

A first difference appears here between λ_v^σ and λ_v (or λ): the normalization strategy for λ_v^σ is not deterministic. Indeed, while the head β_v-reduction (or the call-by-name head reduction) is deterministic (i.e. a partial function), the head v-reduction is non-deterministic and, still worse, non-confluent and there are terms having several head v-normal forms: this might appear disappointing. So, three natural questions arise:

- With respect to head v-reduction, do normalization and strong normalization coincide?[1]
- Can we relate the termination of head β_v-reduction and head v-reduction?
- Can we characterize the terms having a unique head v-normal form?

Our Theorem 21 gives a positive answer to the first two questions. Observe that the lack of any form of confluence for head v-reduction requires a more complex reasoning, passing through a syntactic characterization of head β_v- and head v-normal forms. Theorem 21 not only shows that the head v-reduction and the head β_v-reduction are deeply related (and hence, again, λ_v^σ is conservative with respect to λ_v) but also that both enjoy good properties analogous to the ones of the (call-by-name) head reduction for ordinary λ-calculus.

Our Proposition 27 gives a partial answer to the third question above: it shows that in some cases (of interest) a head v-normalizable term has a unique head v-normal form; in particular, every closed head v-normalizable term has a unique head v-normal form.

So, λ_v^σ appears as an extension of Plotkin's λ_v-calculus that enjoys many meaningful conservation properties with respect to λ_v and therefore it is a useful tool for theoretical and semantic investigations about λ_v and call-by-value setting. See also conclusions in Section 6 for further and more precise motivations for this paper and future work.

Related work. The λ_v^σ-calculus has been recently introduced in [4] and further investigated in [7]. It is an extension of Plotkin's λ_v-calculus inspired by the call-by-value translation of λ-terms into linear logic proof-nets [6]. Other variants of λ_v have been introduced in the literature for modeling the call-by-value computation. We would like to cite here at least the contributions of Moggi [11], Felleisen and Sabry [18], Maraist et al. [10], Herbelin and Zimmerman [8], Accattoli and Paolini [2] (the latter is inspired by the call-by-value translation of λ-terms into linear logic proof-nets, see [1]). All these proposals are based on the introduction of new constructs to the syntax of λ_v, so the comparison between them is

[1] The answer is trivially positive in the case of call-by-name head normalization (for λ) and head β_v-normalization, since these reductions are deterministic.

not easy with respect to syntactical properties (some detailed comparison is given in [2]). We point out that the calculi introduced in [11, 18, 10, 8] present some variants of our σ_1 and/or σ_3 rules, often in a setting with explicit substitutions. Regnier [16, 17] used the rule σ_1 (but not σ_3) in ordinary (i.e. call-by-name) λ-calculus.

The head v-reduction investigated here has been introduced in [7]. Some results of this paper are inspired by the Takahashi's results [19] on the ordinary (i.e. call-by-name) λ-calculus, partially adapted by Crary [5] for λ_v.

Outline. In Section 2 we introduce the syntax and the reduction rules of the λ_v^σ-calculus. In Section 3 we define the head v-reduction and the internal v-reduction, and we recall some results already proven in [7] concerning them. Section 4 is devoted to proving the first main result of our paper: Theorem 21, which studies the normalization for the head v-reduction and relates it to the weak evaluation strategy for Plotkin's λ_v-calculus. In Section 5 we show that the head v-reduction can be used to define a normalization strategy for the λ_v^σ-calculus (Theorem 24), and moreover in some cases the head v-normal form (if any) of a term is unique (Proposition 27). In Section 6 we summarize the findings and suggest future work.

2 The call-by-value lambda calculus with sigma-rules

In this section we present λ_v^σ, a call-by-value λ-calculus introduced in [4] that adds two σ-reduction rules to pure (i.e. without constants) call-by-value λ-calculus defined by Plotkin in [15].

The syntax of terms of λ_v^σ [4] is the same as the one of ordinary λ-calculus and Plotkin's call-by-value λ-calculus λ_v [15] (without constants). Given a countable set $\mathcal{V}$ of *variables* (denoted by $x, y, z, \dots$), the sets Λ of *terms* and Λ_v of *values* are defined by mutual induction:

$$
\begin{array}{lrl}
(\Lambda_v) & V, U ::= x \mid \lambda x.M & \textit{values} \\
(\Lambda) & M, N, L ::= V \mid MN & \textit{terms}
\end{array}
$$

Clearly, $\Lambda_v \subsetneq \Lambda$. All terms are considered up to α-conversion. The set of free variables of a term M is denoted by $\mathsf{fv}(M)$. Given $V_1, \dots, V_n \in \Lambda_v$ and pairwise distinct variables $x_1, \dots, x_n$, $M\{V_1/x_1, \dots, V_n/x_n\}$ denotes the term obtained by the *capture-avoiding simultaneous substitution* of V_i for each free occurrence of x_i in the term M (for all $1 \le i \le n$). Note that, for all $V, V_1, \dots, V_n \in \Lambda_v$ and pairwise distinct variables $x_1, \dots, x_n$, $V\{V_1/x_1, \dots, V_n/x_n\} \in \Lambda_v$.

Contexts (with exactly one hole $(\!|\cdot|\!)$), denoted by $\mathtt{C}$, are defined as usual via the grammar:

$$\mathtt{C} ::= (\!|\cdot|\!) \mid \lambda x.\mathtt{C} \mid \mathtt{C}M \mid M\mathtt{C}\,.$$

We use $\mathtt{C}(\!|M|\!)$ for the term obtained by the capture-allowing substitution of the term M for the hole $(\!|\cdot|\!)$ in the context $\mathtt{C}$.

▶ **Notation.** From now on, we set $I = \lambda x.x$ and $\Delta = \lambda x.xx$.

The reduction rules of λ_v^σ consist of Plotkin's β_v-reduction rule, introduced in [15], and two simple commutation rules called σ_1 and σ_3, studied in [4, 7].

▶ **Definition 1** (Reduction rules). *We define the following binary relations on Λ (for any $M, N, L \in \Lambda$ and any $V \in \Lambda_v$):*

$$
\begin{array}{rll}
(\lambda x.M)V \mapsto_{\beta_v} & M\{V/x\} & \\
(\lambda x.M)NL \mapsto_{\sigma_1} & (\lambda x.ML)N & \textit{with } x \notin \mathsf{fv}(L) \\
V((\lambda x.L)N) \mapsto_{\sigma_3} & (\lambda x.VL)N & \textit{with } x \notin \mathsf{fv}(V).
\end{array}
$$

We set $\mapsto_\sigma = \mapsto_{\sigma_1} \cup \mapsto_{\sigma_3}$ *and* $\mapsto_{\mathsf{v}} = \mapsto_{\beta_v} \cup \mapsto_\sigma$.

For any $\mathsf{r} \in \{\beta_v, \sigma_1, \sigma_3, \sigma, \mathsf{v}\}$*, if* $M \mapsto_{\mathsf{r}} M'$ *then* M *is a* r-redex *and* M' *is its* r-contractum. *In this sense, a term of the shape* $(\lambda x.M)N$ *(for any* $M, N \in \Lambda$*) is a* β-redex.

The side conditions on $\mapsto_{\sigma_1}$ and $\mapsto_{\sigma_3}$ in Definition 1 can be always fulfilled by α-renaming.

Obviously, any β_v-redex is a β-redex but the converse does not hold: $(\lambda x.z)(yI)$ is a β-redex but not a β_v-redex.

▶ **Example 2.** Redexes of different kind may overlap: for example, the term $\Delta I \Delta$ is a σ_1-redex and it contains the β_v-redex ΔI; the term $\Delta(I\Delta)(xI)$ is a σ_1-redex and it contains the σ_3-redex $\Delta(I\Delta)$, which contains in turn the β_v-redex $I\Delta$.

▶ **Notation.** Let R be a binary relation on Λ. We denote by R^* (resp. R^+; $\mathsf{R}^=$) the reflexive-transitive (resp. transitive; reflexive) closure of R.

▶ **Definition 3** (Reductions)**.** *Let* $\mathsf{r} \in \{\beta_v, \sigma_1, \sigma_3, \sigma, \mathsf{v}\}$.

The r-reduction $\to_{\mathsf{r}}$ *is the contextual closure of* $\mapsto_{\mathsf{r}}$*, i.e.* $M \to_{\mathsf{r}} M'$ *iff there is a context* C *and* $N, N' \in \Lambda$ *such that* $M = \mathsf{C}(\!|N|\!)$*,* $M' = \mathsf{C}(\!|N'|\!)$ *and* $N \mapsto_{\mathsf{r}} N'$.

The r-equivalence $\simeq_{\mathsf{r}}$ *is the reflexive-transitive and symmetric closure of* $\to_{\mathsf{r}}$.

Let M *be a term:* M *is* r-normal *if there is no term* N *such that* $M \to_{\mathsf{r}} N$*;* M *is* r-normalizable *if there is a* r*-normal term* N *such that* $M \to_{\mathsf{r}}^* N$*;* M *is* strongly r-normalizable *if there is no sequence* $(N_i)_{i\in\mathbb{N}}$ *of terms such that* $M = N_0$ *and* $N_i \to_{\mathsf{r}} N_{i+1}$ *for any* $i \in \mathbb{N}$.

Obviously, $\to_\sigma = \to_{\sigma_1} \cup \to_{\sigma_3} \subsetneq \to_{\mathsf{v}}$ and $\to_{\beta_v} \subsetneq \to_{\mathsf{v}}$ and $\to_{\mathsf{v}} = \to_{\beta_v} \cup \to_\sigma$.

▶ **Remark 4.** For any $\mathsf{r} \in \{\beta_v, \sigma_1, \sigma_3, \sigma, \mathsf{v}\}$ (resp. $\mathsf{r} \in \{\sigma_1, \sigma_3, \sigma\}$), values are closed under r-reduction (resp. r-expansion): for any $V \in \Lambda_v$, if $V \to_{\mathsf{r}} M$ (resp. $M \to_{\mathsf{r}} V$) then $M \in \Lambda_v$; more precisely, $V = \lambda x.N$ and $M = \lambda x.N'$ for some $N, N' \in \Lambda$ with $N \to_{\mathsf{r}} N'$ (resp. $N' \to_{\mathsf{r}} N$).

For any $\mathsf{r} \in \{\beta_v, \mathsf{v}\}$, values are not closed under r-expansion: $I\Delta \to_{\beta_v} \Delta \in \Lambda_v$ but $I\Delta \notin \Lambda_v$.

▶ **Proposition 5** (See [4])**.** *The* σ*-reduction is confluent and strongly normalizing. The* v*-reduction is confluent.*

The λ_v^σ*-calculus*, λ_v^σ for short, is the set Λ of terms endowed with the v-reduction $\to_{\mathsf{v}}$. The set Λ endowed with the β_v-reduction $\to_{\beta_v}$ is the λ_v-calculus (λ_v for short), i.e. the Plotkin's call-by-value λ-calculus [15] (without constants), which is thus a sub-calculus of λ_v^σ.

▶ **Example 6.** $M = (\lambda y.\Delta)(xI)\Delta \to_{\sigma_1} (\lambda y.\Delta\Delta)(xI) \to_{\beta_v} (\lambda y.\Delta\Delta)(xI) \to_{\beta_v} \dots$ and $N = \Delta((\lambda y.\Delta)(xI)) \to_{\sigma_3} (\lambda y.\Delta\Delta)(xI) \to_{\beta_v} (\lambda y.\Delta\Delta)(xI) \to_{\beta_v} \dots$ are the only possible v-reduction paths from M and N respectively: M and N are not v-normalizable, and $M \simeq_{\mathsf{v}} N$. Meanwhile, M and N are β_v-normal and different, hence $M \not\simeq_{\beta_v} N$ (by confluence of $\to_{\beta_v}$, see [15]).

Informally, σ-rules unblock β_v-redexes which are hidden by the "hyper-sequential structure" of terms. This approach is alternative to the one in [2, 1] where hidden β_v-redexes are reduced using rules acting at a distance (through explicit substitutions). It can be shown that the call-by-value λ-calculus with explicit substitution introduced in [2] can be embedded in λ_v^σ.

It is well-known that the β_v-reduction can be simulated by linear logic cut-elimination via the call-by-value translation $(\cdot)^v$ of λ-terms into proof-nets, called by Girard [6, pp. 81-82] "boring" and defined by $(A \Rightarrow B)^v = {!A^v} \multimap {!B^v}$ (see also [1]). The images under $(\cdot)^v$ of a σ-redex and its σ-contractum are equal modulo some non-structural cut-elimination steps.

3 Head and internal reductions

In this section we introduce the definitions of head v-reduction (which is decomposed in head β_v- and head σ-reductions) and internal v-reduction, then we recall some results proven in [7].

▶ **Notation.** From now on, we always assume that $V, V' \in \Lambda_v$.

Note that the generic form of a term is $VM_1 \dots M_m$ for some $m \in \mathbb{N}$ (in particular, values are obtained when $m = 0$). The sequentialization result is based on a partitioning of v-reduction between head v-reduction and internal v-reduction.

▶ **Definition 7** (Head β_v-reduction). *The* head β_v-reduction $\xrightarrow{h}_{\beta_v}$ *is the binary relation on* Λ *defined inductively by the following rules ($m \in \mathbb{N}$ in both rules):*

$$\frac{}{(\lambda x.M)VM_1 \dots M_m \xrightarrow{h}_{\beta_v} M\{V/x\}M_1 \dots M_m}\,\beta_v \qquad \frac{N \xrightarrow{h}_{\beta_v} N'}{VNM_1 \dots M_m \xrightarrow{h}_{\beta_v} VN'M_1 \dots M_m}\,\textit{right}$$

The head β_v-reduction $\xrightarrow{h}_{\beta_v}$ is exactly the (pure) "left reduction" defined in [15, p. 136] for λ_v and called "(weak) evaluation" in [18, 5]. If $N \xrightarrow{h}_{\beta_v} N'$ then N' is obtained from N by reducing the leftmost-outermost β_v-redex, not in the scope of a λ: thus, the head β_v-reduction is deterministic (i.e. it is a partial function from Λ to Λ) and does not reduce values.

▶ **Definition 8** (Head σ- and head v-reductions). *The* head σ-reduction $\xrightarrow{h}_{\sigma}$ *is the binary relation on* Λ *defined inductively by the following rules ($m \in \mathbb{N}$ in all the rules, $x \notin \mathsf{fv}(L)$ in the rule σ_1, $x \notin \mathsf{fv}(V)$ in the rule σ_3):*

$$\frac{}{(\lambda x.M)NLM_1 \dots M_m \xrightarrow{h}_{\sigma} (\lambda x.ML)NM_1 \dots M_m}\,\sigma_1 \qquad \frac{N \xrightarrow{h}_{\sigma} N'}{VNM_1 \dots M_m \xrightarrow{h}_{\sigma} VN'M_1 \dots M_m}\,\textit{right}$$

$$\frac{}{V((\lambda x.L)N)M_1 \dots M_m \xrightarrow{h}_{\sigma} (\lambda x.VL)NM_1 \dots M_m}\,\sigma_3$$

The head v-reduction *is* $\xrightarrow{h}_{\mathsf{v}} = \xrightarrow{h}_{\beta_v} \cup \xrightarrow{h}_{\sigma}$.

Let $\mathsf{r} \in \{\beta_v, \sigma, \mathsf{v}\}$ *and* $N \in \Lambda$*:* N *is* head r-normal *if there is no* $N' \in \Lambda$ *such that* $N \xrightarrow{h}_{\mathsf{r}} N'$*;* N *is* head r-normalizable *if there is a* r*-normal term* N' *such that* $N \xrightarrow{h}_{\mathsf{r}}^{*} N'$*;* N *is* strongly head r-normalizable *if there is no* $(N_i)_{i \in \mathbb{N}}$ *such that* $N = N_0$ *and* $N_i \xrightarrow{h}_{\mathsf{r}} N_{i+1}$ *for any* $i \in \mathbb{N}$.

Notice that $\mapsto_{\beta_v} \subsetneq \xrightarrow{h}_{\beta_v} \subsetneq \to_{\beta_v}$ and $\mapsto_{\sigma} \subsetneq \xrightarrow{h}_{\sigma} \subsetneq \to_{\sigma}$ and $\mapsto_{\mathsf{v}} \subsetneq \xrightarrow{h}_{\mathsf{v}} \subsetneq \to_{\mathsf{v}}$.

Informally, if $N \xrightarrow{h}_{\sigma} N'$ then N' is obtained from N by reducing "one of the leftmost" σ_1- or σ_3-redexes, not in the scope of a λ: in general, a term may contain several head σ_1- and σ_3-redexes. Indeed, differently from $\xrightarrow{h}_{\beta_v}$, the head σ-reduction $\xrightarrow{h}_{\sigma}$ is not deterministic, for example the leftmost-outermost σ_1- and σ_3-redexes may overlap: if $M = (\lambda y.y')(\Delta(xI))I$ then $M \xrightarrow{h}_{\sigma} (\lambda y.y'I)(\Delta(xI)) = N_1$ by applying the rule σ_1 and $M \xrightarrow{h}_{\sigma} (\lambda z.(\lambda y.y')(zz))(xI)I = N_2$ by applying the rule σ_3. Note that N_1 contains only a head σ_3-redex and $N_1 \xrightarrow{h}_{\sigma} (\lambda z.(\lambda y.y'I)(zz))(xI) = N$ which is head v-normal; meanwhile N_2 contains only a head σ_1-redex and $N_2 \xrightarrow{h}_{\sigma} (\lambda z.(\lambda y.y')(zz)I)(xI) = N'$ which is head v-normal: $N \neq N'$, so the head σ- and head v-reductions are not (locally) confluent and a term may have several head v-normal forms (this example does not contradict the confluence of σ-reduction because $N' \to_{\sigma} N$ but by performing an internal v-reduction step, see next Definition 9).

The head v-reduction $\xrightarrow{h}_{\mathsf{v}}$ is non-deterministic not only because the head σ-reduction $\xrightarrow{h}_{\sigma}$ is non-deterministic, but also because the leftmost-outermost β_v-redex of a term may overlap with "one of its leftmost" σ_1- or σ_3-redexes, as seen in Example 2.

▶ **Definition 9** (Internal v-reduction). *The* internal v-reduction $\xrightarrow{int}_{\mathsf{v}}$ *is the binary relation on* Λ *defined inductively by the following rules:*

$$\frac{(m \in \mathbb{N}) \qquad N \to_{\mathsf{v}} N'}{(\lambda x.N)M_1 \dots M_m \xrightarrow{int}_{\mathsf{v}} (\lambda x.N')M_1 \dots M_m}\ \lambda \qquad \frac{(m \in \mathbb{N}) \qquad N \xrightarrow{int}_{\mathsf{v}} N'}{VNM_1 \dots M_m \xrightarrow{int}_{\mathsf{v}} VN'M_1 \dots M_m}\ \mathit{right}$$

$$\frac{(m \in \mathbb{N}^+) \qquad M_i \to_{\mathsf{v}} M_i' \quad \text{for some } 1 \leq i \leq m}{VNM_1 \dots M_i \dots M_m \xrightarrow{int}_{\mathsf{v}} VNM_1 \dots M_i' \dots M_m}\ @\ .$$

The following fact collects many minor properties which can be easily proved by inspection of the rules of Definitions 7-9.

▶ **Fact 10.**

1. *The head β_v-reduction $\xrightarrow{h}_{\beta_v}$ does not reduce a value (in particular, does not reduce under λ's), i.e., for any $M \in \Lambda$ and any $V \in \Lambda_v$, one has $V \not\xrightarrow{h}_{\beta_v} M$.*
2. *The head σ-reduction $\xrightarrow{h}_{\sigma}$ does neither reduce a value nor reduce to a value, i.e., for any $M \in \Lambda$ and any $V \in \Lambda_v$, one has $V \not\xrightarrow{h}_{\sigma} M$ and $M \not\xrightarrow{h}_{\sigma} V$.*
3. *Values are closed under $\xrightarrow{int}_{\mathsf{v}}$-expansion, i.e., for all $M \in \Lambda$ and $V \in \Lambda_v$, if $M \xrightarrow{int}_{\mathsf{v}} V$ then $M \in \Lambda_v$; more precisely, $M = \lambda x.N$ and $V = \lambda x.N'$ for some $N, N' \in \Lambda$ where $N \to_{\mathsf{v}} N'$.*
4. *If $\mathsf{R} \in \{\xrightarrow{h}_{\beta_v}, \xrightarrow{h}_{\sigma}, \xrightarrow{h}_{\mathsf{v}}, \xrightarrow{int}_{\mathsf{v}}\}$ and $M \mathrel{\mathsf{R}} M'$, then $MN \mathrel{\mathsf{R}} M'N$ for any $N \in \Lambda$.*

Clearly, $\xrightarrow{int}_{\mathsf{v}} \subsetneq \to_{\mathsf{v}}$. Next Proposition 11 (whose proof uses Fact 10.4) relates $\xrightarrow{int}_{\mathsf{v}}$ and $\xrightarrow{h}_{\mathsf{v}}$.

▶ **Proposition 11.** *One has $\xrightarrow{int}_{\mathsf{v}} = \to_{\mathsf{v}} \smallsetminus \xrightarrow{h}_{\mathsf{v}}$.*

Proof.

$\subseteq$: The proof that $\xrightarrow{int}_{\mathsf{v}} \subseteq \to_{\mathsf{v}}$ is trivial. The proof that $M \xrightarrow{int}_{\mathsf{v}} M'$ implies $M \not\xrightarrow{h}_{\mathsf{v}} M'$ is by induction on the derivation of $M \xrightarrow{int}_{\mathsf{v}} M'$. Let us consider its last rule r. If $\mathsf{r} \in \{\lambda, @\}$, then it is evident that there is no last rule to derive $M \xrightarrow{h}_{\mathsf{v}} M'$. If $\mathsf{r} = \mathit{right}$ then $M = VNM_1 \dots M_m$ and $M' = VN'M_1 \dots M_m$ with $m \in \mathbb{N}$ and $N \xrightarrow{int}_{\mathsf{v}} N'$; by induction hypothesis, $N \not\xrightarrow{h}_{\mathsf{v}} N'$ and hence there is no last rule to derive $M \xrightarrow{h}_{\mathsf{v}} M'$.

$\supseteq$: We show that $M \to_{\mathsf{v}} M'$ and $M \not\xrightarrow{h}_{\mathsf{v}} M'$ implies $M \xrightarrow{int}_{\mathsf{v}} M'$, for all $M, M' \in \Lambda$. Since $M \to_{\mathsf{v}} M'$, there exist a context $\mathtt{C}$ and terms N and N' such that $M = \mathtt{C}(\!|N|\!)$, $M' = \mathtt{C}(\!|N'|\!)$ and $N \mapsto_{\beta_v} N'$. We proceed by induction on $\mathtt{C}$.

If $\mathtt{C} = (\!|\cdot|\!)$ then $M = N \mapsto_{\beta_v} N' = M'$ and thus $M \xrightarrow{h}_{\mathsf{v}} M'$ since $\mapsto_{\beta_v} \subseteq \xrightarrow{h}_{\mathsf{v}}$, which contradicts the hypothesis.

If $\mathtt{C} = \lambda x.\mathtt{C}'$ for some context $\mathtt{C}'$, then $M \xrightarrow{int}_{\mathsf{v}} M'$ by applying the rule λ for $\xrightarrow{int}_{\mathsf{v}}$, since $\mathtt{C}'(\!|N|\!) \to_{\mathsf{v}} \mathtt{C}'(\!|N'|\!)$.

If $\mathtt{C} = \mathtt{C}'L$ for some context $\mathtt{C}'$ and term L, then $\mathtt{C}'(\!|N|\!) \to_{\mathsf{v}} \mathtt{C}'(\!|N'|\!)$ and $\mathtt{C}'(\!|N'|\!) \not\xrightarrow{h}_{\mathsf{v}} \mathtt{C}'(\!|N'|\!)$ (by Fact 10.4, since $\mathtt{C}'(\!|N|\!)L \not\xrightarrow{h}_{\mathsf{v}} \mathtt{C}'(\!|N'|\!)L$). By induction hypothesis, $\mathtt{C}'(\!|N|\!) \xrightarrow{int}_{\mathsf{v}} \mathtt{C}'(\!|N'|\!)$, then $M = \mathtt{C}'(\!|N|\!)L \xrightarrow{int}_{\mathsf{v}} \mathtt{C}'(\!|N'|\!)L = M'$ by Fact 10.4.

If $\mathtt{C} = V\mathtt{C}'$ for some context $\mathtt{C}'$ and value V, then $\mathtt{C}'(\!|N|\!) \to_{\mathsf{v}} \mathtt{C}'(\!|N'|\!)$. There are two cases:

- either $\mathtt{C}'(\!|N'|\!) \xrightarrow{h}_{\mathsf{v}} \mathtt{C}'(\!|N'|\!)$, hence $M = V\mathtt{C}'(\!|N|\!) \xrightarrow{h}_{\mathsf{v}} V\mathtt{C}'(\!|N'|\!) = M'$ by the rule *right* for $\xrightarrow{h}_{\beta_v}$ or $\xrightarrow{h}_{\sigma}$, which contradicts the hypothesis;
- or $\mathtt{C}'(\!|N'|\!) \not\xrightarrow{h}_{\mathsf{v}} \mathtt{C}'(\!|N'|\!)$, hence $\mathtt{C}'(\!|N'|\!) \xrightarrow{int}_{\mathsf{v}} \mathtt{C}'(\!|N'|\!)$ by induction hypothesis, thus $M = V\mathtt{C}'(\!|N|\!) \xrightarrow{int}_{\mathsf{v}} V\mathtt{C}'(\!|N'|\!) = M'$ by applying the rule *right* for $\xrightarrow{int}_{\mathsf{v}}$.

Finally, if $\mathtt{C} = L\mathtt{C}'$ for some context $\mathtt{C}'$ and term $L \notin \Lambda_v$, then $L = VN_0 \dots N_n$ for some $n \in \mathbb{N}$, thus $M = VN_0 \dots N_n\mathtt{C}'(\!|N|\!) \xrightarrow{int}_{\mathsf{v}} VN_0 \dots N_n\mathtt{C}'(\!|N'|\!) = M'$ by the rule @ for $\xrightarrow{int}_{\mathsf{v}}$.

◀

We end this section by recalling three results proven in [7] concerning head v-reduction and internal v-reduction: they will be used to prove the main results in Sections 4-5.

The following lemma (proven in [7, Lemma 14]) shows that a head σ-reduction step can be postponed after a head β_v-reduction step, and hence every head v-reduction sequence can be rearranged into a head β_v-reduction sequence followed by a head σ-reduction sequence.

▶ **Lemma 12** (Commutation of head β_v- and head σ-reductions, see [7]).
1. *If $M \xrightarrow{h}_{\sigma} L \xrightarrow{h}_{\beta_v} N$ then there exists $L' \in \Lambda$ such that $M \xrightarrow{h}_{\beta_v} L' \xrightarrow{h}{}^{=}_{\sigma} N$.*
2. *If $M \xrightarrow{h}{}^{*}_{\mathsf{v}} M'$ then there exists $N \in \Lambda$ such that $M \xrightarrow{h}{}^{*}_{\beta_v} N \xrightarrow{h}{}^{*}_{\sigma} M'$.*

Next Lemma 13 (proven in [7, Corollary 21]) says that internal v-reduction can be shifted after head v-reductions.[2]

▶ **Lemma 13** (Postponement, see [7]). *If $M \xrightarrow{int}_{\mathsf{v}} L$ and $L \xrightarrow{h}_{\beta_v} N$ (resp. $L \xrightarrow{h}_{\sigma} N$), then there exist $L', L'' \in \Lambda$ such that $M \xrightarrow{h}{}^{+}_{\beta_v} L' \xrightarrow{h}{}^{*}_{\sigma} L'' \xrightarrow{int}{}^{*}_{\mathsf{v}} N$ (resp. $M \xrightarrow{h}{}^{*}_{\beta_v} L' \xrightarrow{h}{}^{*}_{\sigma} L'' \xrightarrow{int}{}^{*}_{\mathsf{v}} N$).*

Next Theorem 14 is one of the main result proven in [7, Theorem 22] by adapting Takahashi's method [19, 5]: any v-reduction sequence can be sequentialized into a head β_v-reduction sequence followed by a head σ-reduction sequence, followed by an internal v-reduction sequence. In ordinary λ-calculus, the well-known result corresponding to our Theorem 14 states that a β-reduction sequence can be factorized in a head reduction sequence followed by an internal reduction sequence (see for example [19, Corollary 2.6]).

▶ **Theorem 14** (Sequentialization, see [7]). *If $M \to_{\mathsf{v}}^{*} M'$ then there exist $L, N \in \Lambda$ such that $M \xrightarrow{h}{}^{*}_{\beta_v} L \xrightarrow{h}{}^{*}_{\sigma} N \xrightarrow{int}{}^{*}_{\mathsf{v}} M'$.*

The sequentialization of Theorem 14 imposes no order between head σ-reductions. Indeed, the example in [7, p. 10] shows that it is impossible to sequentialize them by giving way to head σ_1- or head σ_3-redexes: a head σ_1-reduction step can create a head σ_3-redex, and vice versa.

In [7, Definition 27 and Corollary 29] it has also been proven that the v-equivalence (and in particular the σ-equivalence) is contained in the call-by-value observational equivalence.

4 Head normalization

In this section we prove the first main result of our paper: Theorem 21, which studies the normalization for head v-reduction and relates it to the head β_v-reduction (i.e. the weak evaluation strategy for Plotkin's λ_v-calculus). Let us start with a preliminary remark.

▶ **Remark 15.** According to Facts 10.1-2, every $V \in \Lambda_v$ is head β_v- and head σ-normal, and hence is head v-normal. The converse does not hold: xI is head v-normal but $xI \notin \Lambda_v$.

First, we give a syntactic characterization of head v- and head β_v-normal forms.

▶ **Definition 16.** *We define the subsets Λ_a, Λ_b and Λ_c (whose elements are denoted by A, B and C respectively) of Λ as follows (for any variable x, any $V \in \Lambda_v$ and any $N \in \Lambda$):*

$$(\Lambda_a) \quad A ::= xV \mid xA \mid AN \qquad (\Lambda_b) \quad B ::= (\lambda x.N)A \qquad (\Lambda_c) \quad C ::= xV \mid VC \mid CN$$

[2] In [7, Corollary 21] there is a more informative statement of our Lemma 13, involving a notion of internal parallel reduction $\overset{int}{\Rightarrow}$. Our Lemma 13 follows immediately from [7, Corollary 21] since $\xrightarrow{int}_{\mathsf{v}} \subseteq \overset{int}{\Rightarrow} \subseteq \xrightarrow{int}{}^{*}_{\mathsf{v}}$.

Notice that $\Lambda_a \cup \Lambda_b \subsetneq \Lambda_c$ and $M, N \in \Lambda_c \smallsetminus (\Lambda_a \cup \Lambda_b)$ where $M = (\lambda y.\Delta)(xI)\Delta$ and $N = \Delta((\lambda y.\Delta)(xI))$ (as in Example 6). Moreover, $\Lambda_v \cap \Lambda_a = \Lambda_v \cap \Lambda_b = \Lambda_v \cap \Lambda_c = \Lambda_a \cap \Lambda_b = \emptyset$ and all terms in $\Lambda_a \cup \Lambda_b \cup \Lambda_c$ are not closed. All terms in Λ_b are β-redexes that are not β_v-redexes; all terms in Λ_a have a free "head variable" and are neither a value nor a β-redex.

▶ **Proposition 17** (Characterization of head β_v-normal forms)**.** *Let M be a term.*
1. *M is head β_v-normal and is not a λ-value if and only if $M \in \Lambda_c$.*
2. *M is head β_v-normal if and only if $M \in \Lambda_v \cup \Lambda_c$.*

Proof. Statement (2) is an immediate consequence of statement (1) and Remark 15.

$\Rightarrow$: We prove the left-to-right direction of statement (1), by induction on $M \in \Lambda$.
The case where $M \in \Lambda_v$ is impossible by hypothesis.
If $M = M_1M_2$ (for some $M_1, M_2 \in \Lambda$) is head β_v-normal then M is not a λ-value and M_1 and M_2 are head β_v-normal, moreover either $M_1 \neq \lambda x.N$ (for any $N \in \Lambda$) or $M_2 \notin \Lambda_v$ (otherwise M would be a head β_v-redex). Therefore, there are only three cases:
- either $M_1 \notin \Lambda_v$, thus $M_1 \in \Lambda_c$ by induction hypothesis, and hence $M \in \Lambda_c$;
- or $M_1 \in \Lambda_v$ and $M_2 \notin \Lambda_v$, so $M_2 \in \Lambda_c$ by induction hypothesis, and thus $M \in \Lambda_c$;
- or M_1 is a variable and $M_2 \in \Lambda_v$, hence $M \in \Lambda_c$ (this is the base case).

$\Leftarrow$: The right-to-left direction of statement (1) can easily be proved by induction on $M \in \Lambda_c$. ◀

A consequence of Proposition 17 is that all closed head β_v-normal forms are abstractions.

▶ **Proposition 18** (Characterization of head v-normal forms)**.** *Let $M \in \Lambda$.*
1. *M is head v-normal and is neither a λ-value nor a β-redex if and only if $M \in \Lambda_a$.*
2. *M is head v-normal and is a β-redex if and only if $M \in \Lambda_b$.*
3. *M is head v-normal if and only if $M \in \Lambda_v \cup \Lambda_a \cup \Lambda_b$.*

Proof. Statement (3) is an immediate consequence of statements (1)-(2) and Remark 15.

$\Rightarrow$: We prove simultaneously the left-to-right direction of statements (1) and (2), by induction on $M \in \Lambda$. The case where $M \in \Lambda_v$ is impossible by hypothesis.
If $M = M_1M_2$ (for some $M_1, M_2 \in \Lambda$) is head v-normal then M is not a λ-value and M_1 and M_2 are head v-normal, moreover M_1 is not a β-redex (otherwise M would be a head σ_1-redex), and either $M_1 \neq \lambda x.N$ (for any $N \in \Lambda$) or $M_2 \notin \Lambda_v$ (otherwise M would be a head β_v-redex), and either $M_1 \notin \Lambda_v$ or M_2 is not a β-redex (otherwise M would be a head σ_3-redex). There are only three cases:
- either M_1 is a variable and M_2 is not a β-redex, so M is not a β-redex; if $M_2 \in \Lambda_v$ then $M \in \Lambda_a$ (this is the base case); otherwise $M_2 \in \Lambda_a$ by induction hypothesis, so $M \in \Lambda_a$;
- or $M_1 \notin \Lambda_v$, thus M is not a β-redex and $M_1 \in \Lambda_a$ by induction hypothesis, so $M \in \Lambda_a$;
- or $M_1 = \lambda x.N$ for some $N \in \Lambda$ and M_2 is neither a λ-value nor a β-redex, so M is a β-redex, furthermore $M_2 \in \Lambda_a$ by induction hypothesis, and thus $M \in \Lambda_b$.

$\Leftarrow$: The right-to-left direction of statement (1) can easily be proved by induction on $M \in \Lambda_a$.
Let us prove the right-to-left direction of statement (2): if $M \in \Lambda_b$ then $M = (\lambda x.N)A$ for some $N \in \Lambda$ and $A \in \Lambda_a$, thus M is a β-redex. For any $M' \in \Lambda$, the last rule of the derivation of $M \xrightarrow{h}_{\mathsf{v}} M'$ might be neither σ_1 nor σ_3 (because A is not a β-redex by statement 1) nor β_v (because $A \notin \Lambda_v$ by statement 1 again) nor *right* (because A is head v-normal, by statement 1 again). Therefore, M is head v-normal. ◀

As a consequence of Proposition 18, all closed head v-normal forms are abstractions.

The sets of terms Λ_a, Λ_b and Λ_c of Definition 16 enjoy the closure properties summarized in Lemma 19 below. Together with the syntactic characterizations of head β_v-normal forms

(Proposition 17) and head v-normal forms (Proposition 18), these closure properties allow one to reason about head v-reduction in spite of its non-confluence: they will be used to prove our main results, Theorems 21 and 24 and Proposition 27.

▶ **Lemma 19** (Closure properties).

1. *The set* Λ_a *is closed under* v*-internal reduction and expansion, i.e., for any* $N' \in \Lambda$ *and* $N \in \Lambda_a$*, if* $N' \stackrel{int}{\to}_{\mathsf{v}} N$ *or* $N \stackrel{int}{\to}_{\mathsf{v}} N'$ *then* $N' \in \Lambda_a$*.*
2. *The set* Λ_b *is closed under* v*-internal reduction and expansion, i.e., for any* $N' \in \Lambda$ *and* $N \in \Lambda_b$*, if* $N' \stackrel{int}{\to}_{\mathsf{v}} N$ *or* $N \stackrel{int}{\to}_{\mathsf{v}} N'$ *then* $N' \in \Lambda_b$*.*
3. *Head* v*-normal forms are closed under* v*-internal reduction and expansion, i.e., for any* $N, N' \in \Lambda$ *where* N *is head* v*-normal, if* $N' \stackrel{int}{\to}_{\mathsf{v}} N$ *or* $N \stackrel{int}{\to}_{\mathsf{v}} N'$ *then* N' *is head* v*-normal.*
4. *Head* β_v*-normal forms are closed under head* σ*-reduction and expansion, i.e., for any* $N, N' \in \Lambda$ *where* N *is head* β_v*-normal, if* $N' \stackrel{h}{\to}_{\sigma} N$ *or* $N \stackrel{h}{\to}_{\sigma} N'$ *then* N' *is head* β_v*-normal.*

Proof.

1. We show that if $N \in \Lambda_a$ and $N' \stackrel{int}{\to}_{\mathsf{v}} N$ (resp. $N \stackrel{int}{\to}_{\mathsf{v}} N'$) then $N' \in \Lambda_a$, by induction on the derivation of $N' \stackrel{int}{\to}_{\mathsf{v}} N$ (resp. $N \stackrel{int}{\to}_{\mathsf{v}} N'$). Let us consider its last rule r.
   Since $N \in \Lambda_a$ (see Definition 16), $N = xLN_1 \dots N_n$ for some $n \in \mathbb{N}$, some variable x, some $L \in \Lambda_v \cup \Lambda_a$ and some $N_1, \dots, N_n \in \Lambda$, thus $\mathsf{r} \neq \lambda$ and hence either $\mathsf{r} = \mathit{right}$ or $\mathsf{r} = @$.
   If $\mathsf{r} = \mathit{right}$ then $N' = xL'N_1 \dots N_n$ where $L' \stackrel{int}{\to}_{\mathsf{v}} L$ (resp. $L \stackrel{int}{\to}_{\mathsf{v}} L'$). Since $L \in \Lambda_v \cup \Lambda_a$, there are two cases:
   - either $L \in \Lambda_a$ and then $L' \in \Lambda_a$ by induction hypothesis, so $N' = xL'N_1 \dots N_n \in \Lambda_a$;
   - or $L \in \Lambda_v$ and then $L' \in \Lambda_v$ by Fact 10.3 (resp. Remark 4, since $\stackrel{int}{\to}_{\mathsf{v}} \subseteq \to_{\mathsf{v}}$), therefore $N' = xL'N_1' \dots N_n' \in \Lambda_a$.

   Finally, if $\mathsf{r} = @$ then $n \in \mathbb{N}^+$ and $N' = xLN_1 \dots N_i' \dots N_n$ for some $1 \leq i \leq n$ with $N_i' \to_{\mathsf{v}} N_i$ (resp. $N_i \to_{\mathsf{v}} N_i'$), hence $N' \in \Lambda_a$ because $xL \in \Lambda_a$.
2. We show that if $N \in \Lambda_b$ and $N' \stackrel{int}{\to}_{\mathsf{v}} N$ (resp. $N \stackrel{int}{\to}_{\mathsf{v}} N'$) then $N' \in \Lambda_b$, by induction on the derivation of $N' \stackrel{int}{\to}_{\mathsf{v}} N$ (resp. $N \stackrel{int}{\to}_{\mathsf{v}} N'$). Let us consider its last rule r. Since $N \in \Lambda_b$, then $N = (\lambda x.M)A$ for some $M \in \Lambda$ and $A \in \Lambda_a$, hence $\mathsf{r} \neq @$ because N has not the shape $VLM_1 \dots M_m$ for any $m \in \mathbb{N}^+$; therefore either $\mathsf{r} = \lambda$ or $\mathsf{r} = \mathit{right}$:
   - if $\mathsf{r} = \lambda$, then $N' = (\lambda x.M')A$ where $M' \to_{\mathsf{v}} M$ (resp. $M \to_{\mathsf{v}} M'$), hence $N' \in \Lambda_b$;
   - if $\mathsf{r} = \mathit{right}$, then $N' = (\lambda x.M)A'$ where $A' \stackrel{int}{\to}_{\mathsf{v}} A$ (resp. $A \stackrel{int}{\to}_{\mathsf{v}} A'$), thus $A' \in \Lambda_a$ by Lemma 19.1, hence $N' \in \Lambda_b$;
3. Thanks to Proposition 18.3, it is sufficient to show that if $N \in \Lambda_v \cup \Lambda_a \cup \Lambda_b$ and $N' \stackrel{int}{\to}_{\mathsf{v}} N$ (resp. $N \stackrel{int}{\to}_{\mathsf{v}} N'$) then $N' \in \Lambda_v \cup \Lambda_a \cup \Lambda_b$. If $N \in \Lambda_v$ then $N' \in \Lambda_v$ by Fact 10.3 (resp. Remark 4, since $\stackrel{int}{\to}_{\mathsf{v}} \subseteq \to_{\mathsf{v}}$). If $N \in \Lambda_a$ then $N' \in \Lambda_a$ by Lemma 19.1. Finally, if $N \in \Lambda_b$ then $N' \in \Lambda_b$ by Lemma 19.2.
4. By Proposition 17.2, $N \in \Lambda_v \cup \Lambda_c$. Since $M \stackrel{h}{\to}_{\sigma} N$ or $N \stackrel{h}{\to}_{\sigma} M$, $N \notin \Lambda_v$ by Fact 10.2. We prove by induction on $N \in \Lambda_c$ that $M \in \Lambda_c$. By Definition 16, there are only two cases:
   - either $N = xVN_1 \dots N_n$ for some $n \in \mathbb{N}$, variable x, $V \in \Lambda_v$ and $N_1, \dots, N_n \in \Lambda$, but this is impossible since the last rule of the derivation of $M \stackrel{h}{\to}_{\sigma} N$ or $N \stackrel{h}{\to}_{\sigma} M$ can be neither σ_1 nor σ_3 (because of the subterm xV) nor *right* (because of Fact 10.2);
   - or $N = VLN_1 \dots N_n$ for some $n \in \mathbb{N}$, $V \in \Lambda_v$, $L \in \Lambda_c$ and $N_1, \dots, N_n \in \Lambda$, and then there are three sub-cases, depending on the last rule r of the derivation of $M \stackrel{h}{\to}_{\sigma} N$ (resp. $N \stackrel{h}{\to}_{\sigma} M$):
     - if $\mathsf{r} = \sigma_1$ then $V = \lambda x.N'N_0$ (resp. $\lambda x.N'$) and $M = (\lambda x.N')LN_0 \dots N_n$ (resp. $M = (\lambda x.N'N_1)LN_2 \dots N_n$ with $n > 0$) for some $N', N_0 \in \Lambda$, hence $M \in \Lambda_c$;
     - if $\mathsf{r} = \sigma_3$ then $V = \lambda x.V'N'$ (resp. $L = (\lambda x.N')L'$) and $M = V'((\lambda x.N')L)N_1 \dots N_n$ (resp. $M = (\lambda x.VN')L'N_1 \dots N_n$) for some $V' \in \Lambda_v$ (resp. $L' \in \Lambda_c$) and $N' \in \Lambda$, thus $(\lambda x.N')L \in \Lambda_c$ (resp. $(\lambda x.VN')L' \in \Lambda_c$) and hence $M \in \Lambda_c$;

- if $\mathsf{r} = \mathit{right}$ then $M = VL'N_1 \dots N_n$ for some $L' \in \Lambda$ such that $L' \xrightarrow{h}_\sigma L$ (resp. $L \xrightarrow{h}_\sigma L'$), so $L' \in \Lambda_c$ by induction hypothesis, and hence $M \in \Lambda_c$. ◀

Lemma 19.4 is a formalization of the two following facts: (a) a head σ-reduction step may create a new β_v-redex but in this case it is not a head β_v-redex; (b) when $M \xrightarrow{h}_\sigma N$, the head β_v-redex of M (if any) has a residual in N which is the head β_v-redex of N.

▶ **Lemma 20.** *There exists no infinite head* v*-reduction sequence with finitely many head* β_v*-reduction steps.*

Proof. Suppose the opposite holds: then there would exist $m \in \mathbb{N}$ and an infinite sequence of terms $(M_i)_{i\in\mathbb{N}}$ such that $M_i \xrightarrow{h}_\mathsf{v} M_{i+1}$ for any $1 \le i \le m$, $M_m \xrightarrow{h}_{\beta_v} M_{m+1}$ and $M_i \xrightarrow{h}_\sigma M_{i+1}$ for any $i > m$ (since $\xrightarrow{h}_\mathsf{v} = \xrightarrow{h}_{\beta_v} \cup \xrightarrow{h}_\sigma$). But this is impossible because $\xrightarrow{h}_\sigma$ is strongly normalizing (by Proposition 5 and since $\xrightarrow{h}_\sigma \subseteq \to_\sigma$). Contradiction. ◀

Now we can state and prove our main result about head β_v- and head v-normalization.

▶ **Theorem 21** (Head normalization)**.** *Let* $M \in \Lambda$*. The following are equivalent:*

1. *there exists a head* β_v*-normal form* N *such that* $M \simeq_{\beta_v} N$*;*
2. *there exists a head* v*-normal form* N *such that* $M \simeq_\mathsf{v} N$*;*
3. M *is head* v*-normalizable;*
4. M *is head* β_v*-normalizable;*
5. *there is no* v*-reduction sequence from* M *with infinitely many head* β_v*-reduction steps;*
6. M *is strongly head* v*-normalizable.*

Proof.

(1)⇒(2) By hypothesis, there exists a head β_v-normal $N \in \Lambda$ such that $M \simeq_{\beta_v} N$, thus $M \simeq_\mathsf{v} N$. Since $\xrightarrow{h}_\sigma$ is strongly normalizing (by Proposition 5 and because $\xrightarrow{h}_\sigma \subseteq \to_\sigma$), there exists a head σ-normal $N' \in \Lambda$ such that $N \xrightarrow{h}{}^*_\sigma N'$, therefore $M \simeq_\mathsf{v} N'$ since $\xrightarrow{h}_\sigma \subseteq \to_\mathsf{v}$. By Lemma 19.4, N' is also head β_v-normal and hence head v-normal.

(2)⇒(3) Since $M \simeq_\mathsf{v} N$, there is $L \in \Lambda$ such that $M \to^*_\mathsf{v} L$ and $N \to^*_\mathsf{v} L$, by confluence of $\to_\mathsf{v}$ (Proposition 5). By Theorem 14, there are $M_1, M_2, N_1, N_2 \in \Lambda$ such that $M \xrightarrow{h}{}^*_{\beta_v} M_1 \xrightarrow{h}{}^*_\sigma M_2 \xrightarrow{int}{}^*_\mathsf{v} L$ and $N \xrightarrow{h}{}^*_{\beta_v} N_1 \xrightarrow{h}{}^*_\sigma N_2 \xrightarrow{int}{}^*_\mathsf{v} L$. As N is head v-normal, $N = N_1 = N_2 \xrightarrow{int}{}^*_\mathsf{v} L$. By Lemma 19.3, L and M_2 are v-head normal. So, $M \xrightarrow{h}{}^*_\mathsf{v} M_2$ with M_2 head v-normal.

(3)⇒(4) By hypothesis, there is $N \in \Lambda$ head v-normal such that $M \xrightarrow{h}{}^*_\mathsf{v} N$. By Lemma 12.2, there is $L \in \Lambda$ such that $M \xrightarrow{h}{}^*_{\beta_v} L \xrightarrow{h}{}^*_\sigma N$. Since N is head v-normal and in particular head β_v-normal, L is head β_v-normal according to Lemma 19.4. So M is head β_v-normalizable.

(4)⇒(5) Lemma 12.1 says that if $N \xrightarrow{h}_\sigma L \xrightarrow{h}_{\beta_v} N'$ then there exists $L' \in \Lambda$ such that $N \xrightarrow{h}_{\beta_v} L' \xrightarrow{h}{}^=_\sigma N'$; Lemma 13 and Fact 10.3 show that if $N \xrightarrow{int}_\mathsf{v} L \xrightarrow{h}_{\beta_v} N'$ then there exist $L', L'' \in \Lambda$ such that $N \xrightarrow{h}{}^+_{\beta_v} L' \xrightarrow{h}{}^*_\sigma L'' \xrightarrow{int}{}^*_\mathsf{v} N'$. Since $\to_\mathsf{v} = \xrightarrow{h}_{\beta_v} \cup \xrightarrow{h}_\sigma \cup \xrightarrow{int}_\mathsf{v}$, this means that if there is an infinite v-reduction sequence from M with infinitely many head β_v-reduction steps, then for any $n \in \mathbb{N}$ there is a head β_v-reduction sequence from M whose length is at least n. Therefore, M is not head β_v-normalizable, since the head β_v-reduction is deterministic.

(5)⇒(6) If M is not strongly head v-normalizable then there exists an infinite head v-reduction sequence. By Lemma 20, this head v-reduction (and hence v-reduction, since $\xrightarrow{h}_\mathsf{v} \subseteq \to_\mathsf{v}$) sequence has infinitely many head β_v-reduction steps.

(6)⇒(1) As M is strongly head v-normalizable, in particular is head v-normalizable, hence there exists $N \in \Lambda$ head v-normal and in particular head β_v-normal such that $M \xrightarrow{h}_{\mathsf{v}}^{*} N$. By Lemma 12.2, there exists $L \in \Lambda$ such that $M \xrightarrow{h}_{\beta_v}^{*} L \xrightarrow{h}_{\sigma}^{*} N$. Therefore $M \simeq_{\beta_v} L$ since $\xrightarrow{h}_{\beta_v} \subseteq \to_{\beta_v}$. According to Lemma 19.4, L is head β_v-normal. ◀

In Theorem 21, the equivalence (3)⇔(6) means that (weak) normalization and strong normalization are equivalent for head v-reduction (for head β_v-reduction they are trivially equivalent since the head β_v-reduction is deterministic), therefore if one is interested in studying the termination of head v-reduction, no difficulty arises from its non-determinism. The equivalence (4)⇔(3) or (4)⇔(6) says that the weak evaluation process defined for Plotkin's λ_v-calculus (the head β_v-reduction) terminates if and only if the weak evaluation process defined for λ_v^σ (the head v-reduction) terminates: σ-rules play no role in deciding the termination of a head v-reduction sequence. The equivalence (3)⇔(2) (resp. (4)⇔(1)) is the version for λ_v^σ (resp. λ_v) of a well-known theorem for ordinary λ-calculus (see for example [3, Theorem 8.3.11]): in some sense, it claims that the head v-reduction (resp. head β_v-reduction) is complete with respect to the v-equivalence (resp. β_v-equivalence). The equivalence (5)⇔(2) (resp. (5)⇔(1)) can be seen as the version for λ_v^σ (resp. λ_v) of the Quasi-Head Reduction Theorem [19, Theorem 2.10] stated by Takahashi for ordinary λ-calculus.

5 Normalization strategy and other results

Theorems 14 and 21 strengthen the idea that, in spite of non-determinism and non-confluence of head v-reduction and non-sequentiability of head σ-reduction steps, the head v-reduction can be used to define a normalization strategy for the λ_v^σ-calculus, as proven in next Theorem 24, the second main result of our paper: given a term M, one starts the (unique) head β_v-head reduction sequence from M as long as a head β_v-normal form N is reached (recall that, according to Theorem 21, a term is (strongly) head v-normalizable if and only if it is head β_v-normalizable); then, one starts a head σ-reduction sequence from N (where head σ_1- and head σ_3-reduction steps can be performed in whatever order) as long as a head σ-normal form N' is reached (such a N' always exists because $\xrightarrow{h}_{\sigma}$ is strongly normalizing, and it is head v-normal by Lemma 19.4); finally, one performs the internal v-reduction steps starting from N' by iterating the head β_v-reduction sequences and then the head σ-reduction sequences as above on the subterms of N', from the left to the right. More precisely:

▶ **Definition 22** (Successors path). *Let $M \in \Lambda$.*
A successor of M *is a $M' \in \Lambda$ defined by induction on $M \in \Lambda$ as follows:*
- *if M is not head β_v-normal, then M' is such that $M \xrightarrow{h}_{\beta_v} M'$;*
- *if M is head β_v-normal but not head σ-normal, then M' is such that $M \xrightarrow{h}_{\sigma} M'$;*
- *if M is head v-normal then:*
  - *if M is a variable then $M' = M$,*
  - *if $M = \lambda x.N$ for some $N \in \Lambda$, then $M' = \lambda x.N'$ for some successor N' of N,*
  - *if $M = NL$ for some $N, L \in \Lambda$, then either N is not v-normal and $M' = N'L$ where N' is a successor of N, or N is v-normal and $M' = NL'$ where L' is a successor of L.*

A successors path of M *is an infinite sequence $(M_i)_{i \in \mathbb{N}}$ of terms such that $M_0 = M$ and M_{i+1} is a successor of M_i, for any $i \in \mathbb{N}$.*

Clearly, for every term M there is at least one successor M' of M; moreover, this successor M' is unique when M is not head β_v-normal, since the head β_v-reduction is deterministic, and $M = M'$ when M is v-normal.

▶ **Remark 23.** Let $M \in \Lambda$ and let $(M_i)_{i\in\mathbb{N}}$ be a successors path of M.

1. For every $i \in \mathbb{N}$, there exist $0 \le j \le k \le i$ such that $M \xrightarrow{h}^*_{\beta_v} M_j \xrightarrow{h}^*_{\sigma} M_k \xrightarrow{int}^*_{\mathsf{v}} M_i$.
2. For every $i \in \mathbb{N}$, if M_i is v-normal then M_j is v-normal for any $j \ge i$.

A successors path of a term M is a call-by-value left-to-right v-evaluation strategy starting from M that can reduce under a λ only when a head v-normal from is reached. Due to the non-determinism of the head σ-reduction, a term M may have several successors paths. We cannot get rid of the non-determinism of the successors path of M because of the non-sequentiability of head σ-reductions, see p. 9 and [7, p. 10].

▶ **Theorem 24** (Normalization strategy)**.** *Let $M \in \Lambda$. Every successors path $(M_i)_{i\in\mathbb{N}}$ of M is a normalization strategy for M, i.e. if M is v-normalizable then there exists $j, k, \ell \in \mathbb{N}$ such that $j \le k \le \ell$, M_j is head β_v-normal, M_k is head v-normal and M_ℓ is v-normal.*

Proof. Let $(M_i)_{i\in\mathbb{N}}$ be a successors path of M and $N \in \Lambda$ be such that N is v-normal and $M \to^*_{\mathsf{v}} N$: we prove by induction on $N \in \Lambda$ that there exist $j, k, \ell \in \mathbb{N}$ such that M_j is head β_v-normal, M_k is head v-normal and M_ℓ is v-normal.

Since M is v-normalizable, then it is head β_v-normalizable (because $\xrightarrow{h}_{\beta_v} \subseteq \to_{\mathsf{v}}$), thus there exists $j \in \mathbb{N}$ such that M_j is head β_v-normal because $\xrightarrow{h}_{\beta_v}$ is deterministic. As $\xrightarrow{h}_{\sigma}$ is strongly normalizing (by Proposition 5, since $\xrightarrow{h}_{\sigma} \subseteq \to_{\sigma}$), there exists $k \in \mathbb{N}$ with $j \le k$ such that M_k is head σ-normal. According to Lemma 19.4, M_k is also head β_v-normal, hence M_k is head v-normal. Certainly, $M_k = VN_1 \dots N_n$ for some $n \in \mathbb{N}$, $V \in \Lambda_v$ and $N_1, \dots, N_n \in \Lambda$. By confluence of $\to_{\mathsf{v}}$ (Proposition 5) and since N is v-normal and M_k is head v-normal, one has $M_k \xrightarrow{int}^*_{\mathsf{v}} N$ and hence $N = V'N'_1 \dots N'_n$ for some v-normal $V' \in \Lambda_v$ and some v-normal $N'_1, \dots, N'_n \in \Lambda$ such that $V \to^*_{\mathsf{v}} V'$ and $N_r \to^*_{\mathsf{v}} N'_r$ for any $1 \le r \le n$. By induction hypothesis, for every successors path $(V_i)_{i\in\mathbb{N}}$ of V and, for any $1 \le r \le n$, for every successors path $(L^r_i)_{i\in\mathbb{N}}$ of N^r there exist $p, p_1, \dots, p_n \in \mathbb{N}$ such that $V_p, L^1_{p_1}, \dots, L^n_{p_n}$ are v-normal: by confluence of $\to_{\mathsf{v}}$ (Proposition 5), $V_p = V'$ and $N'_r = L^r_{p_r}$ for any $1 \le r \le n$.

Let us consider the infinite sequence of terms $s = (M = M_0, \;\dots\dots\; , M_k = VN_1 \dots N_n = V_0N_1 \dots N_n, \;\dots\dots\; , V_pN_1 \dots N_n = V'L^1_0N_2 \dots N_n, \;\dots\dots\; , V'L^1_{p_1}N_2 \dots N_n = V'N'_1L^2_0 \dots N_n, \;\dots\dots\; , V'N'_1N'_2 \dots N'_n = N, N, \dots\dots)$: this is a successors path of M and, for an opportune choice of the successors paths $(V_i)_{i\in\mathbb{N}}$, $(L^1_i)_{i\in\mathbb{N}}$, ..., $(L^n_i)_{i\in\mathbb{N}}$, one has that $s = (M_i)_{i\in\mathbb{N}}$, in particular there exists $\ell \in \mathbb{N}$ such that $j \le k \le \ell$ and $M_\ell = N$. ◀

In ordinary λ-calculus, the well-known theorem corresponding to our Theorem 24 is the Leftmost Reduction Theorem, see [19, Theorem 2.8] or [3, Theorem 13.2.2]. Differently from the leftmost reduction of ordinary λ-calculus, our normalization strategy is not deterministic, i.e., our Theorem 24 provides a family of normalization strategies.

Finally, we have shown at p. 7 that the head σ- and head v-reductions are not (locally) confluent and a term may have several head v-normal forms. Nevertheless, the characterization of head v-normal forms given by Proposition 18 allows us to claim that (see next Proposition 27) in some cases (of interest), more precisely when a term has a head v-normal form which is a value or an element of Λ_a, the head v-normal form is unique (Proposition 27.1): all terms having several head v-normal forms are such that all their head v-normal forms are in Λ_b. In particular, every head v-normalizable closed term has a unique head v-normal form, which is an abstraction and coincides with its head β_v-normal form (Proposition 27.2).

▶ **Remark 25.** By inspection on the rules of Definition 8, it easy to check that the head σ-reduction does not reduce to a term in Λ_a, i.e., for any $M \in \Lambda$ and $N \in \Lambda_a$, one has $M \not\xrightarrow{h}_{\sigma} N$.

Remark 25 does not hold if we replace $\xrightarrow{h}_{\sigma}$ with $\xrightarrow{h}_{\beta_v}$: for instance, $x(II) \xrightarrow{h}_{\beta_v} xI \in \Lambda_a$.

▶ **Fact 26.** *For every $N \in \Lambda_v \cup \Lambda_a$, one has $M \xrightarrow{h}{}^{*}_{\beta_v} N$ if and only if $M \xrightarrow{h}{}^{*}_{\mathsf{v}} N$.*

Proof. The left-to-right direction follows from $\xrightarrow{h}_{\beta_v} \subseteq \xrightarrow{h}_{\mathsf{v}}$. The right-to-left direction is a consequence of Lemma 12.2 and either Fact 10.2 (if $N \in \Lambda_v$) or Remark 25 (if $N \in \Lambda_a$). ◀

Fact 26 means that, given a head v-reduction sequence, the head σ-reduction plays no role not only in deciding its termination (as stated in Theorem 21), but also in reaching a particular value or term in Λ_a. Fact 26 will be used in the proof of Proposition 27.

▶ **Proposition 27** (Uniqueness of "some" head v-normal forms)**.** *Let $M \in \Lambda$ and $M \xrightarrow{h}{}^{*}_{\mathsf{v}} N$.*

1. *If $N \in \Lambda_v \cup \Lambda_a$ then, for every head v-normal $L \in \Lambda$, $M \xrightarrow{h}{}^{*}_{\mathsf{v}} L$ implies $N = L$.*
2. *If M is closed and N is head v-normal, then $M \xrightarrow{h}{}^{*}_{\beta_v} N$ and $N = \lambda x.N'$ for some $N' \in \Lambda$ such that $\mathsf{fv}(N') \subseteq \{x\}$; moreover, for any head v-normal $L \in \Lambda$, $M \xrightarrow{h}{}^{*}_{\mathsf{v}} L$ implies $N = L$.*

Proof.

1. Since $N \in \Lambda_v \cup \Lambda_a$, $M \xrightarrow{h}{}^{*}_{\mathsf{v}} N$ implies $M \xrightarrow{h}{}^{*}_{\beta_v} N$ by Fact 26. According to Proposition 18.3, N is head v-normal.
   Let $L \in \Lambda$ be head v-normal and such that $M \xrightarrow{h}{}^{*}_{\mathsf{v}} L$: by Proposition 18.3, $L \in \Lambda_v \cup \Lambda_a \cup \Lambda_b$. We claim that $L \notin \Lambda_b$. Otherwise, $L \in \Lambda_b$ and then, by confluence of $\to_{\mathsf{v}}$ there would exist $M' \in \Lambda$ such that $N \to_{\mathsf{v}}^{*} M'$ and $L \to_{\mathsf{v}}^{*} M'$. According to Proposition 11 and since N and L are head v-normal, $N \xrightarrow{int}{}^{*}_{\mathsf{v}} M'$ and $L \xrightarrow{int}{}^{*}_{\mathsf{v}} M'$. By Remark 4 (since $\xrightarrow{int}_{\mathsf{v}} \subseteq \to_{\mathsf{v}}$) and Lemma 19.1, $M' \in \Lambda_v \cup \Lambda_a$. By Lemma 19.2, $M' \in \Lambda_b$. But $\Lambda_v \cap \Lambda_b = \emptyset = \Lambda_a \cap \Lambda_b$: contradiction, therefore $L \notin \Lambda_b$.
   So, $L \in \Lambda_v \cup \Lambda_a$ and thus $M \xrightarrow{h}{}^{*}_{\beta_v} L$ by Fact 26, hence $N = L$ since $\xrightarrow{h}_{\beta_v}$ is deterministic.
2. Since M is closed, N is closed too. Hence, by Proposition 18.3, $N \in \Lambda_v$ (since the terms in $\Lambda_a \cup \Lambda_b$ are not closed) and N is not a variable, therefore $N = \lambda x.N'$ for some $N' \in \Lambda$ such that $\mathsf{fv}(N') \subseteq \{x\}$. By Fact 26, $M \xrightarrow{h}{}^{*}_{\beta_v} N$. According to Proposition 27.1, for every head v-normal $L \in \Lambda$, $M \xrightarrow{h}{}^{*}_{\mathsf{v}} L$ implies $N = L$. ◀

Recall that all head v-normal terms are head β_v-normal, since $\xrightarrow{h}_{\beta_v} \subseteq \xrightarrow{h}_{\mathsf{v}}$.

6 Conclusions and future work

In this paper, we have investigated the λ_v^σ-calculus introduced in [4], an extension of Plotkin's call-by-value λ-calculus λ_v [15] with the same syntax as λ_v (without constants) and ordinary (i.e. call-by-name) λ-calculus. The peculiarity of λ_v^σ is in its reduction rules: the v-reduction adds to Plotkin's β_v-reduction two commutation rules called σ_1 and σ_3 which unblock "hidden" β_v-redexes. We have studied the head v-reduction, a non-confluent sub-reduction of the v-reduction already introduced in [7]. We now summarize our main contributions:

1. Theorem 21 is about head v-normalization, it shows that:
   - for the head v-reduction, normalization coincides with strong normalization;
   - the head v-reduction is deeply related to Plotkin's deterministic weak evaluation strategy for λ_v (the former terminates if and only if the latter terminates);
   - both head v-reduction and weak evaluation strategy for λ_v enjoy good properties analogous to the ones of the (call-by-name) head reduction for ordinary λ-calculus.
2. Theorem 24 is about v-normalization: it proves that a top-down extension of the head v-normalization provides a family of normalization strategies for the (full) v-reduction.
3. Proposition 27 is about the uniqueness of the head v-normal form: it shows that, even if there are terms having several head v-normal forms, in some case of interest (for instance, closed terms) the head v-normal form, if any, is unique.

These results, together with the results proven in [4, 7], shows that λ_v^σ is a useful tool to study some theoretical and semantic properties of Plotkin's λ_v-calculus, for instance the notions of call-by-value solvability and potential valuability. This is hard (or impossible) to obtain directly in λ_v because of the "weakness" of Plotkin's β_v-reduction. In the case of ordinary (i.e. call-by-name) λ-calculus, head reduction and solvability are the starting point to investigate separability, semi-separability and Böhm's trees. Hence, it may reasonably be supposed that we have all the ingredients for tackling the question of separability, semi-separability and Böhm's trees in a call-by-value setting. In particular, one may reasonably hope to improve in λ_v^σ the separability theorem already proven by Paolini [12] for λ_v.

Acknowledgements. The author would like to express his gratitude to Luca Paolini and Simona Ronchi Della Rocca for discussions that inspired this work. Also the author wishes to thank Michele Pagani and the anonymous referees for many helpful comments.

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