Lax Bialgebras and Up-To Techniques for Weak Bisimulations

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─── **Abstract** ───────────────────────────────────

Up-to techniques are useful tools for optimising proofs of behavioural equivalence of processes. Bisimulations up-to context can be safely used in any language specified by GSOS rules. We showed this result in a previous paper by exploiting the well-known observation by Turi and Plotkin that such languages form *bialgebras*. In this paper, we prove the soundness of up-to contextual closure for weak bisimulations of systems specified by *cool rule formats*, as defined by Bloom to ensure congruence of weak bisimilarity. However, the weak transition systems obtained from such cool rules give rise to *lax bialgebras*, rather than to bialgebras. Hence, to reach our goal, we extend our previously developed categorical framework to an ordered setting.

1   Introduction

Bisimilarity ($\sim$) is a fundamental equivalence for concurrent systems: two processes are (strongly) bisimilar if they cannot be distinguished by an external observer interacting with them. Formally, it is defined by coinduction, as the greatest fixpoint of a suitable predicate transformer B – a monotone function on binary relations. In particular, to prove processes bisimilar, it suffices to exhibit a *bisimulation* relating them, i.e., a relation R such that $R \subseteq B(R)$; this latter requirement codes for the standard game where the processes must answer to the labelled transitions of each other.

Up-to techniques are enhancements of this coinduction principle; they were introduced by Milner to simplify behavioural equivalence proofs of CCS processes, see [16]. The range of applicability of up-to techniques goes well beyond concurrency theory: they have been used to obtain decidability results [8], to optimise state-of-the-art automata algorithms [7] or in conjuction with parameterized coinduction for mechanizations of coinductive proofs [13].

An up-to technique is a monotone map A on the poset of relations, and a *bisimulation up-to A* is a relation R such that $R \subseteq B(A(R))$. If A is *sound*, that is, if every bisimulation up-to A is included in a bisimulation, one can prove bisimilarity results by exhibiting bisimulations up to A. This may be computationally less expensive than finding actual bisimulations. Typical examples for (strong) bisimilarity include *up-to bisimilarity*, where A is given by $A(R) = \sim R \sim$, and *up-to transitive closure*, where $A(R)$ is the least transitive relation containing R. When the systems at hand are specified in some process algebra, via an algebraic signature, a third example is *up-to context* – where one maps a relation to its

closure with respect to the contexts of the language. Such a technique is sound in CCS, for instance.

In practice, one is often interested in *weak bisimilarity*, a coarser notion allowing to abstract over internal transitions, labelled with the special action τ. When the player proposes a transition $\xrightarrow{a}$, the opponent must answer with a *saturated* transition $\xRightarrow{a}$, which is roughly a transition $\xrightarrow{a}$ possibly combined with internal actions $\xrightarrow{\tau}$. This slight dissymmetry results in a much more delicate theory of up-to techniques. For instance, up-to weak bisimilarity and up-to transitive closure are no longer sound for weak bisimulations. And up-to contextual closure has to be restricted: the external choice from CCS cannot be freely used [20].

Simpler properties also become harder with weak bisimilarity. Consider structural operational semantics [1]: if the semantics of a language is specified by rules adhering to certain formats, then certain well-behavedness properties are automatically inferred. For instance, in languages specified using De-Simone or GSOS rule formats [9, 5], strong bisimilarity is guaranteed to be a congruence. However, those two formats do not ensure congruence of weak bisimilarity, and more advanced formats had to be designed to this end, like Bloom's *cool GSOS format* [4].

Proving soundness of up-to techniques can be rather complicated. To simplify this task, Sangiorgi and Pous devised the stronger notion of *compatible* up-to techniques, which are always sound and, moreover, closed under composition. Proving compatibility of a composite technique can thus be broken into simpler, independent proofs [18, 19]. We recently generalised this framework to a fibrational setting [6], allowing to obtain once and for all the compatibility of a wide range of techniques for strong bisimulation and simulation, for systems modelled as *bialgebras*.

Concerning weak bisimilarity, we proved in [6] that for positive GSOS specification, if the strong $\rightarrow$ and saturated $\Rightarrow$ transition systems form bialgebras, then up-to context is a compatible technique. Unfortunately, the bearing of this result in practical situations is rather limited, since in many important cases the saturated transition system does not form a bialgebra. Intuitively, in a bialgebra *all and only* the transitions of a composite system can be derived by transitions of its components. For $\Rightarrow$, one implication fails: a composite system performs weak transitions which are not derived from transitions of its components (see Example 2). These systems give rise to so called *lax bialgebras*; this is the key observation that lead to the rather involved refinement of the theory we propose here.

Contributions.   In this paper: a) We extend the previously developed framework [6] to an ordered setting, b) we prove that *up-to context* is compatible for lax models of positive [1] GSOS specifications, and, c) as an application, we obtain soundness of up-to context for weak bisimulations of systems specified by the *cool rule format* from [23].

Outline.   We give some necessary preliminaries in Section 2. Then we move to an ordered setting in Section 3, where we use lax bialgebras. In Section 4 we consider the special case of lax bialgebras stemming from (lax models of) positive GSOS specifications. We finally assemble in Section 5 all the technical pieces into our main result, Theorem 20.

2   Preliminaries

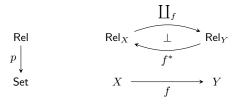### 2.1   Transitions systems and bisimulations

A labelled transition system (LTS) with labels in L consists of a set of states X and a transition function $\xi\colon X \to (\mathcal{P}_\omega X)^L$ that, for every state $x \in X$ and label $a \in L$, assigns a

finite set of possible successor states. We write $x \xrightarrow{a} y$ whenever $y \in \xi(x)(a)$. A *(strong)* *bisimulation* is a relation $R \subseteq X^2$ on the states of an LTS such that for every pair $(x, y) \in R$: (1) if $x \xrightarrow{a} x'$ then $y \xrightarrow{a} y'$ for some y' with $(x', y') \in R$, and (2) vice versa. A *weak bisimulation* is a relation $R \subseteq X^2$ such that for every pair $(x, y) \in R$: (1) if $x \xrightarrow{a} x'$ then $y \xRightarrow{a} y'$ for some y' with $(x', y') \in R$ and (2) if $y \xrightarrow{a} y'$ then $x \xRightarrow{a} x'$ for some x' with $(x', y') \in R$. Here $\Rightarrow$ is the *saturation* [16] of $\rightarrow$, defined by the following rules where τ denotes a special label in L.

$$\frac{x \xrightarrow{a} y}{x \xRightarrow{a} y} \qquad \frac{}{x \xRightarrow{\tau} x} \qquad \frac{x \xRightarrow{\tau} y \xRightarrow{\tau} z}{x \xRightarrow{\tau} z} \qquad \frac{x \xRightarrow{\tau} x' \xRightarrow{a} y' \xRightarrow{\tau} y}{x \xRightarrow{a} y} \tag{1}$$

Transition systems are an instance of the abstract notion of *coalgebras*: given a functor $F\colon C \to C$ on some category C, an F-coalgebra is a pair (X, ξ) where X is an object and $\xi\colon X \to FX$ a morphism. Indeed, LTSs are coalgebras for the functor $(\mathcal{P}_\omega-)^L\colon \mathsf{Set} \to \mathsf{Set}$.

Next, we recall the basic infrastructure of relations that allows us to study both strong and weak bisimulations within a coalgebraic setting. Consider the category Rel whose objects are relations $R \subseteq X^2$ and morphisms from $R \subseteq X^2$ to $S \subseteq Y^2$ are maps from X to Y sending pairs in R to pairs in S. For each set X we denote by Rel_X the category (which in this case is just a preorder) of binary relations on X, ordered by subset inclusion. For a function $f\colon X \to Y$ in Set, we have the following situation in Rel:

$$
\begin{array}{ccc}
\mathsf{Rel} & \qquad & \mathsf{Rel}_X \underset{f^*}{\overset{\coprod_f}{\rightleftarrows}} \mathsf{Rel}_Y \\[2mm]
p \downarrow & & \\[2mm]
\mathsf{Set} & & X \xrightarrow{\;\;f\;\;} Y
\end{array}
$$

where p maps a relation $R \subseteq X^2$ to X, and the functors (monotone maps) f^* and $\coprod_f$, which we will call *reindexing* and *direct image*, are given by inverse and direct image, respectively: $f^*(S) = (f \times f)^{-1}(S)$ for all $S \in \mathsf{Rel}_Y$ and $\coprod_f(R) = (f \times f)[R]$ for all $R \in \mathsf{Rel}_X$. Moreover we have that $\coprod_f$ is a left adjoint for f^*.[1]

A functor $\overline{F}\colon \mathsf{Rel} \to \mathsf{Rel}$ is a *lifting* of $F\colon \mathsf{Set} \to \mathsf{Set}$ whenever $p \circ \overline{F} = F \circ p$; this means that $\overline{F}$ maps a relation on X to a relation on FX. Any lifting $\overline{F}$ can thus be restricted to a functor $\overline{F}_X\colon \mathsf{Rel}_X \to \mathsf{Rel}_{FX}$, which is just a monotone function between posets. For every functor $F\colon \mathsf{Set} \to \mathsf{Set}$, there is a *canonical lifting* denoted by $\mathsf{Rel}(F)\colon \mathsf{Rel} \to \mathsf{Rel}$. In this paper the canonical lifting will play an important role but, for the sake of simplicity, we avoid giving the general definition and refer the interested reader to [14]. As an example, the canonical lifting of $(\mathcal{P}_\omega-)^L$ is defined for all relations $R \subseteq X^2$, and $f, g \in (\mathcal{P}_\omega X)^L$ as

$$f \; \mathsf{Rel}((\mathcal{P}_\omega-)^L)(R) \; g \quad \text{iff} \quad \begin{array}{l} \forall a \in L. \forall x \in f(a).\, \exists y \in g(a).\, xRy \\ \forall a \in L. \forall y \in g(a).\, \exists x \in f(a).\, xRy \end{array} \tag{2}$$

We can now define bisimulations for any Set-functor F in terms of its canonical lifting. For an F-coalgebra (X, ξ), a (Hermida-Jacobs) bisimulation [11] is a coalgebra for the functor

$$\mathsf{Rel}(F)_\xi \;\triangleq\; \xi^* \circ \mathsf{Rel}(F)_X\colon \mathsf{Rel}_X \to \mathsf{Rel}_X\,.$$

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[1] The categorically minded reader may observe that the forgetful functor $p\colon \mathsf{Rel} \to \mathsf{Set}$ is a bifibration, but the concrete definitions given above suffice for understanding the forthcoming technical developments.

The functor $\mathsf{Rel}(F)_\xi$ is also called a predicate transformer. Bisimilarity is defined as the largest bisimulation or, in other terms, as the final $\mathsf{Rel}(F)_\xi$-coalgebra. Since morphisms in Rel_X are just inclusions, a coalgebra for $\mathsf{Rel}(F)_\xi$ is a relation R such that $R \subseteq \mathsf{Rel}(F)_\xi(R)$ and, with (2), it is easy to check that for $FX = (\mathcal{P}_\omega X)^L$ this corresponds to the usual definition of strong bisimulations on transition systems.

The above notion can be further generalised by taking an arbitrary lifting $\overline{F}\colon \mathsf{Rel} \to \mathsf{Rel}$ of F: an $\overline{F}_\xi$-*bisimulation* then is a coalgebra for the endofunctor

$$\overline{F}_\xi \triangleq \xi^* \circ \overline{F}_X \colon \mathsf{Rel}_X \to \mathsf{Rel}_X \,.$$

With this more abstract approach, we can capture various interesting coinductive predicates other than strong bisimilarity, such as simulations [12] and weak bisimulations. Indeed, weak bisimulations are coalgebras for the functor $\overline{F \times F}_\xi \colon \mathsf{Rel}_X \to \mathsf{Rel}_X$ where $F = (\mathcal{P}_\omega -)^L$, $\xi = \langle \to, \Rightarrow \rangle \colon X \to FX \times FX$ is the pairing of the strong transition system $\to$ and its saturation $\Rightarrow$, and the functor $\overline{F \times F}$ is the lifting of $F \times F$ to Rel given for a relation R by

$$(f,g)\ \overline{F \times F}(R)\ (f',g') \quad \text{iff} \quad \begin{array}{l} \forall a \in L.\forall x \in f(a).\ \exists y \in g'(a).xRy \\ \forall a \in L.\forall x \in f'(a).\ \exists y \in g(a).xRy \end{array} \tag{3}$$

2.2   Up-To techniques and Compatible functors

In the previous section we have seen how bisimulations can be regarded as coalgebras (post-fixpoints) for a functor (monotone map) $\overline{F}_\xi \colon \mathsf{Rel}_X \to \mathsf{Rel}_X$. In this perspective, an up-to technique is a functor $A\colon \mathsf{Rel}_X \to \mathsf{Rel}_X$ and an $\overline{F}_\xi$-*bisimulation up to A* is an $\overline{F}_\xi A$-coalgebra. For instance, a bisimulation up to equivalence is a $\mathsf{Rel}(F)_\xi E$-coalgebra, where $E\colon \mathsf{Rel}_X \to \mathsf{Rel}_X$ is the functor mapping a relation to its equivalence closure.

We say that A is $\overline{F}_\xi$-*compatible* if there exists a distributive law (natural transformation) $\rho\colon A\overline{F}_\xi \Rightarrow \overline{F}_\xi A$. If A is $\overline{F}_\xi$-compatible then A is a sound up-to technique: every $\overline{F}_\xi$-bisimulation up-to A is included in an $\overline{F}_\xi$-bisimulation. This is stated in [19, Theorem 6.3.9] for the case of lattices, but it holds more generally in any category with countable coproducts and, rather than considering just endofunctors F on Set and their liftings $\overline{F}$ to Rel, one can take endofunctors and liftings in arbitrary fibrations [6]. For the sake of simplicity we will avoid using fibrations: the reader should only know that the above result holds also for Pre-endofunctors and their liftings to the category $\mathsf{Rel}^\uparrow$ which we introduce in Section 3.

By tuning F, $\overline{F}$ and A one can consider different sorts of, respectively, state-based systems (such as LTSs, deterministic or weighted automata), coinductive predicates (such as bisimilarity, similarity or language equivalence) and up-to techniques (such as up-to transitivity, up-to equivalence, up-to bisimilarity). In [6], we provided several techniques for proving the compatibility of particular techniques. For up-to context, the state space of the coalgebra needs to have some algebraic structure, for instance, the LTSs of process algebras. This is captured systematically by bialgebras: given functors $F, T\colon \mathsf{Set} \to \mathsf{Set}$ and a distributive law $\rho\colon TF \Rightarrow FT$, a ρ-*bialgebra* consists of a set X, an algebra $\alpha\colon TX \to X$ and a coalgebra $\xi\colon X \to FX$ such that the following diagram commutes.

$$\begin{array}{ccccc} TX & \xrightarrow{\ \alpha\ } & X & \xrightarrow{\ \xi\ } & FX \\ {\scriptstyle T\xi}\big\downarrow & & & & \big\uparrow{\scriptstyle F\alpha} \\ TFX & & \xrightarrow[\ \rho_X\ ]{} & & FTX \end{array}$$

The function mapping a relation on X to its contextual closure can be obtained as

$$Ctx \triangleq \coprod\nolimits_\alpha \circ \mathsf{Rel}(T)_X \colon \mathsf{Rel}_X \to \mathsf{Rel}_X \,.$$

To prove the compatibility of *Ctx* w.r.t. different $\overline{F}_\xi$, we showed the theorem below, where we adopt the following terminology: a natural transformation $\overline{\sigma} \colon \overline{F} \Rightarrow \overline{G}$ between Rel-functors is a *lifting* of $\sigma \colon F \Rightarrow G$ when for every $R \in \mathsf{Rel}$ we have that $p(\overline{\sigma}_R) = \sigma_{p(R)}$.

▶ **Theorem 1** (see [6])**.** *Let* (X, α, ξ) *be a ρ-bialgebra and* $\overline{T}, \overline{F} \colon \mathcal{E} \to \mathcal{E}$ *be liftings of T and F. If $\overline{\rho} \colon \overline{T}\,\overline{F} \Rightarrow \overline{F}\,\overline{T}$ is a lifting of ρ, then $\coprod_\alpha \circ \overline{T}$ is $\overline{F}_\xi$-compatible.*

2.3   Abstract GSOS specifications and their models

Abstract GSOS specifications are natural transformations of the form $\lambda \colon S(F \times \mathrm{Id}) \Rightarrow FT$, where T is the free monad over S. As shown in [22], they generalise the concrete GSOS rules, for which $FX = (\mathcal{P}_\omega X)^L$, and S is a polynomial functor – a coproduct of products – representing an algebraic signature, and hence TX is the set of terms over this signature with variables in X. A *model* of a specification λ is a triple (X, α, ξ), where $\xi \colon X \to FX$ and $\alpha \colon SX \to X$, making the following diagram commute:

$$
\begin{array}{ccc}
SX & \xrightarrow{\ \alpha\ } X \xrightarrow{\ \xi\ } & FX \\
{\scriptstyle S\langle \xi, \mathrm{id}\rangle}\Big\downarrow & & \Big\uparrow{\scriptstyle F\alpha^\sharp} \\
S(FX \times X) & \xrightarrow[\ \lambda_X\ ]{} & FTX
\end{array}
\tag{4}
$$

▶ **Example 2.** Consider the parallel operator of CCS [16], whose semantics is defined by the following GSOS rules

$$
\frac{p \xrightarrow{\mu} p'}{p|q \xrightarrow{\mu} p'|q} \qquad
\frac{q \xrightarrow{\mu} q'}{p|q \xrightarrow{\mu} p|q'} \qquad
\frac{p \xrightarrow{a} p' \quad q \xrightarrow{\overline{a}} q'}{p|q \xrightarrow{\tau} p'|q'}
$$

where μ ranges over arbitrary actions, namely inputs $a, b, \dots$ outputs $\overline{a}, \overline{b}, \dots$ or the internal action τ. Take $SX = X \times X$ (for the binary parallel operator) and $F = (\mathcal{P}_\omega -)^L$ where L is the set of all actions. For every set X, the corresponding distributive law $\lambda_X \colon S(FX \times X) \to FTX$ maps $(f, x, g, y) \in (\mathcal{P}_\omega X)^L \times X \times (\mathcal{P}_\omega X)^L \times X$ to the function

$$
\mu \mapsto \begin{cases} \{(x', y) \mid x' \in f(\mu)\} \cup \{(x, y') \mid y' \in g(\mu)\} & \mu \neq \tau \\ \{(x', y) \mid x' \in f(\tau)\} \cup \{(x, y') \mid y' \in g(\tau)\} \cup \{(x', y') \mid \exists a.\, x' \in f(a), y' \in g(\overline{a})\} & \mu = \tau \end{cases}
$$

Now take X to be the set of *all* CCS processes, $\xi \colon X \to (\mathcal{P}_\omega X)^L$ the LTS generated by the standard semantics of CCS [16] and $\alpha \colon X \times X \to X$ to be the algebra mapping a pair of processes (p, q) to their parallel composition $p|q$. It is easy to see that diagram (4) commutes, i.e., (X, α, ξ) is a model for λ.

On the contrary, if we take ξ to be the saturation of the standard CCS semantics, diagram (4) does not commute anymore: take the pairs of CCS processes $(a.b.0, \overline{a}.\overline{b}.0) \in SX$. Following the topmost line, one first maps it to $a.b.0|\overline{a}.\overline{b}.0$ in the weak LTS that, for instance, contains the transition $\xRightarrow{\tau} 0|0$. Following the other path in the diagram one obtains first the tuple $(((a \mapsto \{b.0\}), a.b.0),\ ((\overline{a} \mapsto \{\overline{b}.0\}), \overline{a}.\overline{b}.0))$ where $\mu \mapsto S$ denotes the function assigning to the action μ the set S and to all the others actions the empty set. This tuple is mapped by λ_X to the function

$$
a \mapsto \{(b.0, \overline{a}.\overline{b}.0)\} \qquad \overline{a} \mapsto \{(a.b.0, \overline{b}.0)\} \qquad \tau \mapsto \{(b.0, \overline{b}.0)\}
$$

and then by $F\alpha^\sharp$ to

$$
a \mapsto \{b.0|\overline{a}.\overline{b}.0\} \qquad \overline{a} \mapsto \{a.b.0|\overline{b}.0\} \qquad \tau \mapsto \{b.0|\overline{b}.0\}
$$

Observe that with τ, one cannot reach the state $0|0$.

An abstract GSOS specification λ and a model (X, α, ξ) for it induce respectively a distributive law $\rho\colon T(F \times \mathrm{Id}) \Rightarrow (F \times \mathrm{Id})T$ of the monad T over the copointed functor $F \times \mathrm{Id}$ and a bialgebra $(X, \alpha^\sharp, \langle \xi, \mathrm{id} \rangle)$ for ρ [22, 15]. Using these facts and the characterization of weak bisimulations given in (3) we were able to prove the following result.

▶ **Proposition 3** (see [6]). *Let $FX = (\mathcal{P}_\omega X)^L$, and let $\lambda\colon S(F \times \mathrm{Id}) \Rightarrow FT$ be a GSOS specification with two models (X, α, ξ_1) and (X, α, ξ_2). If λ is positive (see, e.g., [1]) then:*

1. *There exists a distributive law $\rho\colon T(F \times F \times \mathrm{Id}) \Rightarrow (F \times F \times \mathrm{Id})T$ s.t. $(X, \alpha^\sharp, \langle \xi_1, \xi_2, \mathrm{id} \rangle)$ is a ρ-bialgebra.*
2. *There exists $\overline{\rho}\colon \mathsf{Rel}(T)(\overline{F \times F} \times \mathrm{Id}) \Rightarrow (\overline{F \times F} \times \mathrm{Id})\mathsf{Rel}(T)$ lifting ρ, where $\overline{F \times F}$ is defined as in (3).*
3. *By the previous points and Theorem 1, $Ctx = \coprod_{\alpha^\sharp} \circ \mathsf{Rel}(T)_X$ is $(\overline{F \times F} \times \mathrm{Id})_{\langle \xi_1, \xi_2, \mathrm{id} \rangle}$-compatible.*

The above result ensures compatibility w.r.t. $(\overline{F \times F} \times \mathrm{Id})_{\langle \xi_1, \xi_2, \mathrm{id} \rangle}$, which is not exactly $\overline{F \times F}_{\langle \xi_1, \xi_2 \rangle}$. As discussed in [6], weak bisimulations are coalgebras for either of these two predicate transformers. The extra Id is harmless for the above result and for Theorem 20.

Proposition 3 gives us compatibility of up-to Ctx for weak bisimulation whenever ξ_2, given by the saturation of ξ_1, is a model for the GSOS specification. However, Example 2 shows that already for the simple case of parallel composition in CCS, ξ_2 is *not* a model for the GSOS specification. This motivates the need for relaxing the hypothesis of Proposition 3: in the rest of the paper, we will introduce the notions of lax bialgebras and lax models and we will show the analogues of Theorem 1 and Proposition 3 in an ordered setting.

3   Bialgebras and compatibility in an ordered setting

We recalled how to prove soundness of up-to techniques in a modular way, by considering lifting functors and distributive laws along $p\colon \mathsf{Rel} \to \mathsf{Set}$. Now we extend those results to an ordered setting. The first step (Section 3.1) consists in replacing the base category Set with Pre, the category of preorders. (An object in Pre is a set equipped with a preorder, that is, a reflexive and transitive relation; morphisms are monotone maps.) Accordingly, we move from the category Rel of relations to its subcategory $\mathsf{Rel}^\uparrow$ of up-closed relations (Section 3.2). We finally obtain the ordered counterpart to Theorem 1, using the notion of lax bialgebra (Section 3.3, Theorem 15).

3.1   Lifting functors from sets to preorders

We first explain how to lift functors and distributive laws from Set to Pre. Extensions of Set-functors to preorders or posets have been studied via relators as in [12, 21] and using presentations of functors and (enriched) Kan extensions [2, 3]. We are interested in extending not only functors, but also natural transformations to an ordered setting. The description using (lax) relation liftings [12] allows us to leverage some of our results in [6] to extend natural transformations.

For a weak pullback preserving Set-endofunctor T we can consider its canonical relation lifting $\mathsf{Rel}(T)\colon \mathsf{Rel} \to \mathsf{Rel}$. Then, using the following well-known result, we obtain an extension of T to Pre, hereafter called *the canonical Pre-lifting of T* and denoted by $\mathsf{Pre}(T)$.

▶ **Lemma 4.** *If T preserves weak pullbacks then $\mathsf{Rel}(T)$ restricts to a functor $\mathsf{Pre}(T)$ on Pre.*

However, sometimes we are interested in liftings of functors to Pre that are not restrictions of the canonical relation lifting. One such example is the lifting of the LTS functor $(\mathcal{P}_\omega -)^L$

to Pre that maps a preordered set $(X, \leq)$ to $((\mathcal{P}_\omega X)^L, \sqsubseteq)$, where $\sqsubseteq$ is given by

$$f \sqsubseteq g \text{ iff } \forall a \in L : \text{ if } x \in f(a) \text{ then there is } y \in g(a) \text{ such that } x \leq y. \tag{5}$$

This lifting is also a restriction to Pre of relation lifting for $(\mathcal{P}_\omega -)^L$, albeit not the canonical one, but the *lax relation lifting*, as defined in [12]. To describe it, recall from [12] that a Set-functor F is called *ordered* when it factors through a functor $F_\subseteq \colon \mathsf{Set} \to \mathsf{Pre}$.

We denote by $\subseteq_{FX}$ the order on FX given by $F_\subseteq(X)$. The lax relation lifting of F is defined as the functor $\mathsf{Rel}_\subseteq(F) \colon \mathsf{Rel} \to \mathsf{Rel}$ that maps a relation R on X to $\subseteq_{FX} \otimes \mathsf{Rel}(F)(R) \otimes \subseteq_{FX}$, where $\otimes$ denotes composition of relations. In [12, Lemma 5.5] it is shown that $\mathsf{Rel}_\subseteq(F)$ restricts to a functor $\mathsf{Pre}_\subseteq(F)$ on Pre, if the order $\subseteq_{FX}$ has an additional property, namely it is stable, see [12, Definition 4.3]. This property is duly satisfied by all the ordered functors considered in this paper. We call the restriction of $\mathsf{Rel}_\subseteq(F)$ to Pre the *lax Pre-lifting of F* and denote it by $\mathsf{Pre}_\subseteq(F)$.

▶ **Example 5** (see [12]). The LTS functor $(\mathcal{P}_\omega -)^L$ has a stable order $\subseteq_{(\mathcal{P}_\omega X)^L}$ given by pointwise inclusion. The lax Pre-lifting of $(\mathcal{P}_\omega -)^L$ with respect to this order coincides with the lifting described above in (5).

We now show how to lift a natural transformation $\rho \colon F \Rightarrow G$ between Set-functors to a natural transformation $\varrho \colon \mathcal{F} \Rightarrow \mathcal{G}$ between Pre-functors. If F and G preserve weak pullbacks and $\mathcal{F}$ and $\mathcal{G}$ are the canonical Pre-extensions $\mathsf{Pre}(F)$ and $\mathsf{Pre}(G)$, then ϱ is obtained via the restriction of the natural transformation $\mathsf{Rel}(\rho)$ between the corresponding canonical relation liftings ($\mathsf{Rel}(-)$ is functorial, see [14]). The situation is slightly more complex for non-canonical liftings, such as the lax lifting of the LTS functor. In this case we can use Lemma 7 below whenever ρ enjoys the following monotonicity property.

▶ **Definition 6.** Let $F, G \colon \mathsf{Set} \to \mathsf{Set}$ be ordered functors that factor through $F_\subseteq, G_\subseteq \colon \mathsf{Set} \to \mathsf{Pre}$ respectively. We say that a natural transformation $\rho \colon F \Rightarrow G$ is *monotone* if it lifts to a natural transformation $\varrho \colon F_\subseteq \Rightarrow G_\subseteq$ defined by $\varrho_X = \rho_X$.

Spelling out Definition 6 we obtain that ρ is monotone iff for every $t, u \in FX$:

$$t \subseteq_{FX} u \quad \text{implies} \quad \rho(t) \subseteq_{GX} \rho(u)$$

where $\subseteq_{FX}$ and $\subseteq_{GX}$ denote the orders on FX and GX given by $F_\subseteq$ and $G_\subseteq$ respectively.

▶ **Lemma 7.** *Let $F, G \colon \mathsf{Set} \to \mathsf{Set}$ be ordered functors with orders given by $F_\subseteq, G_\subseteq \colon \mathsf{Set} \to \mathsf{Pre}$ respectively, and assume $\rho \colon F \Rightarrow G$ is a monotone natural transformation. Then ρ lifts to a natural transformation $\overline{\rho} \colon \mathsf{Rel}_\subseteq(F) \Rightarrow \mathsf{Rel}_\subseteq(G)$. Furthermore, if the lax relation liftings of F and G restrict to Pre-endofunctors $\mathsf{Pre}_\subseteq(F)$ and $\mathsf{Pre}_\subseteq(G)$ then ρ lifts to a natural transformation $\varrho \colon \mathsf{Pre}_\subseteq(F) \Rightarrow \mathsf{Pre}_\subseteq(G)$.*

3.2 Relation liftings for Pre-endofunctors
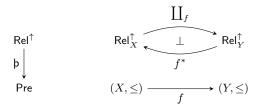
In the previous section we have seen how to extend Set functors, such as those involved in GSOS specifications, to preorders. To reason about relation liftings in this setting we ought to consider a category of relations with a forgetful functor to Pre. On a preorder $(X, \leq)$ we consider relations that are *up-closed* with respect to $\leq$, as defined next.

▶ **Definition 8.** Given a preorder $(X, \leq)$ we define an *up-closed relation* on X as a relation $R \subseteq X^2$ such that for every $x', x, y, y' \in X$ with $x \leq x'$, $y \leq y'$ and xRy we have that $x'Ry'$. A morphism between up-closed relations R and S on $(X, \leq)$, respectively $(Y, \leq)$, is a monotone map $f \colon (X, \leq) \to (Y, \leq)$ such that $R \subseteq (f \times f)^{-1}(S)$.

We denote by $\mathsf{Rel}^\uparrow$ the category of up-closed relations. We have an obvious forgetful functor $\flat \colon \mathsf{Rel}^\uparrow \to \mathsf{Pre}$ mapping every up-closed relation to its underlying preorder. For each preorder $(X, \leq)$ we denote by $\mathsf{Rel}_X^\uparrow$ the subcategory of $\mathsf{Rel}^\uparrow$ whose objects are mapped by $\flat$ to $(X, \leq)$ and morphisms are mapped by $\flat$ to the identity on $(X, \leq)$. Notice that $\mathsf{Rel}_X^\uparrow$ is a category, with morphisms given by inclusions of relations, hence, a preorder.

For a monotone map $f \colon (X, \leq) \to (Y, \leq)$ in Pre, we have the following situation in $\mathsf{Rel}^\uparrow$, similar to the situation described for Rel in Section 2:

$$
\begin{array}{ccc}
\mathsf{Rel}^\uparrow & & \mathsf{Rel}_X^\uparrow \underset{f^*}{\overset{\coprod_f}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} \mathsf{Rel}_Y^\uparrow \\
\flat \downarrow & & \\
\mathsf{Pre} & & (X, \leq) \xrightarrow{\ \ f\ \ } (Y, \leq)
\end{array}
$$

Here, the *reindexing* functor f^* is given by inverse image, i.e., $f^*(S) = (f \times f)^{-1}(S)$ for all $S \in \mathsf{Rel}_Y^\uparrow$ while the *direct image* functor $\coprod_f$ is defined on a up-closed relation $R \in \mathsf{Rel}_X^\uparrow$ as the least up-closed relation containing $(f \times f)[R]$. Just as in the case of Rel, the functor $\coprod_f$ is a left adjoint of f^*, and $\flat \colon \mathsf{Rel}^\uparrow \to \mathsf{Pre}$ is a bifibration. Observe that if the preorder on Y is discrete, then $\coprod_f$ is given simply by direct image.

▶ **Remark 9.** *For every discrete preorder (X, Δ_X), any relation on X is automatically up-closed. We can reformulate this in a conceptual way, using that the forgetful functor $U \colon \mathsf{Pre} \to \mathsf{Set}$ has a left adjoint $D \colon \mathsf{Set} \to \mathsf{Pre}$ mapping a set X to the discrete preorder (X, Δ_X). Then the adjunction $D \dashv U$ lifts to an adjunction $\overline{D} \dashv \overline{U} \colon \mathsf{Rel}^\uparrow \to \mathsf{Rel}$.*

Pre has an enriched structure, in the sense that the homsets are equipped with an order themselves. Given morphisms $f, g \colon (X, \leq) \to (Y, \leq)$ we say that $f \leq g$ if $f(x) \leq_Y g(x)$ for every $x \in X$. This order is preserved by the reindexing functors:

▶ **Lemma 10.** *For any Pre-morphisms $f, g \colon (X, \leq) \to (Y, \leq)$ such that $f \leq g$ there exists a (unique) natural transformation $f^* \Rightarrow g^*$.*

We now show how to port liftings of functors from Rel and Pre to $\mathsf{Rel}^\uparrow$.

▶ **Lemma 11.** *For a weak pullback preserving Set-functor T, the canonical Pre-lifting $\mathsf{Pre}(T)$ has a lifting $\overline{\mathsf{Pre}(T)}$ to $\mathsf{Rel}^\uparrow$ acting on a relation as the canonical relation lifting $\mathsf{Rel}(T)$.*

Some of the liftings used in Section 5 to describe weak bisimulations are not canonical, nor lax relation liftings. In Equation (3) we saw how to obtain the weak bisimulation game via a relation lifting $\overline{F \times F}$ of the functor $F \times F$ with $FX = (\mathcal{P}_\omega X)^L$. The next example gives a lifting of $F \times F$ to Pre, such that the relation lifting (3) restricts to up-closed relations, thus yielding a functor on $\mathsf{Rel}^\uparrow$ for the weak bisimulation game.

▶ **Example 12.** For $F = (\mathcal{P}_\omega -)^L$ we consider the Pre-endofunctor $\mathsf{Pre}(F) \times \mathsf{Pre}_\subseteq(F)$, where $\mathsf{Pre}(F)$ is the canonical Pre-lifting of F and $\mathsf{Pre}_\subseteq(F)$ is the lax Pre-lifting of Example 5. In Appendix A, we show that for any preorder $(X, \leq)$ and $R \in \mathsf{Rel}_{(X, \leq)}^\uparrow$ we have that $\overline{F \times F}(R)$ as defined in (3) is an up-closed relation on $\mathsf{Pre}(F)(X, \leq) \times \mathsf{Pre}_\subseteq(F)(X, \leq)$.

Thus we obtain a lifting $\overline{\mathsf{Pre}(F) \times \mathsf{Pre}_\subseteq(F)}$ of $\mathsf{Pre}(F) \times \mathsf{Pre}_\subseteq(F)$ to $\mathsf{Rel}^\uparrow$ such that $\overline{U}\,\overline{\mathsf{Pre}(F) \times \mathsf{Pre}_\subseteq(F)} = \overline{(F \times F)}\,\overline{U}$. This means that coalgebras for $\overline{\mathsf{Pre}(F) \times \mathsf{Pre}_\subseteq(F)}_{\langle \xi_1, \xi_2 \rangle}$ correspond to weak bisimulations, whenever ξ_2 is the saturation of ξ_1.

In Theorem 20 we will need liftings of natural transformations to $\mathsf{Rel}^\uparrow$. We show next how to obtain them leveraging existing liftings to Rel and Pre introduced in Sections 2 and 3.1.

▶ **Lemma 13.** *Consider* Set*-functors* F, T *with respective liftings* $\overline{F}, \overline{T}$ *on* Rel*;* $\mathcal{F}, \mathcal{T}$ *on* Pre*. Assume that* $\mathcal{F}$ *and* $\mathcal{T}$ *lift to* $\overline{\mathcal{F}}$ *and* $\overline{\mathcal{T}}$ *on* $\mathsf{Rel}^\uparrow$*, such that* $\overline{U}\overline{\mathcal{T}} = \overline{T}\overline{U}$ *and* $\overline{U}\overline{\mathcal{F}} = \overline{F}\overline{U}$*, as in the diagram*

$$
\begin{array}{ccc}
\overline{\mathcal{F}},\overline{\mathcal{T}} \;\circlearrowright\; \mathsf{Rel}^\uparrow & \xrightarrow{\;\overline{U}\;} & \mathsf{Rel} \;\circlearrowright\; \overline{F},\overline{T} \\
\downarrow & & \downarrow \\
\mathcal{F},\mathcal{T} \;\circlearrowright\; \mathsf{Pre} & \xrightarrow[\;U\;]{} & \mathsf{Set} \;\circlearrowright\; F,T
\end{array}
$$

Assume further that we have a natural transformation $\rho \colon TF \Rightarrow FT$ *that lifts to both* $\varrho \colon \mathcal{T}\mathcal{F} \Rightarrow \mathcal{F}\mathcal{T}$ *and* $\overline{\rho} \colon \overline{T}\,\overline{F} \Rightarrow \overline{F}\,\overline{T}$*. Then* ϱ *also lifts to a natural transformation* $\overline{\varrho} \colon \overline{\mathcal{T}}\,\overline{\mathcal{F}} \Rightarrow \overline{\mathcal{F}}\,\overline{\mathcal{T}}$*.*

In the sequel, we use notations for liftings as in the above lemma: for a functor F, we denote by calligraphic $\mathcal{F}$ a lifting along $\mathsf{Pre} \to \mathsf{Set}$ and by $\overline{\mathcal{F}}$ a lifting of $\mathcal{F}$ along $\mathsf{Rel}^\uparrow \to \mathsf{Pre}$; for natural transformations, we use ϱ for a lifting of ρ to Pre and $\overline{\varrho}$ for a lifting of ϱ to $\mathsf{Rel}^\uparrow$.

3.3 Lax bialgebras and compatibility of contextual closure

As explained in the Introduction, we moved to an order enriched setting because we want to reason about systems for which the saturated transition system forms a lax bialgebra:

▶ **Definition 14.** *Given* $\mathcal{T}, \mathcal{F} \colon \mathsf{Pre} \to \mathsf{Pre}$ *such that there is a distributive law* $\varrho \colon \mathcal{T}\mathcal{F} \Rightarrow \mathcal{F}\mathcal{T}$*, a* lax bialgebra *for* ϱ *consists of a preorder* X*, an algebra* $\alpha \colon \mathcal{T}X \to X$ *and a coalgebra* $\xi \colon X \to \mathcal{F}X$ *such that we have the next lax diagram, with* $\leq$ *denoting the order on* $\mathcal{F}X$*.*

$$
\begin{array}{ccc}
\mathcal{T}X & \xrightarrow{\;\alpha\;} X \xrightarrow{\;\xi\;} & \mathcal{F}X \\
{\scriptstyle \mathcal{T}\xi}\downarrow & {\scriptstyle |\vee} & \uparrow{\scriptstyle \mathcal{F}\alpha} \\
\mathcal{T}\mathcal{F}X & \xrightarrow[\;\varrho_X\;]{} & \mathcal{F}\mathcal{T}X
\end{array}
$$

In this setting, the contextual closure of an up-closed relation is defined by the functor

$$
Ctx \;\triangleq\; \coprod\nolimits_\alpha \circ \,\overline{\mathsf{Pre}(T)}_X \colon \mathsf{Rel}^\uparrow_X \to \mathsf{Rel}^\uparrow_X
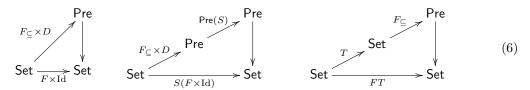$$

where $\overline{\mathsf{Pre}(T)}$ is the lifting of $\mathsf{Pre}(T)$ to $\mathsf{Rel}^\uparrow$ that, by Lemma 11, exists whenever T preserves weak-pullbacks. For any Pre-functor $\mathcal{F}$ and lifting $\overline{\mathcal{F}}$, we can prove $\overline{\mathcal{F}}_\xi$-compatibility of up-to Ctx using the following result which extends Theorem 1 to a lax setting.

▶ **Theorem 15.** *Let* $\mathcal{T}, \mathcal{F}$ *be* Pre*-endofunctors with liftings* $\overline{\mathcal{T}}, \overline{\mathcal{F}}$ *to* $\mathsf{Rel}^\uparrow$*. Assume that* $\varrho \colon \mathcal{T}\mathcal{F} \Rightarrow \mathcal{F}\mathcal{T}$ *is a natural transformation such that there exists a lifting* $\overline{\varrho} \colon \overline{\mathcal{T}}\,\overline{\mathcal{F}} \Rightarrow \overline{\mathcal{F}}\,\overline{\mathcal{T}}$ *of* ϱ*. If* (X, α, ξ) *is a lax* ϱ*-bialgebra, then the functor* $\coprod_\alpha \circ \overline{\mathcal{T}}$ *is* $\overline{\mathcal{F}}_\xi$*-compatible.*

4 Monotone GSOS in an ordered setting

In this section we describe how to obtain a distributive law in Pre and a lax bialgebra from an abstract GSOS specification in Set and a *lax* model for it. The key property is monotonicity (Definition 6) of the abstract GSOS specification.

Let $\lambda\colon S(F \times \mathrm{Id}) \Rightarrow FT$ be an abstract GSOS specification. Suppose F has a stable order given by a factorization through $F_\subseteq \colon \mathsf{Set} \to \mathsf{Pre}$ and let $\subseteq_{FX}$ denote the induced order on FX. Then the functors $F \times \mathrm{Id}$, $S(F \times \mathrm{Id})$ and FT are ordered, as follows:

(6)

where $D\colon \mathsf{Set} \to \mathsf{Pre}$ is the functor assigning to a set the discrete order (Remark 9). Recall that $\mathsf{Pre}_\subseteq(F)$ is the lax Pre-lifting of F with respect to the order given by $F_\subseteq$ and consider the canonical Pre-lifting $\mathsf{Pre}(T)$ of the monad T; then the lax Pre-liftings of the functors $F \times \mathrm{Id}$, $S(F \times \mathrm{Id})$ and FT with respect to the orders in (6) are given by $\mathsf{Pre}_\subseteq(F) \times \mathrm{Id}$, $\mathsf{Pre}(S)(\mathsf{Pre}_\subseteq(F) \times \mathrm{Id})$, respectively $\mathsf{Pre}_\subseteq(F)\mathsf{Pre}(T)$.

If the GSOS specification λ is *monotone* with respect to the orders in (6) (recall Definition 6) then, by Lemma 7, λ lifts to $\dot\lambda\colon \mathsf{Pre}(S)(\mathsf{Pre}_\subseteq(F) \times \mathrm{Id}) \Rightarrow \mathsf{Pre}_\subseteq(F)\mathsf{Pre}(T)$.

If S is a polynomial functor representing a signature, then λ is monotone if and only if for any operator σ (of arity n) we have

$$\frac{b_1 \subseteq_{FX} c_1 \quad \ldots \quad b_n \subseteq_{FX} c_n}{\lambda_X(\sigma(\mathbf{b}, \mathbf{x})) \subseteq_{FTX} \lambda_X(\sigma(\mathbf{c}, \mathbf{x}))} \tag{7}$$

where $\mathbf{b}, \mathbf{x} = (b_1, x_1), \ldots, (b_n, x_n)$ with $x_i \in X$ and similarly for $\mathbf{c}, \mathbf{x}$. When $F = (\mathcal{P}_\omega -)^L$ with the pointwise inclusion order $\subseteq_{(\mathcal{P}_\omega X)^L}$ from Example 5, then condition (7) corresponds to the *positive GSOS* format [10] which, as expected, is GSOS without negative premises.

▶ **Lemma 16.** *A monotone GSOS specification induces a distributive law $\rho\colon T(F \times \mathrm{Id}) \Rightarrow (F \times \mathrm{Id})T$ that lifts to a distributive law $\varrho\colon \mathsf{Pre}(T)(\mathsf{Pre}_\subseteq(F) \times \mathrm{Id}) \Rightarrow (\mathsf{Pre}_\subseteq(F) \times \mathrm{Id})\mathsf{Pre}(T)$.*

▶ **Definition 17.** Let $\lambda\colon S(F \times \mathrm{Id}) \Rightarrow FT$ be a monotone abstract GSOS specification. A lax model for λ is a triple (X, α, ξ) such that the next diagram is lax w.r.t. the order $\subseteq_{FX}$.

(8)

▶ **Example 18.** Consider the GSOS specification λ given in Example 2. Since in the corresponding rules there are no negative premises, it conforms to condition (7), namely it is a positive GSOS specification. Lemma 16 ensures that we have a distributive law $\varrho\colon \mathsf{Pre}(T)(\mathsf{Pre}_\subseteq(F) \times \mathrm{Id}) \Rightarrow (\mathsf{Pre}_\subseteq(F) \times \mathrm{Id})\mathsf{Pre}(T)$.

Recall that ξ_2 is the saturation of the standard semantics of CCS and that (X, α, ξ_2) is not a model for λ, since not *all* the weak transitions of a composite process $p|q$ can be deduced by the ones of the components p and q. However, (X, α, ξ_2) is a lax model. Intuitively, the fact that the inequality (8) holds means that *only* the weak transitions of $p|q$ can be deduced by those of p and q, i.e., $p|q$ contains all the weak transitions that can be deduced from those of p and q and the rules for parallel composition.

By unfolding the definitions of α and $\subseteq_{(\mathcal{P}_\omega X)^L}$, (8) is equivalent to

$$F\alpha^\sharp \lambda_X(\xi_2(p), p, \xi_2(q), q)(\mu) \subseteq \xi_2(p|q)(\mu)$$

for all CCS processes p, q and actions $\mu \in L$. When $\mu = \tau$ (the others cases are simpler) this is equivalent to

$$\{p'|q \mid p \overset{\tau}{\Rightarrow} p'\} \cup \{p|q' \mid q \overset{\tau}{\Rightarrow} q'\} \cup \{p'|q' \mid p \overset{a}{\Rightarrow} p', \, q \overset{\bar{a}}{\Rightarrow} q'\} \subseteq \{r \mid p|q \overset{\tau}{\Rightarrow} r\} \tag{9}$$

which holds by simple calculations. Notice that (9) means exactly that the weak transition system should be closed w.r.t. the rule of the GSOS specification: whenever $\Rightarrow$ satisfies the premises of a rule, then it should also satisfy its consequences.

For a non-example, consider the GSOS rules for the non-deterministic choice of CCS.

$$\frac{p \overset{\mu}{\to} p'}{p + q \overset{\mu}{\to} p'} \qquad \frac{q \overset{\mu}{\to} q'}{p + q \overset{\mu}{\to} q'}$$

This specification is also positive, but the saturated transition system ξ_2 is not a lax model. Intuitively, *not only* the weak transitions of $p + q$ can be deduced by the weak transitions of p and q: indeed from $p \overset{\tau}{\Rightarrow} p$ one can infer that $p + q \overset{\tau}{\Rightarrow} p$ which is not a transition of $p + q$.

The inclusion (9) in the previous example suggests a more concrete characterization for the validity of (8): every transition that can be derived by instantiating a GSOS rule to the transitions in ξ should be already present in ξ, namely, the transition structure is closed under the application of GSOS rules. In contrast to (strict) models (see (4)), in a lax model the converse does not hold: not all the transitions are derivable from the GSOS rules.

Lax models for a monotone GSOS specification λ induce lax bialgebras for the distributive law ϱ obtained as in Lemma 16.

▶ **Lemma 19.** *Let (X, α, ξ) be a lax model for a monotone specification $\lambda \colon S(F \times \mathrm{Id}) \Rightarrow FT$. Then we have a lax bialgebra in* Pre *for the induced distributive law ϱ carried by (X, Δ_X), i.e., the set X with the discrete order, with the algebra map given by $\alpha^\sharp \colon \mathsf{Pre}(T)X \to X$ and the coalgebra map given by $\langle \xi, \mathrm{id} \rangle \colon X \to \mathsf{Pre}_\subseteq(F)X \times X$.*

5   Weak bisimulations up-to context for cool GSOS

We put together the results of Sections 3 and 4 to obtain our main result: if the saturation of a model of a positive GSOS specification is a lax model, then up-to context is compatible for weak bisimulation.

▶ **Theorem 20.** *Let $\lambda \colon S(F \times \mathrm{Id}) \Rightarrow FT$ be a positive GSOS specification. Let ξ_2 be the saturation of an LTS ξ_1. If (X, α, ξ_1) and (X, α, ξ_2) are, respectively, a model and a lax model for λ, then Ctx is $\overline{(\mathsf{Pre}(F) \times \mathsf{Pre}_\subseteq(F)} \times \mathrm{Id})_{\langle \xi_1, \xi_2, \mathrm{id} \rangle}$-compatible.*

Proof. We apply Theorem 15. To this end we have to provide the following ingredients:
1. a distributive law ϱ between Pre-endofunctors;
2. a lax bialgebra for ϱ;
3. a lifting $\overline{\varrho}$ of ϱ between $\mathsf{Rel}^\uparrow$-liftings of the aforementioned functors.

We will explain each step in turn.
1. From a monotone $\lambda \colon S(F \times \mathrm{Id}) \Rightarrow FT$ we first obtain a natural transformation $\tilde{\lambda} \colon S(F \times F \times \mathrm{Id}) \Rightarrow (F \times F)T$ by pairing the natural transformations $\lambda \circ S\langle \pi_1, \pi_3 \rangle \colon S(F \times F \times \mathrm{Id}) \Rightarrow FT$ and $\lambda \circ S\langle \pi_2, \pi_3 \rangle \colon S(F \times F \times \mathrm{Id}) \Rightarrow FT$. Let $G \colon \mathsf{Set} \to \mathsf{Set}$ denote the functor $F \times F \times \mathrm{Id}$. From the GSOS specification $\tilde{\lambda}$ we obtain a distributive law $\rho \colon TG \Rightarrow GT$ in Set. Since λ is monotone w.r.t. the order given by $F_\subseteq$, we have that $\tilde{\lambda}$ can be seen as a monotone abstract GSOS specification for the functor $F \times F$ with the order $\Delta_{FX} \times \subseteq_{FX}$

on $FX \times FX$ given by the product of the discrete order and the one obtained from $F_{\subseteq}$. We consider the Pre-lifting $\mathcal{G}$ of G defined as $\mathcal{G} = \mathsf{Pre}_{\subseteq}(F \times F) \times \mathsf{Id}$ where $\mathsf{Pre}_{\subseteq}(F \times F)$ is the lax Pre-lifting of $F \times F$ w.r.t. the order given above.[2] By Lemma 16 we get a lifting $\varrho : \mathsf{Pre}(T)\mathcal{G} \to \mathcal{G}\mathsf{Pre}(T)$ of ρ, with $\mathsf{Pre}(T)$ the canonical Pre-extension of T.

2. Since (X, α, ξ_1) and (X, α, ξ_2) are, respectively, a model and a lax model for λ, we have

$$
\begin{array}{ccc}
SX \xrightarrow{\ \alpha\ } X \xrightarrow{\ \xi_1\ } FX & & SX \xrightarrow{\ \alpha\ } X \xrightarrow{\ \xi_2\ } FX \\
{\scriptstyle S\langle\xi_1,\mathrm{id}\rangle}\downarrow \qquad\qquad\quad \uparrow{\scriptstyle F\alpha^\sharp} & & {\scriptstyle S\langle\xi_2,\mathrm{id}\rangle}\downarrow \qquad |\vee \qquad \uparrow{\scriptstyle F\alpha^\sharp} \\
S(FX \times X) \xrightarrow{\ \lambda_X\ } FTX & & S(FX \times X) \xrightarrow{\ \lambda_X\ } FTX
\end{array} \quad (10)
$$

Notice that the left model is strict, yet we can also see it as a lax model for the discrete order on F. Hence we can pair the two coalgebra structures to obtain a lax model

$$
\begin{array}{c}
SX \xrightarrow{\ \alpha\ } X \xrightarrow{\ \langle\xi_1,\xi_2\rangle\ } FX \times FX \\
{\scriptstyle S\langle\xi_1,\xi_2,\mathrm{id}\rangle}\downarrow \qquad |\vee \qquad\qquad \uparrow{\scriptstyle F\alpha^\sharp \times F\alpha^\sharp} \\
S(FX \times FX \times X) \xrightarrow{\ \tilde{\lambda}_X\ } (F \times F)TX
\end{array} \quad (11)
$$

for the monotone GSOS specification $\tilde{\lambda}$ considered above. We apply Lemma 19 for the lax model in (11) to obtain a lax bialgebra as in the next diagram with the carrier (X, Δ_X).

$$
\begin{array}{c}
\mathsf{Pre}(T)X \xrightarrow{\ \alpha^\sharp\ } X \xrightarrow{\ \langle\xi_1,\xi_2,\mathrm{id}\rangle\ } \mathcal{G}X \\
{\scriptstyle \mathsf{Pre}(T)\langle\xi_1,\xi_2,\mathrm{id}\rangle}\downarrow \qquad |\vee \qquad\qquad \uparrow{\scriptstyle \mathcal{G}\alpha^\sharp} \\
\mathsf{Pre}(T)\mathcal{G}X \xrightarrow{\ \varrho_X\ } \mathcal{G}\mathsf{Pre}(T)X
\end{array}
$$

3. We consider the $\mathsf{Rel}^\uparrow$ liftings $\overline{\mathsf{Pre}(T)}$ and $\overline{\mathcal{G}}$ of $\mathsf{Pre}(T)$ and $\mathcal{G}$ obtained from Lemma 11, respectively Example 12. Using Proposition 3 we know that the distributive law ρ lifts to a distributive law $\overline{\rho} : \overline{TG} \Rightarrow \overline{GT}$ in Rel. To obtain the lifting of $\overline{\varrho}$ to $\mathsf{Rel}^\uparrow$ we apply Lemma 13 for the liftings $\overline{T}$, $\overline{G}$, $\overline{\mathsf{Pre}(T)}$ and $\overline{\mathcal{G}}$ and the liftings $\overline{\rho}$ and ϱ of ρ to Rel, respectively Pre. ◀

By Remark 9, since the order on X is discrete, we have that $\mathsf{Rel}^\uparrow_X \cong \mathsf{Rel}_X$. Hence the functor *Ctx* is indeed the usual predicate transformer for contextual closure and coalgebras for $(\overline{\mathsf{Pre}(F)} \times \mathsf{Pre}_{\subseteq}(F) \times \mathsf{Id})_{\langle\xi_1,\xi_2,\mathrm{id}\rangle}$ correspond to the usual weak bisimulations.

▶ **Example 21.** Recall from Example 18 that $\to$ and $\Rightarrow$ are, respectively, a model and a lax model for the positive GSOS specification of Example 2. By Theorem 20, it follows that up-to context (for the parallel composition of CCS) is compatible for weak bisimulation.

We can apply Theorem 20 to prove analogous results for the other operators of CCS with the exception of $+$ which is not part of a lax model, see Example 18. More generally, for any process algebra specified by a positive GSOS, one simply needs to check that the saturated transistion systems is a lax model. As explained in Section 4, this means that whenever $\Rightarrow$ satisfies the premises of a rule, it also satisfies its consequence. By [23, Lemma WB],

---

[2] Notice that $\mathcal{G} = \mathsf{Pre}(F) \times \mathsf{Pre}_{\subseteq}(F) \times \mathsf{Id}$ where $\mathsf{Pre}(F)$ and $\mathsf{Pre}_{\subseteq}(F)$ are the canonical, respectively the lax Pre-liftings of F w.r.t. the order given by $F_{\subseteq}$.

this holds for all calculi that conform to the so-called *simply WB cool* format [4], amongst which it is worth mentioning the fragment of CSP consisting of action prefixing, internal and external choice, parallel composition, abstraction and the 0 process ([23, Example 1]).

▶ **Corollary 22.** *For a simply WB cool GSOS language, up-to context is a compatible technique for weak bisimulation.*

6   Conclusion

We have shown that up-to context is compatible (and thus sound) for weak bisimulation whenever the strong and the weak transition systems are a model and a lax model for a positive GSOS specification, as it is the case for calculi adhering to the cool GSOS format [4, 23]. For our proof, we construct a tool-kit of abstract results that can be safely reused for proving compatibility for other coinductive notions. For instance, with our technology it is trivial to show that up-to context is compatible for bisimilarity and similarity for lax models of positive GSOS specifications, while in [6] this was proved just for (strict) models. For dynamic bisimilarity [17], one can use the lifting in (3) with a different saturated transition system that is obtained as in (1) but without the axiom $x \overset{\tau}{\Rightarrow} x$. Then for all the rules of CCS (including +), whenever this system satisfies the premises, it also satisfies its consequence, so it is a lax model; hence up-to context is compatible for dynamic bisimulation. We leave branching bisimilarity [24] for future work.

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A Details for Example 12

Assume we have the following situation

$$
\begin{array}{ccc}
(h,k) & \xrightarrow{\ \overline{F\times F}(R)\ } & (h',k') \\
{\scriptstyle \mathsf{Rel}(F)(\leq)\times\mathsf{Rel}_{\subseteq}(F)(\leq)}\Big| & & \Big|{\scriptstyle \mathsf{Rel}(F)(\leq)\times\mathsf{Rel}_{\subseteq}(F)(\leq)} \\
(f,g) & \xrightarrow{\ \overline{F\times F}(R)\ } & (f',g')
\end{array}
$$

This means that for all $a \in L$ we have the following

$$
\begin{aligned}
(f,g)\ \overline{F\times F}(R)\ (f',g') \quad \Leftrightarrow \quad & \forall x \in f(a).\ \exists y \in g'(a).xRy \\
& \forall x \in f'(a).\ \exists y \in g(a).xRy
\end{aligned}
$$

$$
\begin{array}{cc}
(f,g)\ \mathsf{Rel}(F)(\leq)\times\mathsf{Rel}_{\subseteq}(F)(\leq)\ (h,k) & (f',g')\ \mathsf{Rel}(F)(\leq)\times\mathsf{Rel}_{\subseteq}(F)(\leq)\ (h',k') \\
\Updownarrow & \Updownarrow \\
\forall x \in f(a).\ \exists y \in h(a).x \leq y & \forall x \in f'(a).\ \exists y \in h'(a).x \leq y \\
\forall y \in h(a).\ \exists x \in f(a).x \leq y & \forall y \in h'(a).\ \exists x \in f'(a).x \leq y \\
\forall x \in g(a).\ \exists y \in k(a).x \leq y & \forall x \in g'(a).\ \exists y \in k'(a).x \leq y
\end{array}
\tag{12}
$$

and we need to show

$$
\begin{aligned}
& \forall x \in h(a).\ \exists y \in k'(a).xRy \\
& \forall x \in h'(a).\ \exists y \in k(a).xRy
\end{aligned}
\tag{13}
$$

Using the fact the R is up-closed we can prove this using (12).

▶ Remark. Notice that some of the relations in (12) were not actually used in the proof. In order for the lifting $\overline{F\times F}(R)$ to restrict to up-closed relations, we need to carefully choose the Pre-liftings for $F\times F$. Indeed, we could replace the lifting $\mathsf{Pre}(F)$ with the lax relation lifting given by pointwise reverse inclusion $\mathsf{Pre}_{\supseteq}(F)$. However the proof would break if we would consider instead the Pre-lifting of $F\times F$ given by $\mathsf{Pre}_{\subseteq}(F)\times\mathsf{Pre}_{\subseteq}(F)$, since the functor $\mathsf{Pre}_{\subseteq}(F)\times\mathsf{Pre}_{\subseteq}(F)$ does not have a $\mathsf{Rel}^{\uparrow}$ lifting that also extends $\overline{F\times F}$.